EECS 126 Discussion 10

→ HW9 self grades today!
→ Midterm soon - please feel free to email me for specific advice, etc.
→ Today's problems:
- Gaussians and the MSE
- Bernoulli Hypothesis Testing
- Bayesian Hypothesis Testing

1. Gaussians and the MSE

$$MLE[W \mid (x_i, y_i)] = \arg\max_{w} f_{(X_i, Y_i)}((x_i, y_i) \mid w)$$

$$= \arg\max_{w} \prod_{i=1}^{n} f((x_i, y_i) \mid w) \qquad Y_i \sim \mathcal{N}(wX_i, \sigma^2)$$

$$= \arg\max_{w} \prod_{i=1}^{n} \exp\left(\frac{-(\overset{\mu}{wx_i} - \overset{x}{y_i})^2}{2\sigma^2}\right) \to (y_i - wx_i)^2$$

$$= \arg\max_{w} \sum_{i=1}^{n} \frac{-(wx_i - y_i)^2}{2\sigma^2}$$

$$= \arg\min_{w} \sum_{i=1}^{n} (y_i - wx_i)^2$$

2. Bernoulli Hypothesis Testing

with probability

$$X=0 \rightarrow Y \sim \text{Bern}(\tfrac{1}{4}) \longrightarrow \hat{x}(y) = \begin{cases} 0 \text{ wp } ?, & y=0 \\ 1 \text{ wp } ?, & y=1 \end{cases}$$

$$X=1 \rightarrow Y \sim \text{Bern}(\tfrac{3}{4}) \longrightarrow \hat{x}(y) \Rightarrow \begin{cases} 0 \text{ wp } ?, & y=0 \\ 1 \text{ wp } ?, & y=1 \end{cases}$$

Likelihood ratio $L = \dfrac{P(Y=y \mid X=1)}{P(Y=y \mid X=0)} = \begin{cases} \frac{1/4}{3/4} = \frac{1}{3}, & y=0 \\ \frac{3/4}{1/4} = 3, & y=1 \end{cases}$

Monotonically increasing: $L(x) > c$ declare H_0 $x > c'$

form of decision rule? $L(x) \leq c$ declare H_1 $x \leq c'$

$$\hat{x}(y) = \text{(A)} \begin{cases} 0, & y=0 \\ 1 \text{ wp. } \gamma, & y=1 \end{cases} \quad \text{or} \quad \hat{x}(y) = \text{(B)} \begin{cases} 0 \text{ wp } \gamma, & y=0 \\ 1, & y=1 \end{cases}$$

(A) when we're trying to reduce FPP.

(B) when we're trying to reduce FNP

FPP: $P(\hat{x}(Y)=1 \mid X=0)$

FNP: $P(\hat{x}(Y)=0 \mid X=1)$

$\beta \leq P(Y=1 \mid X=0)$? → yes: FPP too much; → no: bring FNP down

no

$P(Y=1 \mid X=0)$ ~~FPP~~ $\leq \beta$: $\hat{x}(y) = \begin{cases} 0 \text{ wp } \gamma, & y=0 \\ 1, & y=1 \end{cases}$

Set γ s.t. FPP $= \beta$

$P(\hat{x}(Y)=1 \mid X=0) = \beta$

$\frac{1}{4} P(\hat{x}(Y)=1 \mid X=1) + \frac{3}{4} P(\hat{x}(Y)=1 \mid Y=0) = \beta$

$\frac{1}{4} + \frac{3}{4} \cdot (1-\gamma) = \beta \Rightarrow \gamma = \frac{4-4\beta}{3}$

$P(Y=1 \mid X=0) \ngtr \beta$ $\qquad$ $g(y) = \begin{cases} 0, & y=0 \\ 1 \text{ wp } \gamma, & y=1 \end{cases}$

$P(g(Y)=1 \mid X=0) = \beta$

$\frac{1}{4} P(g(Y)=1 \mid \cancel{X}=1) + \frac{3}{4} P(g(Y)=1 \mid \cancel{X}=0) = \beta$

$\frac{1}{4} \gamma + \frac{3}{4} \cdot 0 = \beta$

$\gamma = 4\beta$

$$g(Y) = \begin{cases} \begin{cases} 0 \text{ wp } \frac{4-4\beta}{3}, & y=0 \\ 1, & y=1 \end{cases}, & P(Y=1 \mid X=0) \leq \beta \\ \begin{cases} 0, & y=0 \\ 1 \text{ wp. } 4\beta, & y=1 \end{cases}, & P(Y=1 \mid X=0) > \beta \end{cases}$$

3. $E[I\{g(Y) \neq X\}] = P(g(Y) \neq X)$

$= \frac{1}{2} P(g(Y)=1 \mid X=0) + \frac{1}{2} P(g(Y)=0 \mid X=1)$

$X = 0 \rightarrow Y \sim \text{Bern}\left(\frac{1}{4}\right)$

$X = 1 \rightarrow Y \sim \text{Bern}\left(\frac{3}{4}\right)$

$\hat{x}: \{0,1\} \rightarrow \{0,1\}$: min FNP given FPP $\leq \beta$.

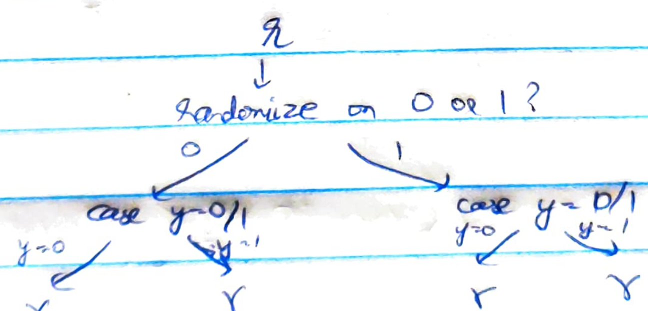
$\hat{x}(Y)$ is as close as possible to X?

N-P lemma: says it turns on $L(y) = \dfrac{f(y|H_1)}{f(y|H_0)}$

$x^3 > 27 \iff x > 3$

example 3 in notes: $L(x) = \exp\left(\dfrac{2x-1}{2\sigma^2}\right) > c$

might as well say $x > t$.



$$\hat{x}_A(y) = \begin{cases} \begin{cases} 0 & \text{wp } 1 \\ 1 & \text{wp } 0 \end{cases}, & y = 0 \\ \begin{cases} 0 & \text{wp } 1-\gamma \\ 1 & \text{wp } \gamma \end{cases}, & y = 1 \end{cases}$$