EECS 126 Discussion 13

now in high-tech form!

1. Balls in Bins Estimation
2. Gaussian Estimation
3. Forrest Gump (Kalman Filter Derivation)

1. Balls in Bins

a) $E[Y|X]$

$n-X$ balls left over

$m-1$ bins possible

$E[Y|X] = aX^3+b$

$$E[Y|X] = \frac{n-X}{m-1}$$

$L[Y|X] \neq E[Y|X]$

b) $L[Y|X] = (n-X)/(m-1)$

$Q[Y|X] = L[Y|X] = E[Y|X]$

c) $E[X]$, $E[Y]$

$$E[X] = E[Y] = \frac{n}{m}$$

d) $var(X)$

$$X \sim Binom\left(n, \frac{1}{m}\right)$$

$$var(X) = n \cdot \frac{1}{m}\left(1 - \frac{1}{m}\right)$$

e) $cov(X, Y)$

indicator variables

$$E[XY] - E[X]E[Y]$$

$X_i : 1\{i \text{ in bin } 1\}$

$Y_i : 1\{i \text{ in bin } 2\}$

$$E[XY] = E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_i Y_j\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} E[X_i Y_j]$$

$i=j$: 0 , ball i can't be in 1 and 2.

$$= (n^2-n)\cdot\frac{1}{m^2}$$

$$\mathrm{cov}(X,Y) = \frac{n^2-n}{m^2} - E[X]^2$$

$$= \frac{n^2-n}{m^2} - \left(\frac{n}{m}\right)^2 = \boxed{\frac{-n}{m^2}}$$

f) $L[Y|X] = E[Y] + \dfrac{\mathrm{cov}(X,Y)}{\mathrm{var}\, X}(X-E[X])$

$$= \frac{n}{m} + \frac{-\cancel{n}/m^{\cancel{2}}}{\cancel{n}\cdot\frac{1}{\cancel{m}}\cdot\left(1-\frac{1}{m}\right)}\left(X-\frac{n}{m}\right)$$

$$= \left(\frac{n}{m} - \frac{n}{m}\cdot\frac{1}{m\left(1-\frac{1}{m}\right)}\right) - \frac{X}{\cancel{m\left(1-\frac{1}{m}\right)}}$$

$m-1$

$$= \frac{n-X}{m-1}$$

LLSE : $\text{proj}_{\{1, Y\}} X$

QLSE : $\text{proj}_{\{1, Y, Y^2\}} X$ - never really comes up other than if QLSE = MMSE

3. Forrest Gump

a) $E[X(0) \mid Y(0)] = L[X(0) \mid Y(0)]$

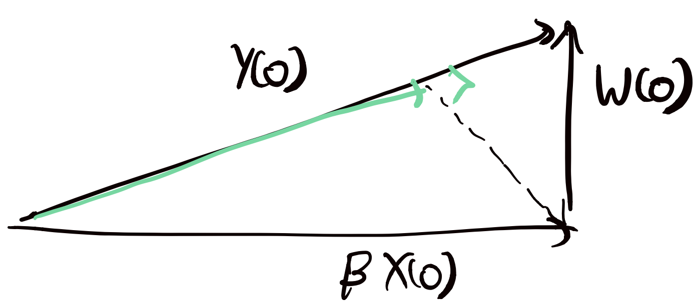
$$= E[X(0)] + \frac{\text{cov}(X(0), Y(0))}{\text{var}\, Y(0)} (Y(0) - E[Y(0)])$$

$$\text{cov}(X_0, Y_0) = E[X_0 Y_0] - \underbrace{\cancel{E[X_0]}}_{0} E[Y_0]$$

$$= E[X_0(\beta X_0 + W_0)]$$

$$= \beta E[X_0^2] + \underbrace{\cancel{E[X_0 W_0]}}_{0}$$

$$= \beta \sigma_X^2$$



$$\text{var } Y_0 = \beta^2 \text{ var } X_0 + \text{var } W_0 = \beta^2 \sigma_X^2 + \sigma_W^2$$

$$E[X_0 | Y_0] = \frac{\beta \sigma_X^2}{\beta^2 \sigma_X^2 + \sigma_W^2} Y(0)$$