EE 120: Signals and Systems **Fall 2018**

Lecture 1: Introduction

Lecturer: Babak Ayazifar *23 August* *Aditya Sengupta*

1.1 Logistics

Should do fine without EE16AB, if any Fourier experience.

Contact information: `babakayazifar@berkeley.edu`, office in 517 Cory. Email to set up appointments.

1.2 Grading

HW	10%
Quizzes (x3)	15%
Midterms (x3)	75%

Work on homework in groups of 3-5 people, strictly less than 6. The lowest 2 homeworks are dropped.

The first quiz is on 6 September. The other two are TBD. Quizzes are 30-35 minutes, at the end of lecture. The lowest quiz score is dropped.

The class has three midterms, the last of which will be in the last lecture slot before RRR week. Each is worth 25% of the grade. The first midterm is on 20 September, and the second is on 25 October. There will be no final!

Looking for a formula to throw at every problem is a bad idea. Here's how to go about it. For each problem on homework, etc., list the following:

1. List the concepts being tested
2. List the address (HW 1.1(a), or MT1.2(b) Fa17)
3. List the techniques that work.

Keep this log updated, and you'll have a good idea of which subset of these concepts you are/aren't comfortable with, so you know what to focus on.

Optional-but-recommended, inexpensive (or electronically free) textbook: *Signals and Systems, 2nd Edition, Hwei P. Hsu, Schaum's Outline Series.* Another recommended book is *Signals and Systems, 2nd Edition, Oppenheim, Willsky, Nawab.* This one is expensive. To approach this book, first go to the in-chapter examples, then to the end-of-chapter problems. These come in several types: with answers in the back, without answers in the back, challenge problems (which are mostly old MIT exam problems) and extension problems. The challenge problems are representative of exam problems.

Group work is incredibly important. Even if you're able to solve each problem, you'll never get to all the techniques. You won't learn as much that way.

There is no extra credit, but the curve will be generous. Get the top 35% and you should be good for an A. Also, after the last midterm, students will be able to write two paragraphs on what grade they think they deserve and why, and whether there were any extenuating circumstances. Class participation can help.

1.3 Continuous- and Discrete-Time Signals

This class will involve two kinds of switching around. The first is between continuous time (CT) and discrete time (DT). You can have a voice recording, which is a continuous function of intensity with respect to time, or a description of the stock market at the end of every day, which is a discrete function. We make no distinction between signals and functions. Signals have a domain and range: $x : \mathbb{R} \to \mathbb{R}$. Mostly, the range of every signal will be the reals or the complex numbers, and the domain will be the reals or the integers.

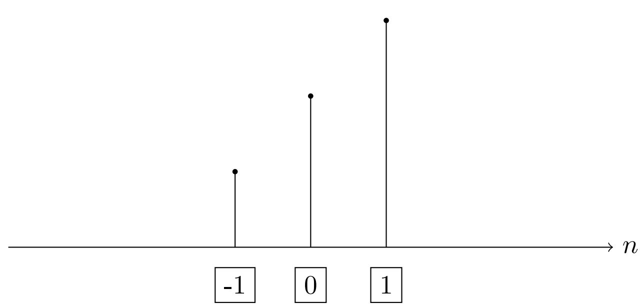
A CT signal will usually be of the form $x : \mathbb{R} \to \mathbb{R}$ (a real-valued CT signal) or it will be $x : \mathbb{R} \to \mathbb{C}$ (a complex-valued CT signal). A DT signal is one that goes from the integers to the reals or complex numbers: $x : \mathbb{Z} \to \mathbb{R}$ or $x : \mathbb{Z} \to \mathbb{C}$.

Definitions of analog/digital signals (usually CT/DT respectively) vary throughout EE, so we'll stick with CT and DT as the fundamental distinction.

The input variable is not necessarily time; a signal can also be of the form $x(n)$ (continuous) or $x[n]$ (discrete). A DT signal is represented as a set of "lollipops" on a graph. Where there is no lollipop, the signal is undefined as it is discrete, so the input is not in the domain, e.g. $x[\frac{1}{2}]$ is not defined.

The most fundamental DT signal is the DT unit impulse or Kronecker delta $\delta(n)$, which is 0 everywhere except at the origin, where it is 1. The equivalent in CT is the Dirac delta. The Dirac delta is not a function; 1 is the area under the curve, not the height of the graph. It's still as weird as it was in 5B.

We can always represent a signal as a linear combination of impulses in DT. For example, the following diagram:



This can be expressed as a sum of delta functions:

$$x[n] = 1 \cdot \delta[n+1] + 2 \cdot \delta[n] + 3 \cdot \delta[n-1] \tag{1.1}$$

The other kind of switching around is from frequency-domain to time-domain signals. The fundamental way to express a frequency in CT is the complex exponential:

$$x(t) = e^{i\omega t} \tag{1.2}$$

where ω has units of rad/s. Alternatively, if a frequency description in Hertz is desired, the following is an equivalent formula:

$$x(t) = e^{i2\pi ft} \tag{1.3}$$

The 2π shows up elsewhere even if you use the ω formula, so both are equally difficult to use and have their own use cases.

In discrete-time domains, this becomes

$$x[n] = e^{i\omega n} \tag{1.4}$$

where ω has units of $\frac{rad}{sample}$, and n is a number of samples. Sometimes, the ωs in CT and DT will both be used at the same time, such as when we take samples of a CT signal to make a DT one. In such cases, Ω will be used for DT instead.

1.4 Real Exponentials

The exponential is the function $x(t)$ with the following fundamental property:

$$\frac{dx(t)}{dt} = x(t) \tag{1.5}$$
$$x(0) = 1 \tag{1.6}$$

The unique signal with these properties is $x(t) = e^t$. This can be generalized:

$$\dot{x}(t) = \sigma x(t), \sigma \in \mathbb{R} \tag{1.7}$$
$$x(0) = 1 \tag{1.8}$$
$$\therefore x(t) = e^{\sigma t} \tag{1.9}$$

1.5 Complex Exponentials

This can be extended to complex numbers fairly easily. Consider a signal $x(t) = e^{zt}$, where $z \in$, i.e. $z = \sigma + i\omega$.

