EECS 126 Discussion 8

→ HW 8 self-grades tonight

→ today's problems:

- Expected Squared Arrival Times (Poisson processes)
- Bounds on Entropy (Info theory)
- Exponential MLE and MAP

Recall: $MLE[X|Y=y] = \arg\max_x P(Y=y \mid X=x)$

$MAP[X|Y=y] = \arg\max_x P(X=x \mid Y=y)$

$H(X) = \sum_{x \in \mathcal{X}} -p_x \log p_x \qquad (p_x = P(X=x))$

$= E[-\log p_x]$ — "amount of surprise"

1. Expected Squared Arrival Times.

$T_1, T_2, T_3 \,|_{N(1)=3} \sim U[0,1]$?

identically distributed : $E[\sum_{i=1}^{3} T_i^2 \mid N(1)=3]$

$= 3\, E[T_1^2 \mid N(1)=3]$

$= 3 \int_0^1 t^2 \cdot 1 \, dt = 3 \left[\frac{t^3}{3} \right]_0^1$

$= 3 \cdot \frac{1}{3} = 1$

Entropy

→ "amount of surprise"

→ predictable → not surprising

→ # of bits required to encode a sequence

$X = 0 \Rightarrow H(X) = 0$

$X \sim \text{Bern}(\tfrac{1}{2}) \Rightarrow H(X) = 1$

Suppose RV X has support $\mathcal{X}$.

Let $P_X(x) = \mathbb{P}(X = x)$

$$H(X) = -\sum_{x \in \mathcal{X}} P_X(x) \log_2 P_X(x) = \sum_{x \in \mathcal{X}} P_X(x) \log_2 \frac{1}{P_X(x)}.$$

2. $H(X) \leq \log |\mathcal{X}|$.

f concave, Z is an RV:

$f(E(Z)) \geq E[f(Z)]$.

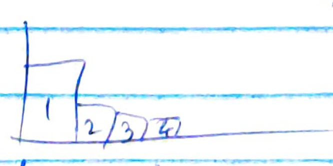
$\log_2$ is a concave function.

Can we write $H(X)$ as $E[\text{something}]$?

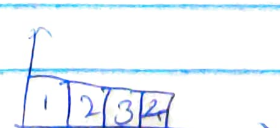
$Z = \frac{1}{P_X(x)}$ w.p. $P(X = x)$

$$H(X) = E\left[\log_2 \frac{1}{P_X(x)}\right] \overset{\text{Jensen's}}{\leq} \log_2 E\left[\frac{1}{P_X(x)}\right]$$

$$H(X) \leq \log_2 \sum_{x \in \mathcal{X}} \frac{1}{P_X(x)} P_X(x) = \log \sum_{x \in \mathcal{X}} 1 = \log |\mathcal{X}|.$$



less surprise



more surprise.

$$X = \begin{cases} 1 & \text{wp } 1/k \\ 2 & \text{wp } 1/k \\ \vdots & \\ k & \text{wp } 1/k . \end{cases}$$

$$H(X) = \sum_{x=1}^{K} \underbrace{\frac{1}{K}}_{P_X(x)} \underbrace{\log_2 K}_{\log_2 \frac{1}{P_X(x)}} = K \cdot \frac{1}{K} \cdot \log_2 K$$

$$= \log_2 K.$$

f concave : $f(E(Z)) \geq E[f(Z)]$

f convex : $f(E(Z)) \leq E[f(Z)]$

$f(x) = x^2$, convex

$$E[Z]^2 \leq E[Z^2]$$

$$E[Z^2] - E[Z]^2 \geq 0$$

$$\text{var}(Z) \geq 0.$$

3. $MLE[X|Y=y] = \arg\max_x f_Y(y|X=x)$

$MAP[X|Y=y] = \arg\max f_X(x|Y=y)$

maximizing f $\Longleftrightarrow$ maximizing $\ln f$. (Recall MCMC lab!)

$MLE[X|Y=y] = \hat{\arg\max_x} \underbrace{f_Y(y|X=x)}_{Y\sim Exp(x)} = \arg\max_x x e^{-xy}$

$= \arg\max_x \ln(x e^{-xy}) = \arg\max_x (\ln x - xy)$.

$\frac{d}{dx}(\ln x - xy) = 0$

$\frac{1}{x} - y = 0 \Rightarrow x = \frac{1}{y}$

$$\boxed{MLE[X|Y=y] = \frac{1}{y}.}$$

$MAP[X|Y=y] = \arg\max_x f_{Y|X}(y|x) f_X(x)$

$= \arg\max_x x e^{-xy} e^{-x}$

$= \arg\max_x (\ln x - xy - x)$

$\frac{d}{dx}(\ln x - xy - x) = \frac{1}{x} - y - 1 = 0$

$$\boxed{MAP[X|Y=y] = \frac{1}{y+1}}$$