EECS 126 Discussion 4

→ remember self-grades today!

→ Good luck on the midterm! Come to OH for help! (Mine: W 2–5)

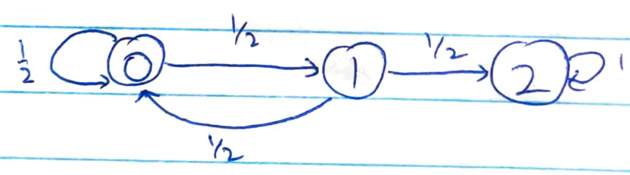
→ Today's problems:

- Hitting Time with Coins — application
- Reducible Markov Chains — definitions
- Random Walk on an Undirected Graph — theory/properties

1. Hitting Time with Coins.

a) First, set up Markov chain: want a state definition
+ rules for state transitions.

Next, we're interested in: expected time to one of the states,
starting from some other state.



HT $T\boxed{HH}$

$\beta(i) = E[T_2 \mid X_0 = i]$

$\beta(2) = 0$

$\beta(1) = 1 + \frac{1}{2}\beta(0) + \frac{1}{2}\beta(2)$

$\beta(0) = 1 + \frac{1}{2}\beta(0) + \frac{1}{2}\beta(1)$

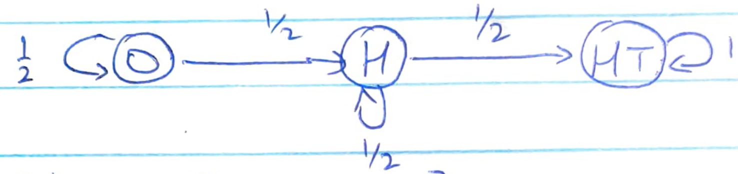
$\beta(0) = 6, \quad \beta(1) = 4.$

b) Markov chain + transition rules

Is the state space the same as (a)?

... HH HT ✓

... TT HT



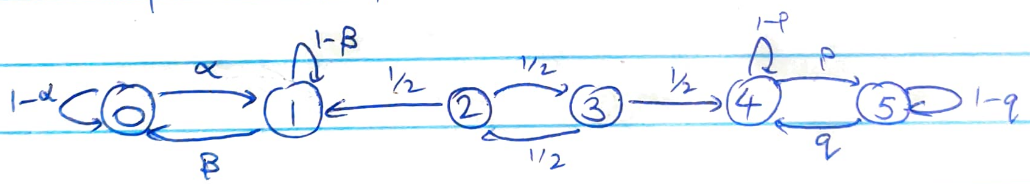
$\beta(i) = E[T_{HT} \mid X_0 = i]$

$\beta(HT) = 0$

$\beta(H) = 1 + \frac{1}{2}\beta(H) + \frac{1}{2}\beta(HT)$

$\beta(0) = 1 + \frac{1}{2}\beta(0) + \frac{1}{2}\beta(H)$

$\beta(H) = 2, \ \beta(0) = 4.$

2. 

a) 0 recurrent, 1 recurrent

{0, 1} recurrent class

{4, 5} recurrent class

{2, 3} transient class.

b) $\alpha(i) = P(T_0 < T_5 \mid X_0 = i)$; $\alpha(0) = \alpha(1) = 1$

$\alpha(4) = \alpha(5) = 0$

$\alpha(3) = \frac{1}{2}\alpha(2)$

$\alpha(2) = \frac{1}{2}\alpha(1) + \frac{1}{2}\alpha(3) = \frac{1}{2} + \frac{1}{2}\alpha(3)$

~~α~~

$$\alpha(2) = \frac{1}{2} + \frac{1}{2}\cdot\left(\frac{1}{2}\alpha(2)\right) \Rightarrow \frac{3}{4}\alpha(2) = \frac{1}{2} \Rightarrow \alpha(2) = \frac{2}{3}$$

$$\alpha(3) = \frac{1}{2}\alpha(2) = \frac{1}{3}$$

c) Recall: $0, 1, 4, 5$ recurrent

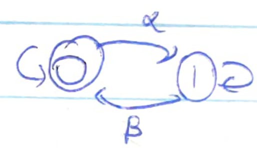
$2, 3$ transient

Let the stationary dist. be π.

$\pi(2) = 0$

$\pi(3) = 0$

$\{0,1\}$: $\pi_{(0,1)} = \frac{1}{\alpha+\beta}[\beta, \alpha]$



$\{4,5\}$: $\pi_{(4,5)} = \frac{1}{p+q}[q, p]$ "conditioned stationary dist."

"gluing/parameter" c : $0 \le c \le 1$
weighting

$$\pi = c\,\pi_{(0,1)} + (1-c)\,\pi_{(4,5)}$$

$P(A) = \sum_B P(A|B)P(B)$

d) γ specifies the undetermined weight c.

$c = P(\text{being in } \{0,1\}) = \frac{2}{3}\gamma + \frac{1}{3}(1-\gamma) = \boxed{\frac{1}{3}\gamma + \frac{1}{3}}$

$1-c = P(\text{being in } \{4,5\}) =$

$$\left[\frac{c\beta}{\alpha+\beta}, \frac{c\alpha}{\alpha+\beta}, 0, 0, \frac{(1-c)q}{p+q}, \frac{(1-c)p}{p+q}\right]$$

3. Random Walk on an Undirected Graph

Intuition: $\pi(v) \propto$ # of ways to get to v

$= \deg v$

Movement is uniform so there's $\deg v$ paths into v, one per neighbor.

Normalize:

$$\pi(v) = \frac{\deg v}{\sum_{v'} \deg v'}.$$

How do we show this is the stationary dist? Want to show it doesn't change over a single timestep, i.e.

$\pi_{n-1}(u)$ is the dist at time $n-1$ $\Rightarrow$ $\pi(u)$ is the dist at time n

Want: $P(X_n = v) = \dfrac{\deg v}{\sum_{v'} \deg v'}$.

Use total probability:

$$P(X_n = v) = \sum_{u \in \mathcal{X}} \underbrace{P(X_n = v \mid X_{n-1} = u)}_{\frac{1}{\deg u} \text{ if } u - v,\ 0 \text{ otherwise.}} \underbrace{P(X_{n-1} = u)}_{\pi(u) \text{ by assumption}}$$

$$= \sum_{\substack{u \mid u - v, \\ u \in \mathcal{X}}} \frac{1}{\deg u} \pi(u) = \sum_{\substack{u \mid u - v, \\ u \in \mathcal{X}}} \frac{1}{\deg u} \cdot \frac{\deg u}{\sum_{v'} \deg v'}$$

$$= \frac{1}{\sum_{v'} \deg v'} \boxed{\sum_{\substack{u \mid u - v, \\ u \in \mathcal{X}}} 1} \leftarrow \deg v \text{ by definition}$$

$$= \frac{\deg v}{\sum_{v'} \deg v'}, \text{ what we wanted!}$$