Then, the fundamental equation characterizing the exponential is:

$$\dot{x}(t) = zx(t) \tag{1.10}$$
$$x(0) = 1 \tag{1.11}$$

To start with, consider the case $\sigma = 0$, so that $x(t) = e^{i\omega t}$. Take $\omega = 1$, that is, the angular frequency of the signal is $1 rad/s$. Then, the signal becomes $x(t) = e^{it}$. We're interested in the behavior of this signal on the complex plane, based on the fact that $x(t)$ is the instantaneous position of a particle on the complex plane at time t.

At time $t = 0$, $x(t) = x(0) = 1 + 0i$. The velocity can be found by taking a time derivative of x: $\dot{x}(t) = ix(t)$. This will help us predict the behavior of the signal at later times. The speed is the magnitude of $\dot{x}(t)$, which is always 1.

Wait, how?

$$|ix(t)| = |i|\,|x(t)| = 1 \cdot |x(t)| \tag{1.12}$$
$$\tag{1.13}$$

We don't know that $|x(t)| = 1$ for $t \neq 0$. To further analyze this, we use the fact that multiplication by i is equivalent to a counterclockwise rotation of 90^o. Since the velocity is i times the position, the velocity is always 90^o counterclockwise to $\vec{x}(t)$ for all t. Also, since

$$|\dot{x}(t)| = |x(t)| \tag{1.14}$$

we can analyze the magnitude of $x(t)$ to verify the above.

Let

$$x(t) = e^{it} = a(t) + ib(t) \tag{1.15}$$

$$\left(\exists a(t), b(t) \in \mathbb{R}, a, b : \mathbb{R} \to \mathbb{R}\right)$$

We take a time derivative,

$$\dot{x}(t) = ix(t) \tag{1.16}$$
$$\dot{a}(t) + i\dot{b}(t) = -b(t) + ia(t) \tag{1.17}$$

Therefore,

$$\dot{a} + i\dot{b} = -b + ia \tag{1.18}$$
$$\therefore \dot{a} = -b, a = \dot{b} \tag{1.19}$$
$$\therefore a\dot{a} = -b\dot{b} \tag{1.20}$$
$$2a\dot{a} + 2b\dot{b} = 0 \tag{1.21}$$
$$\frac{d}{dt}\left(a^2 + b^2\right) = 0 \tag{1.22}$$

Since the time derivative is zero, $a^2 + b^2$ is a constant for all t. This is the equation for a circle of radius $\sqrt{C}$, that constant. We can show that it is 1 by plugging in $t = 0$:

$$
\begin{aligned}
a^2(0) + b^2(0) &= C && (1.23)\\
a(0) + ib(0) = 1 \implies a(0) &= 1, b(0) = 0 && (1.24)\\
\therefore a^2(0) + b^2(0) &= 1 + 0 = 1 && (1.25)\\
\therefore C &= 1 && (1.26)
\end{aligned}
$$

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Lecture 2: Euler's Formula and Linear Systems

Lecturer: Babak Ayazifar *August 28* *Aditya Sengupta*

2.1 Euler's Formula

We established before that the speed of a particle traversing the unit circle in the complex plane is 1 rad/s. Therefore, it takes 2π seconds to circumambulate (wow) because it has to cover 2π radians. Therefore, after t seconds, the particle will be at t rad from the positive real line. From geometry, we can infer the complex and real components of the complex exponential:

$$e^{it} = \cos t + i \sin t \tag{2.1}$$

This is Euler's formula. We can use it to get complex exponential representations of $\sin t$ and $\cos t$:

$$e^{it} = \cos t + i \sin t \tag{2.2}$$
$$e^{-it} = \cos t - i \sin t \tag{2.3}$$

Adding these together,

$$e^{it} + e^{-it} = 2 \cos t \tag{2.4}$$
$$\cos t = \frac{e^{it} + e^{-it}}{2} \tag{2.5}$$

Similarly, we can get $\sin t$:

$$e^{it} = \cos t + i \sin t \tag{2.6}$$
$$e^{-it} = \cos t - i \sin t \tag{2.7}$$

Subtracting these,

$$e^{it} - e{-}it = 2i \sin t \tag{2.8}$$
$$\sin t = \frac{e^{it} - e^{-it}}{2i} \tag{2.9}$$

These are the *inverse Euler's formulas.*

2.2 ω

Introduce ω, a fixed angular speed, into the complex exponential expression:

$$x(t) = e^{i\omega t}, \omega > 0 \tag{2.10}$$
$$\dot{x}(t) = i\omega e^{i\omega t} \implies |\dot{x}(t)| = \omega \tag{2.11}$$

In this case, the angle between the positive reals and the vector in complex space becomes ωt, and the components become $\cos(\omega t)$ and $i\sin(\omega t)$. Then, we can get similar inverse Euler's relations:

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2} \tag{2.12}$$

Note here that this tells us that $\cos(\omega t)$ is not a fundamental signal; it is a linear combination of two complex exponentials, one of frequency ω and one of frequency $-\omega$ (which can be interpreted as a frequency of ω clockwise). Vector addition can show you that adding these two phasors (rotating vectors starting from the origin on the unit circle) gives you a vector along the real line. Similarly with $\sin(\omega t)$, you get a vector along the imaginary line.

In discrete time, $x(n) = e^{i\omega n}$ where ω is a number of radians per sample and $n \in \mathbb{Z}$. Here we have a phasor that hops between discrete points. The signal has the same form as the continuous case; it's like evaluating the continuous one at specific points in time.

