EECS 126 Discussion 6

→ self-grades ~~today!~~ Wednesday!

→ I'll leave the discussion zoom open a little late today, so you can stay and do homework together!

→ today's problems:

- Generating Random Variables
- Gaussian Tail Bounds
- Revisiting Facts Using Transforms

intermediate results

- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$: exp Taylor series.
- $\int_a^b f'(x)\,dx = f(b) - f(a)$: FTC
- $\int_0^\infty e^{-\alpha x}\,dx = \frac{1}{\alpha}$ if $Re(\alpha) > 0$.
- $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$: geometric sum

1. Generating Random Variables

CDF of $F^{-1}(U)$?

Say $Y = F^{-1}(U)$

$$F_Y(y) = P(Y \le y) = \int_{-\infty}^{y} f_Y(y')\,dy'$$

don't know PDF, can't proceed

$$= P(F^{-1}(U) \le y)$$ strictly increasing

$$= P(U \le F(y))$$

$$= F(y)$$

2. a) $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

$$\frac{d\phi}{dy}(y) = \frac{1}{\sqrt{2\pi}} \frac{d}{dy}\left(e^{-y^2/2}\right) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \cdot \left(\frac{-y^2}{2}\right)'$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \cdot (-y)$$

$$\phi' = \phi \cdot (-y) \Rightarrow \phi = \frac{-1}{y} \phi'$$

b) $P(Y \geq t) = 1 - P(Y < t)$

$$\hookrightarrow \int_t^\infty \phi(y)\,dy = \int_t^\infty \frac{-1}{y}\phi'(y)\,dy$$

$$\frac{-1}{y} \qquad \frac{-1}{t} \qquad t \leq y$$

$$\frac{1}{t} \geq \frac{1}{y}$$

$$P(Y \geq t) = \int_t^\infty \frac{-1}{y}\phi'(y)\,dy \leq \int_t^\infty \frac{-1}{t}\phi'(y)\,dy$$

$$P(Y \geq t) \leq \frac{+1}{t}\int_\infty^t \phi'(y)\,dy \qquad \text{FTC}$$

$$= \frac{1}{t}\left(\phi(t) - \underbrace{\cancel{\phi(\infty)}}_{0}\right)$$

$$= \frac{1}{t}\frac{e^{-t^2/2}}{\sqrt{2\pi}}.$$

3. a) $\phi_X(u) = E[e^{iuX}]$

$X \sim Pois(\lambda)$: $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$$\phi_X(u) = \sum_{x=0}^{\infty} e^{iux} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{iu})^x}{x!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^{iu}} = \exp(\lambda(e^{iu}-1))$$

$Y \sim Pois(\mu)$, $\phi_Y(u) = \exp(\mu(e^{iu}-1))$

WTG : $X+Y \sim Pois(\lambda+\mu)$, i.e.

$$\phi_{X+Y}(u) = \underline{\exp((\lambda+\mu)(e^{iu}-1))}$$

$$\phi_X(u)\phi_Y(u) = \exp(\lambda(e^{iu}-1)) \exp(\mu(e^{iu}-1)) \quad \text{WTG}$$

$$= \phi_{X+Y}(u)$$

X, Y indep $\Rightarrow E[e^{iuX}] E[e^{iuY}] = E[e^{iu(X+Y)}]$

note: stronger than linearity of expectation, 'u' is a variable.

b) $X \sim Exp(\lambda)$

$$\int_0^{\infty} e^{-\alpha x} dx = \frac{1}{\alpha} \text{ if } Real(\alpha) > 0.$$

$$E[e^{iuX}] = \int_0^{\infty} e^{uix} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-(\lambda - ui)x} dx = \frac{\lambda}{\lambda - ui} = \frac{1}{1 - u i/\lambda}$$

converges if $|u| < \lambda$

$$E[e^{iuX}] = \sum_{k=0}^{\infty} \left(\frac{ui}{\lambda}\right)^k .$$

$$\sum_{k=0}^{\infty} \frac{E[(iuX)^k]}{k!} = \sum_{k=0}^{\infty} \left(\frac{ui}{\lambda}\right)^k \Rightarrow \frac{(iu)^k E[X^k]}{k!} = \frac{(ui)^k}{\lambda^k} \Rightarrow \boxed{E[X^k] = \frac{k!}{\lambda^k}}$$

3. $\overset{c}{a})$ $E[e^{iuX}]$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$\phi_X(u) = e^{-u^2/2}$, see Piazza 250 - $f2/f3$.

$$E[e^{iuX}] = \sum_{k=0}^{\infty} \frac{(iu)^k E[X^k]}{k!} = e^{-u^2/2}$$

complex real.

All imaginary components must be 0, so
$E[X^k] = 0$ for k odd. (corresponding to $i^1, i^3, i^5, \dots$).

So for even k: $E[e^{iuX}] = \sum_{k=0}^{\infty} \frac{(iu)^{2k} \cancel{E[X^{2k}]} E[X^{2k}]}{(2k)!}$.

Also $E[e^{-u^2/2}] = \sum_{k=0}^{\infty} \frac{\left(\frac{-u^2}{2}\right)^k}{k!}$

So matching terms: $\frac{\cancel{(iu)^{2k}} E[X^{2k}]}{(2k)!} = \frac{\cancel{(-1)^k} u^{2k}}{2^k k!}$ $(i^{2k} = (-1)^k)$

$$\boxed{E[X^{2k}] = \frac{2k!}{2^k k!}} = (2k-1)!!$$