EECS 126 Discussion 5

→ ~~self grades today!~~
→ thanks for post-MT1 survey feedback! + ask for a TA one-on-one if you want!
→ problems:

- convergence of exponentials — conv. in prob. definition
- Breaking a Stick — conv. a.s. application
- r^{th} moment convergence — definitions / theory

1. Convergence of Exponentials

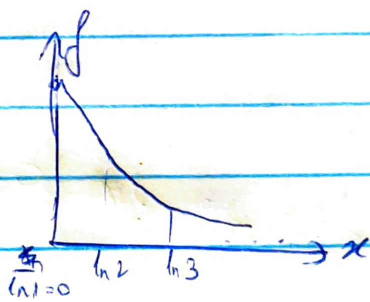
hint: CDF of $Exp(\lambda)$ is $F(x) = 1 - e^{-\lambda x}$

Def. of convergence in probability:

$$X_n \to X \text{ i.p. if } \lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0 \quad \forall \epsilon > 0$$

WTS:

$$\frac{X_n}{\ln n} \to 0, \quad X_n \sim Exp(\lambda)$$



WTS:

$$\lim_{n\to\infty} P\left(\left|\frac{X_n}{\ln n}\right| \geq \epsilon\right) = 0 \quad \forall \epsilon > 0$$

$$= \lim_{n\to\infty} P(X_n \geq \epsilon \ln n) = 0 \quad \forall \epsilon > 0$$

Recall $F_{X_n}(x) = P(X_n \leq x) = 1 - e^{-\lambda x}$

$$\Rightarrow P(X_n \geq x) = e^{-\lambda x}$$

$a \ln b = \ln b^a$

$$\therefore P(X_n \geq \epsilon \ln n) = e^{-\lambda \epsilon \ln n} = n^{-\lambda\epsilon}$$

$$\lim_{n\to\infty} P(X_n \geq \epsilon \ln n) = \lim_{n\to\infty} n^{-\lambda\epsilon} = 0 \Rightarrow \frac{X_n}{\ln n} \to 0 \text{ i.p.}$$

2. Breaking a Stick

a) $P_n^{1/n} \to ?$ $\quad P_n = X_1 \cdot X_2 \cdot \ldots \cdot X_{n-1} \cdot X_n$, $X_i \sim U[0,1]$ iid.

Continuous mapping theorem: $X_n \to X$ a.s., g continuous (w.p. 1)

$\Rightarrow g(X_n) \to g(X)$ a.s.

use natural log: turns products to sums.

$\ln a^b = b \ln a$

$$\ln P_n^{1/n} = \ln\left(\prod_{i=1}^n X_i^{1/n}\right) = \sum_{i=1}^n \ln X_i^{1/n} = \frac{1}{n}\sum_{i=1}^n \ln X_i$$

SLLN: $\frac{1}{n}\sum_{i=1}^n Y_i \xrightarrow{a.s.} \underline{E[Y_1]}$ if Y_i iid

$$E[\ln X_1] = \int_0^1 \ln x \cdot 1 \, dx = -1.$$

$$\ln P_n^{1/n} \xrightarrow{a.s.} -1 \Rightarrow P_n^{1/n} \xrightarrow{a.s.} e^{-1}.$$

$$\frac{1}{n} \to 0, \quad (\cdot)^{1/n} \to (\cdot)^0 \to 1$$

b) $$E[P_n]^{1/n} = E\left[\prod_{i=1}^n X_i\right]^{1/n} \underset{\text{indep.}}{=} \prod_{i=1}^n \left(E[X_i]\right)^{1/n}$$

$$\underset{\text{ident.}}{=} \left(\left(E[X_i]\right)^n\right)^{1/n} = E[X_i] = \frac{1}{2}.$$

$$e^{E[\ln P_n]} \neq E[P_n], \text{ in general}$$

f convex

$$f(E[X]) \leq E[f(X)] \quad \text{Jensen's inequality.}$$

3. Convergence in r^{th} moment.

$$\begin{tikzcd}
\text{a.s.} \arrow[dr] & & r^{th}\text{ moment} \arrow[dl, "\leftarrow \text{WTS this}"'] \\
& \text{in prob} \arrow[d] & \\
& \text{in dist} &
\end{tikzcd}$$

$$\lim_{n\to\infty} E[|X_n - X|^r] = 0.$$

Fix $\epsilon > 0$

WTS: $\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$

WTS: $\lim_{n\to\infty} P(|X_n - X|^r \geq \epsilon^r) = 0$

Markov's:

$$\lim_{n\to\infty} P(|X_n - X|^r \geq \epsilon^r) \leq \lim_{n\to\infty} \frac{E[|X_n - X|^r]}{\epsilon^r} = 0.$$

$$\lim_{n\to\infty} P(|X_n - X|^r \geq \epsilon^r) = 0 \quad \blacksquare$$

alt. intuition $\quad |X_n - X|^r \geq 0$

0 in expectation $\Rightarrow$ 0 w.p. 1.