2.3 Periodicity

In CT, we say that x is periodic if

$$x(t+T) = x(t) \forall t, \exists T \neq 0 \tag{2.13}$$

The smallest positive T is called the fundamental period of a signal. There are many possible values of T; any integer multiple of the fundamental period is valid.

Consider the following function:

$$x(t) = e^{i\frac{3\pi}{5}t} \tag{2.14}$$

We want to find the fundamental period of this signal, that is, find the smallest possible T such that $x(t+T) = x(t)$.

$$x(t+T) = e^{i\frac{3\pi}{5}t}e^{i\frac{3\pi}{5}T} = e^{i\frac{3\pi}{5}t} \tag{2.15}$$

We can cancel the $e^{i\frac{3\pi}{5}t}$ term, as it is on the unit circle:

$$e^{i\frac{3\pi}{5}T} = 1 \tag{2.16}$$

We know that $e^{i\theta} = 1$ for any $\theta = 2\pi k, k \in \mathbb{Z}$. Therefore, we get

$$\frac{3\pi}{5}T = 2\pi k \frac{3}{5}T = 2k \implies T = \frac{10k}{3} \tag{2.17}$$

Since the smallest positive k is 1, the smallest positive $T = \frac{10}{3}s$.

We can always find a fundamental period unless $\omega = 0$, because that corresponds to the signal $x(t) = 1$. This is because there is no smallest positive shift after which the signal repeats; any shift will do this. In general, a signal $x(t) = e^{i\omega t}$ has fundamental period $T = \frac{2\pi}{\omega}$, which is why it doesn't make sense to do this for $\omega = 0$.

For CT signals, T need not be an integer; it can be any positive real number.

(bonsai)

In DT, we say a signal $x(n)$ is periodic if $x(n+N) = x(n) \forall n \in Z \exists N \in Z$. Here, the fundamental period has to be an integer, because the signal is not defined elsewhere. The analysis is similar to the continuous-time case; consider a signal $x(n) = e^{i\frac{\pi}{4}n}$. We get the following:

$$e^{i\frac{\pi}{4}N} = 2\pi k N = 8k \implies N = 8 \tag{2.18}$$

Consider the signal $x(n) = e^{in}$. The required N has the property that $e^{iN} = 1$. This gives us $N = 2\pi k$. No k produces an integer N, so this is not a periodic signal. (It is quasiperiodic.) In general, $e^{i\omega n}$ is periodic in n if and only if ω is a rational multiple of π.

Another DT vs CT distinction is in ω. In DT:

$$e^{i(\omega+2\pi)n} = e^{i\omega n}e^{i2\pi n} = e^{i\omega n} \tag{2.19}$$

Therefore $e^{i\omega n}$ is 2π-periodic in ω. This implies that all the frequency-domain action for a discrete time signal is over a length-2π portion of the frequency axis. Usually, we pick the interval from $-\pi$ to π, or that from 0 to 2π.

Lowest frequency for CT signals: $\omega = 0$.

Lowest frequency for DT signals: $\omega = 2\pi k, k \in \mathbb{Z}$

Highest frequency for CT signals: (does not exist - infinity)

Highest frequency for DT signals: a signal that changes polarity between samples, alternating. This is $e^{i\pi n} = \cos(\pi n) + i\sin(\pi n) = \cos(\pi n) = (-1)^n$. This corresponds to frequencies $\omega = (2k+1)\pi$.

2.4 Linear Time-Invariant Systems

Consider a system represented as a black box, that takes in an input signal x, and returns an output signal $y = G(x)$. X is the input signal space and Y is the output signal space. The input in continuous time is usually $\mathbb{R} \to \mathbb{R}$ (e.g. $\cos \omega t$) or $\mathbb{R} \to$ (e.g. $e^{i\omega t}$). In discrete time, this is $x : \mathbb{Z} \to \mathbb{R}$ or $x : \mathbb{Z} \to$.

DT -¿ DT (discrete time system) CT -¿ DT (sampling)

A linear system is a system such that the following properties hold:

$$x \to G \to y \implies \alpha x \to G \to \alpha y \tag{2.20}$$

$$x_1 \to G \to y_1, x_2 \to G \to y_2 \implies x_1 + x_2 \to G \to y_1 + y_2 \tag{2.21}$$

X and Y are vector spaces, albeit ones on which some properties such as inner products are not necessary (yet).

A continuous-time example would be a current source that produces a voltage drop across a resistor. Let $x(t)$ be a current source, and let $y(t)$ be the corresponding voltage. Then $y(t) = Rx(t)$; this is a linear system. Another possible linear system is $y(t) = \cos(\omega_0 t)x(t)$. This is useful for AM radio; a signal is multiplied by a cosine to modulate its amplitude in order to reduce transmission loss, and the multiplication is reversed on the other end.

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Lecture 3: Linear Time-Invariant Systems

Lecturer: Babak Ayazifar *30 August* *Aditya Sengupta*

3.1 Relation between fundamental frequency and fundamental period

Consider $x(t) = \cos \omega t$, a continuous-time signal. Then, we have this relation between the fundamental period and frequency:

$$T = \frac{2\pi}{\omega} \tag{3.1}$$

$$\omega = \frac{2\pi}{T} \tag{3.2}$$

If ω decreases, T increases and vice versa. This inverse monotonic relationship doesn't hold for discrete-time signals. For example, consider the following DT signals:

$$x_1[n] = e^{i\frac{\pi}{4}n} \implies N_1 = 8 \tag{3.3}$$

$$x_2[n] = e^{i\frac{3\pi}{4}n} \implies N_2 = 8 \tag{3.4}$$

In discrete time, it is only safe to use $\omega = \frac{2\pi}{N}$, not the inverse.

