EECS 126 Discussion 12

→ fill out the post MT2 survey! (Due tomorrow night)

→ Today: a lot on LLSE.

1. MMSE for Jointly Gaussian RVs
2. Orthogonal LLSE
3. Linear Regression

orthogonality: $X \perp Y \leftrightarrow E[XY] = 0$.

$L[X|Y]$ is LLSE $\leftrightarrow X - L[X|Y] \perp \{1, Y\}$.

$g(Y) = L[X|Y]$

MMSE is orthogonal to all functions of Y.

$MMSE[X|Y] = E[X|Y]$

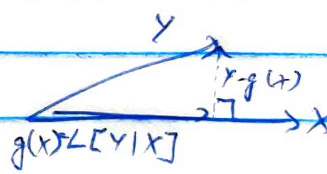
and $E[E[X|Y]\, g(Y)] = 0$

$\forall g(\cdot)$.

1. $g(X) = L[Y|X] = \cancel{E[X]}\ E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X])$

(1) WTS: $E[(Y - g(X))X] = 0 \leftrightarrow Y - g(X) \perp X$

True because LLSE: residual $\perp X$.



(2) WTS $cov(Y - g(X), X) = 0$.

$cov(X,Y) = E[XY] - E[X]E[Y]$

$= E[(Y - g(X))X] - E[Y - g(X)]E[X]$ $\neq 0$ in general

$= \underset{\text{by eq 1.}}{0} - 0 = 0$

LLSE is unbiased

$E[g(X)] = E[Y]$

$E[E[Y] + (\ldots)(\cancel{E[X] - E[X]})]$

$= E[Y]$.

$\perp$ othogonal , $\perp\!\!\!\perp$ indep.

(3) $\mathrm{cov}(Y - g(X), X) = 0 \longrightarrow Y - g(X) \perp\!\!\!\perp X$

true for any RVs? No.

true for J Gaussians? Yes.

X, Y are JG

Are $X, \underline{Y - g(X)}$ JG? Yes

linear combination of X, Y.

$X = A_1 Z_1 + \mu_1$

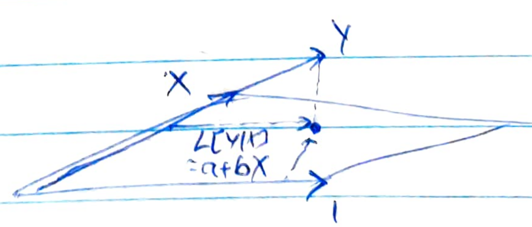
$Y = A_2 Z_2 + \mu_2$

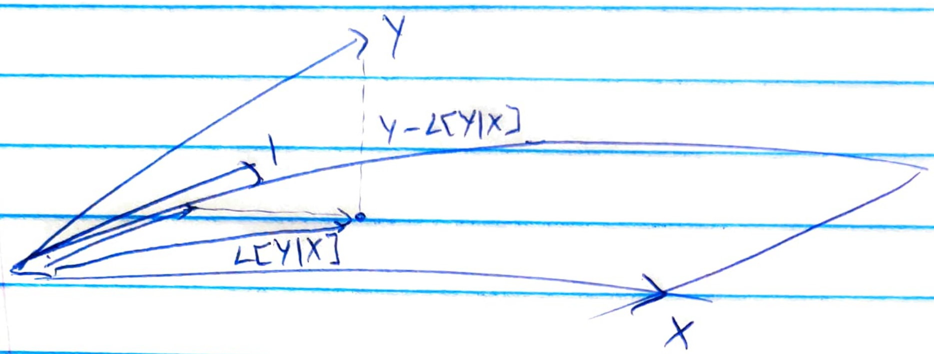
(4) $Y - g(X) \perp f(X)$ for all $f(\cdot)$.

$$E[(Y-g(X))f(X)] = \underbrace{E[(Y-g(X))]}_{0} \, E[f(X)] = 0.$$

(5) MMSE is the unique function $h(X)$ such that

$Y - h(X) \perp f(X)$ for all $f(\cdot)$





$Y - L[Y|X] \perp 1$

$Y - L[Y|X] \perp X$.

2. Orthogonal LLSE.

$X - L[X|Y, Z]$ has to be orthogonal to all of $\{1, Y, Z\}$.

idea: Y and Z give you "independent information" about X.

