EECS 126 Discussion 3

→ feedback and quick poll

→ this week (+ prob. next week) I've got an extra OH, so Wed 2-5 PM

→ remember to do self-grades today!

→ problem-solving! 3 problems, about 15 mins each.

- Poisson Merging - related to HW3 Q3, Q4
- Triangle Density - related to HW3 Q1
- Change of Variables - good PDF/CDF practice, e.g. for HW3 Q5

1. Poisson: "binomial with low probability, and every instant is a trial"

$X \sim Pois(\lambda) \Rightarrow P(X=k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

$Y \sim Pois(\mu)$

Want to show $X+Y \sim Pois(\lambda+\mu)$, i.e. $P(X+Y=z) = \dfrac{e^{-(\lambda+\mu)}(\lambda+\mu)^z}{z!}$

$$P(X+Y=z) = \sum_{i=0}^{z} P(X=i)\,P(Y=z-i)$$

$$= \sum_{i=0}^{z} \frac{e^{-\lambda}\lambda^i}{i!} \frac{e^{-\mu}\mu^{z-i}}{(z-i)!}$$

$$= e^{-(\lambda+\mu)} \sum_{i=0}^{z} \frac{\lambda^i \mu^{z-i}}{i!(z-i)!} \cdot \frac{z!}{z!}$$

$$= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{i=0}^{z} \lambda^i \mu^{z-i} \frac{z!}{i!(z-i)!} = \frac{e^{-(\lambda+\mu)}}{z!} \sum_{i=0}^{z} \lambda^i \mu^{z-i} \binom{z}{i}$$

$z = 3$

$P(X+Y = 3)$

X	Y
0	3
1	2
2	1
3	0
~~4~~	~~-1~~

$\boxed{\text{xkcd.com/121}}$

$\boxed{\text{Poisson is} \geq 0!}$

$$P(X+Y=z) = \frac{e^{-(\lambda+\mu)}}{z!}(\lambda+\mu)^z \underbrace{\sum_{i=0}^{z} \frac{\lambda^i \mu^{z-i}}{(\lambda+\mu)^z}\binom{z}{i}}$$

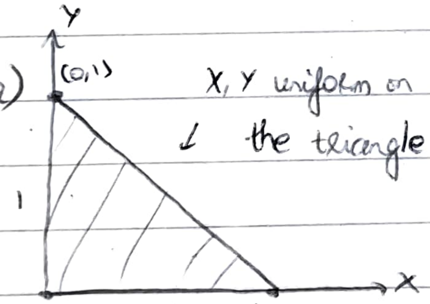
$$\sum_{i=0}^{z} \left(\frac{\lambda}{\lambda+\mu}\right)^i \left(\frac{\mu}{\lambda+\mu}\right)^{z-i} \binom{z}{i}$$

$$= 1$$

$$= \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^z}{z!}$$

$\therefore\ X+Y \sim \text{Pois}(\lambda+\mu)$

2. a)



X, Y uniform on the triangle

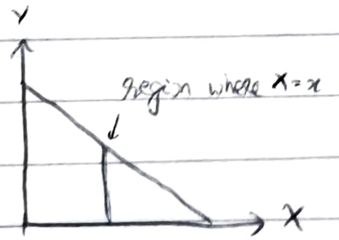
$\Rightarrow f_{X,Y}(x,y)$ is a constant

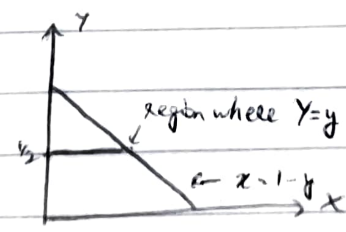
$$\iint f_{X,Y}(x,y)\,dx\,dy = 1$$

$$f_{X,Y}(x,y) \underbrace{\iint dx'\,dy'}_{1/2} = 1 \quad \forall x,y$$

$$\boxed{f_{X,Y}(x,y) = 2 \quad \forall x,y.}$$

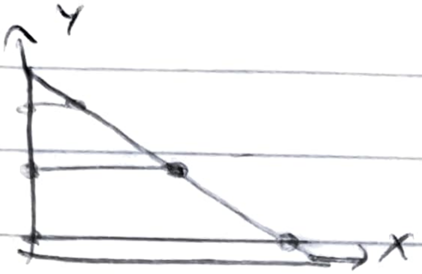
b) Marginal PDF of Y : $f_Y(y)$





$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = 2\int_0^{1-y} dx = \boxed{2(1-y)}$$

c) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{2}{2(1-y)} = \dfrac{1}{1-y}$, $0 \le x \le 1-y$