3.2 Linearity and Time-Invariance

3.2.1 Linearity

A system G that takes in a signal $x \in X$ and yields a signal $y \in Y$ is linear if

$$G(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 y_1 + \alpha_2 y_2 \forall \alpha_1, \alpha_2 \in \tag{3.5}$$

3.2.2 Time Invariance

If for every $x \in X$ and every shift $T \in \mathbb{R}$,

$$G(x(t-T)) = y(t-T) \tag{3.6}$$

then we say G is time-invariant.

3.2.3 The Magic

LTI (linear and time-invariant) systems are magical. In general, any DT signal can be expressed as a linear combination of its values times the impulse that is active at that value.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \tag{3.7}$$

This is a decomposition of any signal in terms of DT impulses.

3.2.4 Bold Claim

Given the response of an LTI system to an impulse, its response to any input can be found.

Let $x(n) = \delta(n)$. Then, we can feed this through the LTI system G: $y(n) = g(n) = G(x(n)) = G(\delta(n))$.

More simply, $g(n) = G(\delta(n))$.

We can show this using the properties of linearity and time invariance.

$$\delta(n) \to G \to g(n) \tag{3.8}$$
$$x(0)\delta(n) \to G \to x(0)g(n)(\text{scaling}) \tag{3.9}$$
$$\delta(n+1) \to G \to g(n+1)(\text{TI}) \tag{3.10}$$
$$x(-1)\delta(n+1) \to G \to x(-1)g(n+1) \tag{3.11}$$

This can be generalized:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \to G \to g(n) = \sum_{k=-\infty}^{\infty} x(k)g(n-k) = (x * g)(n) \tag{3.12}$$

$(x * g)$ is referred to as the *convolution* of x and g.

3.3 Frequency-Domain LTI Systems

In the frequency domain, we know that the complex exponential is the purest form of signal. We want to know the LTI system's response to $x(n) = e^{i\omega n}$. We know that it can be expressed as a linear combination of impulse responses, $y(n) = \sum_k x(k)g(n-k)$.

Let $l = n - k$. Then,

$$y(n) = \sum_{l=-\infty}^{\infty} g(l)x(n-l) \tag{3.13}$$

The output signal is the convolution of x and g, as well as that of g and x: $x * g = g * x$. (Convolution is commutative). Therefore, we can plug in the input signal to the convolution expression.

$$x(n) = e^{i\omega n} \implies x(n-l) = e^{i\omega(n-l)} = e^{-i\omega l}e^{i\omega n} \tag{3.14}$$

$$y(n) = \sum_l g(l)e^{-i\omega l}e^{i\omega n} = e^{i\omega n}G(\omega) \tag{3.15}$$

The sum over l of a number of exponentials in ω (weighted by $g(l)$) is a function of ω, which we call $G(\omega)$. Therefore the system response is $G(\omega)e^{i\omega n}$, where $G(\omega) = \sum_{l=-\infty}^{\infty} g(l)e^{-i\omega l}$.

This is reminiscent of $A\vec{v} = \lambda\vec{v}$, where the complex exponential corresponds to the eigenvector, the function $G(\omega)$ corresponds to the eigenvalue, and the system G corresponds to the matrix A.

$$G(\omega) = \sum_{l=-\infty}^{\infty} g(l)e^{-i\omega l} \tag{3.16}$$

In general, this is a complex-valued function, so it can be characterized completely by its magnitude (the magnitude response of the system) $|G(\omega)|$ and its phase (the phase response of the system) $\sphericalangle G(\omega)$. Then,

$$G(\omega) = |G(\omega)|e^{i\sphericalangle G(\omega)} \tag{3.17}$$

For example, consider the two-point moving average filter.

$$y(n) = \frac{x(n) + x(n-1)}{2} \tag{3.18}$$

The impulse response to this is $\frac{1}{2}\delta(n) + \frac{1}{2}\delta(n-1)$.

To find the frequency response, we can use the definition:

$$G(\omega) = \sum_{l=0}^{1} g(l)e^{-i\omega l} = g(0) + g(1)e^{-i\omega} = \frac{1 + e^{-i\omega}}{2} \tag{3.19}$$

Another way is to use the eigenfunction property of the complex exponential. We know that an input of $e^{i\omega n}$ yields an output of $G(\omega)e^{i\omega n}$. Let $x(n) = e^{i\omega n}$, then we get:

$$y(n) = \frac{e^{i\omega n} + e^{i\omega n}e^{-i\omega}}{2} \tag{3.20}$$

$$y(n) = e^{i\omega n}\frac{1 + e^{-i\omega}}{2} \tag{3.21}$$

$$\therefore G(n) = \frac{1 + e^{-i\omega}}{2} \tag{3.22}$$

The magnitude response of this filter is

$$|G(\omega)| = \left|\frac{1+e^{-i\omega}}{2}\right| = \left|\frac{e^{i\omega/2}+e^{-i\omega/2}}{2} \times e^{-i\omega/2}\right| \tag{3.23}$$

$$|G(\omega)| = |\cos(\omega/2)| \left|e^{-i\omega/2}\right| = \cos(\omega/2) \tag{3.24}$$

which can be graphed as an arc between $\omega = -\pi$ and $\omega = \pi$. This is a low-pass filter. For example, $x(n) = 1$ passes unchanged, but $x(n) = (-1)^n$ is completely killed, as $(-1)^n = e^{i\pi n}$ implies $\omega = \pi$, and $G(\pi) = \cos(\pi/2) = 0$.

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Lecture 4: Phase Response, FIR/IIR

Lecturer: Babak Ayazifar *4 September* *Aditya Sengupta*

4.1 Magnitude and Angle of Frequency Response

Since I don't think I covered this method of finding a frequency response before, here:

$$G(\omega) = \sum_{n=-\infty}^{\infty} g(n)e^{-i\omega n} \tag{4.1}$$

For the two-point moving average, we plug in the impulse response and get

$$G(\omega) = g(0) + g(1)e^{-i\omega} = \frac{1 + e^{-i\omega}}{2} \tag{4.2}$$

which is the same as the other method.