$L[X|Y]$ should be independent of / orthogonal to Z

and $L[X|Z]$ should be independent of / orthogonal to Y.

Is their sum's residual orthogonal to both?

$(a + bZ)$

$E[Z \cdot L[X|Y]] = 0$, $E[Y \cdot L[X|Z]] = 0$.

$g(Y, Z) = L[X|Y] + L[X|Z]$.

WTS $g(Y, Z) = L[X|Y, Z] \leftrightarrow X - g(Y, Z) \perp \{1, Y, Z\}$

$X - g(Y, Z) \perp 1 \leftrightarrow E[(X - g(Y, Z)) \cdot 1] = 0$.

$A \perp B$ $\quad E[X - L[X|Y] - L[X|Z]] = 0$.

$E[AB] = 0$ $\quad E[X] - E[L[X|Y]] - E[L[X|Z]] = 0$

$0 - 0 - 0 = 0$

$$g(Y, Z) = L[X|Y] + L[X|Z]$$

$$X - g(Y, Z) \perp Y \iff E[(X - L[X|Y] - L[X|Z])Y] = 0$$

$$E[(X - L[X|Y]) \cdot Y] - E[L[X|Z]\, Y] = 0$$

↳ 0 because it's an LLSE

↳ 0 because $Y \perp\!\!\!\perp Z$.

Similarly, $X - g(Y, Z) \perp Z$.

$\therefore$ $X - g(Y,Z) \perp \{1, Y, Z\}$

$g(Y, Z) = L[X|Y, Z]$.

$$X \perp Y \iff E[XY] = 0$$

$$X - g(Y) \perp 1 \iff E[(X - g(Y)) \cdot 1] = 0$$

$$E[X] = E[g(Y)]$$

b) $L[X|Y, Z] = L[X|Y] + L[X|\underbrace{Z - L[Z|Y]}_{W}]$

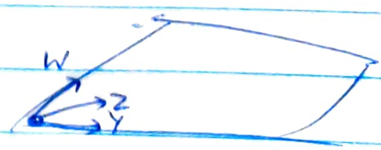
if Y, Z indep $L[Z|Y] = E[Z]$.

$\therefore L[X|Y, Z] = L[X|Y] + L[X|Z - E[Z]]$

$= L[X|Y] + L[X|Z]$.

Say $W = Z - L[Z|Y]$

$Y \perp W$ (LLSE property).

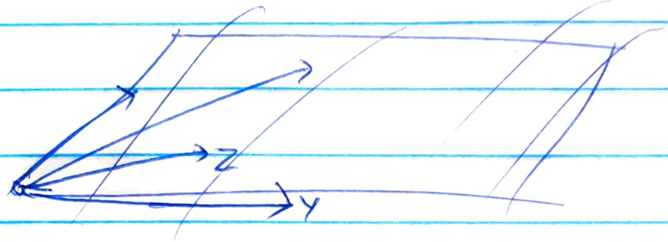


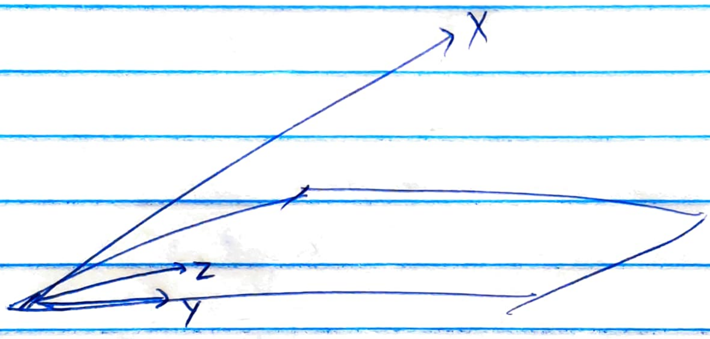
$$L[X|Y, W] = L[X|Y] + L[X|W]$$

$$= \text{"} + L[X|Z - L[Z|Y]]$$

why is this $L[X|Y, Z]$? Yes: $(Y, W) \iff (Y, Z)$ reversible.

$Z \to$ something w/ only the info unique to Z.





$E[YZ] \neq 0$ - correlated as RVs
not orthogonal as vectors

$W = Z - L[Z|Y]$.

$L[X|Y,W] = L[X|Y] + L[X|W]$

$L[X|Y,Z] = L[X|Y,W]$.