EECS 126 Dis 11

Today: ~~either Fa19 MT2 walkthrough,~~

or disc worksheet.

1. Exponential Hypothesis
2. JG + Characteristic Fns
3. JG Probability

1. $X \in \{1, a\}$, $a > 1$ const.

max $P(\hat{X}=1 \mid X=1)$

sub. to $P(\hat{X}=1 \mid X=a) < 0.05$

likelihood ratio $\longrightarrow$ form of N-P test $\rightarrow$ equality on PFA $\rightarrow$ solve

$$L = \frac{f_{Y|X}(y \mid X=1)}{f_{Y|X}(y \mid X=a)} = \cancel{\frac{ae^{-ay}}{e^{-y}}} \quad \frac{e^{-y}}{ae^{-ay}} = \frac{1}{a} e^{-y(1-a)}$$

$$\hat{X}(y) = \begin{cases} a, & y < y^* \\ 1, & y \geq y^* \end{cases}$$

$$\begin{aligned} P(\hat{X}=1 \mid X=a) &= P(Y \geq y^* \mid X=a) \\ &= 1 - (1 - e^{-ay^*}) \\ &= e^{-ay^*} \quad = 0.05 \end{aligned}$$

$$y^* = \frac{\ln(20)}{a} = \frac{-\ln(0.05)}{a}$$

2. a) $\varphi_Z(t) = E[\exp(i\langle t, Z\rangle)] = E[\exp(i\sum_{j=1}^n t_j Z_j)]$

$$Z = \begin{bmatrix} N(0,1) \\ N(0,1) \\ \vdots \\ N(0,1) \end{bmatrix}_{n\times 1} \quad \text{- independent}$$

$E[Y_1 Y_2 \cdots Y_n] = E[Y_1]E[Y_2]\cdots E[Y_n]$

$$\varphi_Z(t) = \prod_{j=1}^n E[\exp(i t_j Z_j)]$$

$$= \prod_{j=1}^n \exp\left(\frac{-1}{2} t_j^2\right)$$

$E[\exp(i\sum_{j=1}^n t_j Z_j)]$

$= E[\exp(i(t_1 Z_1 + t_2 Z_2 + \cdots + t_n Z_n))]$

$= E[\exp(it_1 Z_1)\exp(it_2 Z_2)\cdots\exp(it_n Z_n)]$

$= E[\exp(it_1 Z_1)]\,E[\exp(it_2 Z_2)]\cdots$

$$= \exp\left(\frac{-1}{2}\sum_{j=1}^n t_j^2\right) = \exp\left(\frac{-1}{2}\langle t,t\rangle\right) = \exp\left(\frac{-1}{2}t^T t\right).$$

b) $X = \overset{\text{const}}{A}\overset{N(0,1)\text{s}}{Z} + \overset{\text{constant}}{\mu}$

$$\varphi_X(t) = E[\exp(i\langle t, X\rangle)]$$

$$= E[\exp(i\langle t, \mu + AZ\rangle)]$$

$$= E[\exp(i\langle t,\mu\rangle)] \cdot E[\exp(i\langle t, AZ\rangle)]$$

$\langle t, AZ\rangle = \langle A^T t, Z\rangle$

$= \sum_i t_i (AZ)_i = \sum_i \sum_j t_i A_{ij} Z_j = \sum_j \underbrace{\sum_i (A^T_{ji} t_i)}_{A^T t} Z_j$

$$\varphi_X(t) = \exp(i\langle t,\mu\rangle)\, E[\exp(i\langle A^T t, Z\rangle)] \quad (A^T t)^T = t^T A$$

$$= \exp(i\langle t,\mu\rangle)\exp\left(\frac{-1}{2} t^T \underbrace{AA^T}_{C} t\right)$$

From (a)

$E[\exp(i\langle t, Z\rangle)] = \exp(\frac{-1}{2} t^T t)$

$E[\exp(i\langle A^T t, Z\rangle)]$

$t \to A^T t$

$t^T t \to (A^T t)^T A^T t = t^T \underbrace{AA^T}_{C} t = t^T C t$

3. $X \sim N(1,1)$, $Y \sim N(0,1)$ <u>marginally</u>.

i.e. there's some $f_{X,Y}(x,y)$; $X, Y \sim N\left(\begin{bmatrix}1\\0\end{bmatrix}, \Sigma\right)$

$X \sim N(1,1)$ if you integrate over Y : $f_X(x) = \int_y f_{X,Y}(x,y)dy$
and same for Y.
$= \text{PDF of } N(1,1)$

$\bar{X} = X - 1 \sim N(0,1)$

$\text{cov}(\bar{X}, Y) = \rho$

$Y = \rho \bar{X} + \sqrt{1-\rho^2}\, Z$ → to check : $E[Y] = 0$ as $E[X] = E[Z] = 0$.

$\text{var}(Y) = \rho^2 \cdot \text{var}(X) + (1-\rho^2)\text{var}(Z)$

$= \rho^2 + 1 - \rho^2 = 1.$

indep $N(\mu_1, \sigma_1^2) + N(\mu_2, \sigma_2^2)$

$= N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$P(X > Y) = \cancel{P(\bar{X} > Y+1)}\ P(\bar{X} > Y - 1)$

$= P(\bar{X} > \rho \bar{X} + \sqrt{1-\rho^2}\, Z - 1)$

$= P((1-\rho)\bar{X} > \sqrt{1-\rho^2}\, Z - 1)$

$\rightarrow P\left((1-\rho)\bar{X} > \sqrt{1-\rho^2}\, Z - 1\right)$

$\bar{X} \sim N(0,1)$ $\qquad$ $Z \sim N(0,1)$

$(1-\rho)\bar{X} \sim N(0, (1-\rho)^2)$ $\qquad$ $\sqrt{1-\rho^2}\, Z \sim N(0, 1-\rho^2)$

$(1-\rho)\bar{X} - \sqrt{1-\rho^2}\, Z \sim N(0, 2(1-\rho))$

$\rightarrow P\left(N(0, 2(1-\rho)) > -1\right)$

$= P\left(N(0,1) > \dfrac{-1}{\sqrt{2(1-\rho)}}\right)$

$= \Phi\left(\dfrac{1}{\sqrt{2(1-\rho)}}\right)$