To find the magnitude of the frequency response to the two-point moving average, we pull out an $e^{-i\omega/2}$ to balance the exponents, and we therefore get

$$G(\omega) = \cos\left(\frac{\omega}{2}\right) e^{-\frac{i\omega}{2}} \tag{4.3}$$

This is almost polar, but not quite; in general cos can go negative. Fortunately, it doesn't in this area of interest. $\cos(\frac{\omega}{2}) \geq 0 \forall \omega \in [-\pi, \pi]$.

The phase is the coefficient of the i in the exponent, which here is $-\frac{\omega}{2}$. The phase here is a periodically replicating linear function:

$$\sphericalangle G(\omega) = \begin{cases} -\frac{\omega}{2} & -\pi < \omega \leq \pi \\ 2\pi & \text{periodically replicates} \end{cases} \tag{4.4}$$

Phases are weird, because magnitudes always have to be positive, which means corrections may have to be made to te phase. For example, consider

$$H(\omega) = \sin\left(\frac{\omega}{2}\right) e^{-\frac{i\omega}{2}} \tag{4.5}$$

Because the sine term will go negative in the area of interest in $[-\pi, 0]$, we take an absolute value on the negative region, and define the signal over $[-\pi, 0]$ as

$$H(\omega) = -\left|\sin\left(\frac{\omega}{2}\right)\right| e^{-\frac{-i\omega}{2}} \tag{4.6}$$
$$H(\omega) = \left|\sin\left(\frac{\omega}{2}\right)\right| e^{\frac{-i\omega}{2}} e^{\pm i\pi} \tag{4.7}$$

Consider a filter defined as follows:

$$h(n) = (-1)^n g(n) \tag{4.8}$$
$$H(\omega) = \sum (-1)^n g(n) e^{-i\omega n} \tag{4.9}$$
$$\tag{4.10}$$

which can be rewritten in terms of complex exponentials:

$$H(\omega) = \sum_n g(n) e^{-i\omega n} e^{i\pi n} H(\omega) = \sum_n g(n) e^{-i(\omega - \pi)n} = G(\omega - \pi) \tag{4.11}$$

This is a high-pass filter; for an input signal peaking at 0, such as a sine wave, the filter yields a signal peaking at odd multiples of π.

4.2 FIR Filters

A finite-duration impulse response (FIR) filter has an impulse response that is zero outside a finite set of values of n.

Consider an input $x(n) = \cos(\omega_0 n)$ to the filter $G(\omega) = |\cos(\omega_0 n)| e^{-i\omega/2}$. We can rewrite the input as complex exponentials:

$$\cos(\omega_0 n) = \frac{e^{i\omega_0 n} + e^{-i\omega_0 n}}{2} \tag{4.12}$$
$$\therefore y(n) = \frac{1}{2} G(\omega_0) e^{i\omega_0 n} + \frac{1}{2} G(-\omega_0) e^{-i\omega_0 n} \tag{4.13}$$

We can substitute in the $G(\omega_0)$ terms and rewrite the expression to get

$$y(n) = \frac{1}{2} \cos\left(\frac{\omega_0}{2}\right) \left[e^{i(\omega_0 n - \frac{\omega_0}{2})} + e^{-i(\omega_0 n - \frac{\omega_0}{2})}\right] \tag{4.14}$$
$$y(n) = \cos\left(\frac{\omega_0}{2}\right) \cos\left(\omega_0 n - \frac{\omega_0}{2}\right) \tag{4.15}$$
$$y(n) = |G(\omega_0)| \cos\left[\omega_0 \left(n - \frac{1}{2}\right)\right] \tag{4.16}$$

Therefore, under only the condition that the filter is real-valued, the response of the filter to $\cos(\omega_0 n)$ is given by the magnitude of the frequency response evaluated at ω_0, multiplied by the original cosine after a shift.

In general, $h(n) = (-1)^n g(n) \implies H(\omega) = G(\omega - \pi)$.

4.3 Infinite Impulse Response Filter

Consider a recursive linear constant-coefficient difference equation:

$$y(n) = \alpha y(n-1) + x(n) \tag{4.17}$$

with initial condition $y(n) = 0 \forall n < 0$.

In general, a set of these recursive functions can be represented in matrices:

$$q_{N\times 1}(n+1) = A_{N\times N} q(n) + B_{N\times M} u_{M\times 1} u(n) \tag{4.18}$$

Let $0 < \alpha < 1$; then, find the impulse and frequency responses, and plot the magnitude and the phase of the frequency response.

To find the impulse response, we set $x(n) = \delta(n)$. Then:

$$\begin{aligned} h(n) &= 0 \forall n < 0 & (4.19)\\ h(0) &= \alpha h(-1) + \delta(0) = 1 & (4.20)\\ h(1) &= \alpha h(0) = \alpha & (4.21)\\ &\dots & (4.22)\\ h(n) &= \alpha^n & (4.23) \end{aligned}$$

Therefore,

$$h(n) = \alpha^n u(n) \tag{4.24}$$

where $u(n)$ is the unit step function.

We can find the frequency response:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-i\omega n} \tag{4.25}$$

$$H(\omega) = \sum_{n=0}^{\infty} (\alpha e^{-i\omega})^n = \sum_{n=0}^{\infty} \frac{1}{1 - \alpha e^{-i\omega}} \tag{4.26}$$

This holds as long as $|\alpha| < 1$.

In general, a filter defined by a recurrence relation $ay(n) + \alpha y(n-1) = bx(n)$ has a frequency response $H(\omega) = \frac{b}{a+\alpha e^{-i\omega}}$.

For example, for a system that delays its input by 1 sample, the frequency response is $H(\omega) = \frac{1}{1-e^{-i\omega}}$ (check).