$y = 0 \Rightarrow X \sim U[0,1]$

$y = 1 \Rightarrow X = 0$

a) but easier to read

$$\iint f_{X,Y}(x,y)\,dx\,dy = 1$$

$$f_{X,Y}(x,y) \underbrace{\iint dx'dy'}_{\text{area of the triangle}} = 1 \quad \forall x, y$$

$$f_{X,Y}(x,y) \cdot \frac{1}{2} = 1 \quad \forall x, y$$

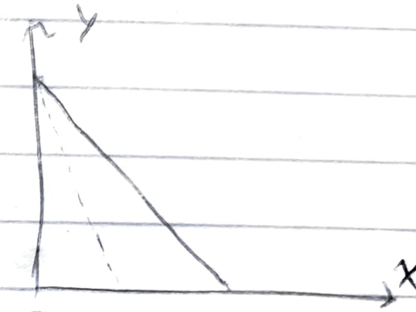
$$f_{X,Y}(x,y) = 2 \quad \forall x, y.$$

2. d), e)

Two relationships b/w $E[X]$ and $E[Y]$

The easy one: $E[X] = E[Y]$ – the triangle is symmetric.

The harder one: iterated expectation.



The dotted line indicates $E[X|Y]$, want an avg of that weighted by X's prob. mass

$$E[X] = E[E[X|Y]]$$

Let's unpack what this means: isn't an expectation a number?

What does $E[E[\cdots]]$ mean?

$E[X|Y]$: "given we know Y, best guess at X" – MMSE, later
"given we know Y, expectation of X"

Critical point: $E[X|Y]$ is a function of Y!
Expectation of that averages over all Y.

In the discrete case it's just $E[E[X|Y]] = \sum_{y \in \mathcal{Y}} E[X|Y=y]\, P(Y=y)$

Continuous: $E[E[X|Y]] = \int_0^1 dy\, E[X|Y=y]\, f_Y(y)$

From (b), $f_Y(y) = \frac{1-y}{2}$

$$E[X] = \int_0^1 dy \left(\frac{1}{2} - \frac{y}{2}\right) f_Y(y) = \frac{1}{2}\underbrace{\int_0^1 dy \cdot f_Y(y)}_{1,\ \text{pdf norm.}} - \frac{1}{2}\underbrace{\int_0^1 dy \cdot y \cdot f_Y(y)}_{E[Y]}$$

$\therefore \boxed{E[X] = \frac{1}{2} - \frac{E[Y]}{2}}$ (d) and $E[X] = E[Y]$.

$\frac{3}{2} E[X] = \frac{1}{2} \Rightarrow \boxed{E[X] = E[Y] = \frac{1}{3}}$ (e)

3. Change of Variables

a) $$f_{\exp X}(x) = \frac{d}{dx} P(\exp X \leq x)$$

$$\boxed{\text{FTC: } \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x)}$$

$$= \frac{d}{dx} P(X \leq \log x)$$

$$= \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\log x} \exp\left(\frac{-x'^2}{2}\right) dx'\right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{d(\log x)}{dx} \frac{d}{d\log x} \int_{-\infty}^{\log x} \exp\left(\frac{-x'^2}{2}\right) dx'$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot \exp\left(-\frac{(\log x)^2}{2}\right)$$

$$f_{X^2}(x) = \frac{d}{dx} P(X^2 \leq x) = \frac{d}{dx} P(-\sqrt{x} \leq X \leq \sqrt{x})$$

$$= \frac{d}{dx}\left(F_X(\sqrt{x}) - F_X(-\sqrt{x})\right) = \frac{d\sqrt{x}}{dx} \cdot \frac{d}{d\sqrt{x}}\left(F_X(\sqrt{x}) - F_X(-\sqrt{x})\right)$$

$$= \frac{1}{2\sqrt{x}}\left(f_X(\sqrt{x}) + f_X(-\sqrt{x})\right).$$

$(-$ from outside$) \times (-$ from $\boxed{-\sqrt{x}}\,)$

c) use the answer to (b) with the standard normal pdf.

$$f_{X^2}(x) = \frac{1}{2\sqrt{x} \cdot \sqrt{2\pi}}\left(\exp\left(-\frac{(\sqrt{x})^2}{2}\right) + \exp\left(-\frac{(-\sqrt{x})^2}{2}\right)\right)$$

$$= \frac{1}{\sqrt{2\pi x}} \cdot \frac{1}{2} \cdot 2 \exp\left(\frac{-x}{2}\right)$$

$$= \boxed{\frac{1}{\sqrt{2\pi x}} \exp\left(\frac{-x}{2}\right)}$$