An alternative way of looking at this is the eigenfunction approach. Let $y(n) = H(\omega)e^{i\omega n}$, and substitute this into the recurrence relation:

$$H(\omega)e^{i\omega n} = \alpha H(\omega)e^{i\omega(n-1)} + e^{i\omega n} \tag{4.27}$$

To simplify the frequency response, we can multiply the previous answer by $e^{i\omega}$:

$$H(\omega) = \frac{1}{1-\alpha e^{-i\omega}} = \frac{e^{i\omega}}{e^{i\omega}-\alpha} \tag{4.28}$$

The magnitude response is the quotient of the magnitudes of the two vectors, one being the regular vector from the origin and the other being a vector to the point $e^{i\omega}$ from $+\alpha$ on the real line:

$$H(\omega) = \frac{||e^{i\omega}||}{||e^{i\omega}-\alpha||} \tag{4.29}$$

$$H(\omega) = \frac{1}{||e^{i\omega}-\alpha||} \tag{4.30}$$

At $-\pi$ or π, the magnitude of the frequency response is $\frac{1}{1+\alpha}$, the minimum. At 0, it is $\frac{1}{1-\alpha}$. The closer α is to 1, the sharper the low-pass filter. If we let α go negative, a high-pass filter can be created.

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Lecture 5: Phase Response, Sampling, Frequency/Time Uncertainty

Lecturer: Babak Ayazifar *6 September* *Aditya Sengupta*

5.1 Phase Response

The phase response for $H(\omega)$ as defined in the previous lecture is

$$\sphericalangle H(\omega) = \sphericalangle e^{i\omega} - \sphericalangle(e^{i\omega} - \alpha) \tag{5.1}$$
$$\tag{5.2}$$

Heuristic-based geometric analysis (the angle at $\alpha + 0i$ being greater than that at the origin on $[0, \pi]$, etc) shows us that the phase response is zero at multiples of π, positive on $(-\pi, 0)$ and negative on $(0, \pi)$.

5.2 Example with delay > 1

Let $y(n) = \alpha y(n - N) + x(n)$, where $N \geq 2$. Computing the impulse response for a few input set shows us that

$$g(n) = \begin{cases} 0 & n \text{odd} \\ \alpha^{n/2} & n \text{even} \end{cases} \tag{5.3}$$

The filter increases the spacing between successive lollipops to N, which is called *up-sampling*. If we have a filter $h(n)$ (say, one that is α^n for all n) and we want to up-sample it by a factor of N, then the up-sampling filter is characterised by

$$g(n) = \begin{cases} 0 & n mod N \neq 0 \\ h\left(\frac{n}{N}\right) & n mod N = 0 \end{cases} \tag{5.4}$$

Linearity holds, but time-invariance is only true for intervals of N (a shift of 1 in the input causes a shift of N in the output), so in general G is not time-invariant.

What about the frequency response? Consider $N = 2$,

$$G(\omega) = \sum_n g(n) e^{-i\omega n} \tag{5.5}$$
$$G(\omega) = g(0)e^{-i\omega 0} + g(2)e^{-2i\omega} + g(4)e^{-4i\omega} + \dots \tag{5.6}$$
$$G(\omega) = h(0) + h(1)e^{-2i\omega} + h(2)e^{-4i\omega} + \dots \tag{5.7}$$
$$\tag{5.8}$$

Compare this with $H(\omega) = h(0) + h(1)e^{-i\omega} + h(2)e^{-2i\omega} + \dots$. We see that $G(\omega) = H(2\omega)$. This represents contraction along the ω axis. We can generalize this to $G(\omega) = H(N\omega)$; there are N peaks/troughs per 2π interval.

In general, we've seen that if you dilate in time, you compress in frequency, and vice versa. This is why sped-up voices have a higher pitch. As time increases, frequency is compressed and becomes more "wiggly" (that's the technical term) and so the peaks are pitched higher. (Verify)

5.3 Causality

A system F is causal if $\forall x_1, x_2 \in X$ such that $x_1(n) = x_2(n) \forall n \leq N$, the responses $y_1(n) = y_2(n) \forall n \leq N$. Essentially, this means F is causal if it does not compute anything using $x(n+1), x(n+2), \dots,$ etc. to compute $y(n)$.

In LTI systems, the generalized output is

$$y(n) = \dots + f(-2)x(n+2) + f(-1)x(n+1) + f(0)x(n) + f(1)x(n-1) + f(2)x(n-2) + \dots \tag{5.9}$$

For the system to be causal, $f(n)$ must be 0 for all $n < 0$, because then the coefficients become 0 and the generalized response is causal.

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Lecture 6: BIBO Stability, LTI System Interconnections

Lecturer: Babak Ayazifar *11 September* *Aditya Sengupta*

6.1 BIBO Stability

BIBO is short for "Bounded-Input, Bounded-Output". We say that a system H whose response is $y = H(x)$ is BIBO stable if every bounded input produces a bounded output.

A signal $x(n)$ is bounded if there exists some finite quantity B_x (that is, $0 \leq B_x < \infty$) such that $|x(n)| \leq B_x \forall n$.

If for every $x \in X$ such that $|x(n)| \leq B_x$ the corresponding output y is bounded, i.e. $|y(n)| \leq B_y$ for some $0 \leq B_y < \infty$, then we say H is BIBO stable.

Consider a system H with impulse response $h(n)$ that is DT-LTI. Then H is BIBO stable if and only if

$$\sum_n |h(n)| < \infty \tag{6.1}$$

If the impulse response is absolutely summable, then every bounded input produces a bounded output.

We know that the response of an LTI system can be found using convolution,

$$y(n) = \sum_k h(k)x(n-k) \tag{6.2}$$

$$\therefore |y(n)| = \left|\sum_k h(k)x(n-k)\right| \leq \sum_k |h(k)||x(n-k)| \tag{6.3}$$

(using the Triangle Inequality above)

Given that $|x(n)| \leq B_x \forall n$, we can replace the x term with:

$$|y(n)| \leq B_x \sum_k |h(k)| \tag{6.4}$$

Since it is given that $h(n)$ is bounded (by some $H_0 < \infty$), we can say

$$|y(n)| \leq B_x H_0 = B_y \tag{6.5}$$

Therefore, if the impulse response is absolutely summable, it is BIBO stable.

We can prove this in reverse too: if every bounded input produces a bounded output, then the impulse response is absolutely summable. We proceed by contraposition: if the impulse response is not absolutely summable, not every bounded input produces a bounded output.

Suppose $\sum_n |h(n)|$ diverges. Then $\exists x \in X$ (bounded), such that the corresponding output y is unbounded.

Let

$$\hat{x}(n) = \begin{cases} \frac{h(n)}{|h(n)|} & h(n) \neq 0 \\ 0 & h(n) = 0 \end{cases} \tag{6.6}$$

Every nonzero lollipop gets mapped to -1 or 1 (and 0 gets mapped to 0).

Let $x(n) = \hat{x}(-n)$. Then, apply x to a DT-LTI system H whose impulse response is not absolutely summable. We find that the output blows up at $n = 0$.

$$y(n) = \sum_k h(k)x(n-k) \tag{6.7}$$

$$y(0) = \sum_k h(k)x(-k) = \sum_k h(k)\hat{x}(k)y(0) = \sum_{k \mathrm{s.t.} h(k)\neq 0} h(k)\frac{h(k)}{|h(k)|}y(0) = \sum_{k \mathrm{s.t.} h(k)\neq 0} \frac{|h(k)|^2}{|h(k)|} = \sum_k |h(k)| \tag{6.8}$$

We know that this diverges. Therefore, we have found a bounded input such that the output is not bounded.

Therefore, a DT-LTI system H is BIBO stable iff $\sum |h(n)| < \infty$. FIR filters are BIBO stable, whereas IIR filters may not be.

Example

Consider the causal system

$$y(n) = \alpha y(n-1) + x(n) \tag{6.9}$$

which we previously calculated had the impulse response

$$h(n) = \alpha^n u(n) \tag{6.10}$$

Suppose $\alpha \geq 1$. Then, $h(n)$ is at least the unit step, which is not absolutely summable, therefore it is not BIBO stable. Let $x(n) = 1 \forall n$. Then,

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} 1 = \infty \tag{6.11}$$

If a system H is BIBO stable, then $|H(\omega)| < \infty \forall \omega$. (We can also say that $H(\omega)$ is a smooth function of ω, but this is enough for now).

We know that

$$|H(\omega)| = |\sum_n h(n)e^{-i\omega n}| \leq \sum_n |h(n)||e^{-i\omega n} = \sum_n |h(n)| < \infty \tag{6.12}$$

Consider a filter whose frequency response is 1 for $|n| \leq B$ (for some $Bs.t.|B| \leq \pi$), and 0 elsewhere. Don't know what we're doing with that.

6.2 Dirac delta

The Dirac delta is the continuous-time version of $\delta(n)$. It has the property that

$$\int_{-\infty}^{\infty} \delta_\Delta(t)dt = 1 \tag{6.13}$$

We don't care exactly what the function is, because that's kind of weird. We care what it does to signals that it comes in contact with. We can get to the Dirac delta by letting $\Delta \to \infty$ in a Gaussian distribution, $\delta_\Delta(t) = \frac{1}{\Delta\sqrt{2\pi}}e^{-t^2/2\Delta^2}$.

Example

Consider a rubber ball, $m = 0.2kg$, hitting a rigid floor. It hits the floor at time $-\varepsilon/2$ and bounces back up at time $\varepsilon/2$ (for an arbitrarily small ε. Given that $v(0^-) = -8m/s$ and $v(0^+) = +8m/s$, find the average force exerted on the ball by the floor.

$$F = M\frac{dv}{dt} = \frac{dP(t)}{dt} \tag{6.14}$$

$$\int_{-\varepsilon/2}^{\varepsilon/2} F(t)dt = P(t_1) - P(t_0) \tag{6.15}$$

The shape of the F/t curve over $[-\varepsilon/2, \varepsilon/2]$ doesn't matter, only the initial momentum and the final momentum do.

$$P(-\frac{\varepsilon}{2}) = -0.2 \times 8 = -1.6kgm/s \tag{6.16}$$

$$P(\frac{\varepsilon}{2}) = 1.6kgm/s \tag{6.17}$$

$$\tag{6.18}$$

Therefore, $\int_{-\varepsilon/2} \varepsilon/2F(t)dt = 3.2kgm/s$. This is the area under the force-time curve. F_{avg} is the equivalent flat curve:

$$F_{avg} \cdot \varepsilon = 3.2kgm/s \implies F_{avg} = 3200N \tag{6.19}$$

Consider $\delta_\Delta() = 0$ for $\notin [0, \Delta]$ and $/Delta$ otherwise. Then, consider the integral

$$u_\Delta(t) = \int_{-\infty}^{t} \delta_\Delta()d \tag{6.20}$$

Below 0, this is 0; above 0, this is linear between 0 and Δ, and greater than Δ, it is 1. Therefore, this is almost a unit-step function.

In discrete time, we can do the same analysis with an infinite summation: $u(n) = \sum_{k=-\infty}^{n} \delta(k) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$, which gives us the correct unit-step. Therefore $\delta(n) = \dot{u}(n)$, and in CT similarly, $\delta_\Delta(n) = \dot{u}(t)$.

6.3 Dirac Delta Operations

Multiply a continuous function by the Dirac delta over a small enough range $-\Delta/2, \Delta/2$, and we get a function slice that can be averaged to get $\phi(0)/\Delta$, which as $\Delta \to 0$ becomes a (basically) DT-delta multiplied by $\phi(0)$. Therefore,

$$\phi()\delta_\Delta() = \phi(0)\delta() \tag{6.21}$$

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Lecture 7: LTI Interconnections and CT LTI Systems

Lecturer: Babak Ayazifar *13 September* *Aditya Sengupta*

7.1 Dirac Delta Wrap-Up

The Dirac delta has the sampling property, that

$$\phi(t)\delta(t-T) = \phi(T)\delta(t-T) \tag{7.1}$$

and integrating this allows us to derive the sifting property:

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-T)dt = \phi(T) \tag{7.2}$$

The sifting property allows us to define an entire signal in terms of time-shifted Dirac deltas:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \tag{7.3}$$

which is analogous to a similar expression in discrete time:

$$x(n) = \sum_{m=-\infty}^{\infty} x(m)\delta(n-m) \tag{7.4}$$

7.2 LTI Interconnections

For two LTI systems placed in parallel, such that a signal is split and fed through LTI systems F and G and added to get a response $y(n)$, we can easily show that $h(n) = f(n) + g(n)$. The frequency responses are also additive:

$$y(n) = |F(\omega) + G(\omega)|\, e^{i\omega n} = H(\omega)e^{i\omega n} \tag{7.5}$$

Series or cascading interconnections are more difficult to deal with. Let H be a system in which the output is the output of G if its input is the output of F (if its input is the original input signal). This is mathematically represented with convolution:

$$h(n) = (f \circledast g)(n) \tag{7.6}$$
$$H(\omega) = F(\omega)G(\omega) \tag{7.7}$$
$$\sphericalangle H(\omega) = \sphericalangle F(\omega) + \sphericalangle G(\omega) \tag{7.8}$$

7.3 Feedback

Consider a system with two filters: a plant filter $P(\omega)$ leading from the input to the output, and a controller filter $K(\omega)$ leading from the output that is added into the input before it is passed into P. Let $y(n) = H(\omega)e^{i\omega n}$; we want to find an expression for $H(\omega)$ in terms of $P(\omega)$ and $K(\omega)$.

The output $H(\omega)e^{i\omega n}$ is passed through the controller filter and added to the input signal to make the input to P equal to $(H(\omega)K(\omega) + 1)e^{i\omega n}$. Then it is passed through P:

$$P(\omega)\left(H(\omega)K(\omega) + 1\right)e^{i\omega n} = H(\omega)e^{i\omega n} \tag{7.9}$$

Cancelling the exponential and isolating $H(\omega)$ yields:

$$P(\omega)K(\omega)H(\omega) + P(\omega) - H(\omega) = 0 \tag{7.10}$$
$$H(\omega) = -\frac{P(\omega)}{P(\omega)K(\omega) - 1} = \frac{P(\omega)}{1 - P(\omega)K(\omega)} \tag{7.11}$$

This is Black's formula. It can also be expressed as

$$H(\omega) = \frac{\text{Forward Gain}}{1 - \text{Loop Gain}} \tag{7.12}$$

The expression can be changed by subtracting the response of the controller filter instead of adding it. In that case, we get

$$H(\omega) = \frac{P(\omega)}{1 + K(\omega)P(\omega)} \tag{7.13}$$

What if $K(\omega)P(\omega) >> 1$? In this case, we can say $H(\omega) \approx \frac{1}{K(\omega)}$; if the loop gain of a feedback loop is very large, the plant does not matter.

The controller can compensate for non-ideal behaviour; if $K(\omega) = K$, then $H(\omega)$ is roughly constant as well, even if the plant is nonoptimal.

If H is cascaded with K in this case, the net filter will be characterized by $y(n) = x(n)$.

7.4 CT-LTI Systems

We know that in discrete time, any signal is a convolution of its inputs at specific points with the Kronicker delta:

$$x(n) = \sum_{m \in \mathbb{R}} x(m)\delta(n-m) = (x \circledast \delta)(n) = (\delta \circledast x)(n) \tag{7.14}$$

In continuous time, we have seen that this is equivalent to

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \tag{7.15}$$

Just like in discrete time, if the impulse response is known then the response to any input is known, because of linearity and time-invariance:

$$F(\delta(t)) = f(t) \tag{7.16}$$

$$F(\delta(t-\tau)) = f(t-\tau) \tag{7.17}$$

$$F(x(\tau)\delta(t-\tau)) = x(\tau)f(t-\tau) \tag{7.18}$$

$$\therefore x(t) = \int_{-\infty}^{\infty} x(\tau)f(t-\tau)d\tau \implies y(t) = \int_{-\infty}^{\infty} x(\tau)f(t-\tau)d\tau y(t) = (x \circledast f)(t) \tag{7.19}$$

where convolution in continuous time is defined as follows:

$$(x \circledast f)(t) = \int_{-\infty}^{\infty} x(\tau)f(t-\tau)d\tau \tag{7.20}$$

Let $\lambda = t - \tau \implies \tau = t - \lambda \implies d\tau = -d\lambda$. Then the integral becomes

$$y(t) = \int_{\infty}^{-\infty} f(\lambda)x(t-\lambda)(-d\lambda) = (f \circledast x)(t) \tag{7.21}$$

So, CT convolution is also commutative.

In the frequency domain, we let $x(t) = e^{i\omega t}$; then

$$y(t) = \int_{-\infty}^{\infty} f(\lambda)e^{i\omega(t-\lambda)}d\lambda \tag{7.22}$$

$$y(t) = \left(\int_{-\infty}^{\infty} f(\lambda)e^{-i\omega\lambda}d\lambda\right)e^{i\omega t} = F(\omega)e^{i\omega t} \tag{7.23}$$

This is almost exactly the same as the discrete-time case.

As an example, consider an RC circuit where $x(t)$ is a voltage source and $y_c(t)$ is the voltage across the capacitor.

$$y_c(t) = \frac{1}{C}\int_{-\infty}^{t} I(\tau)d\tau = \frac{1}{C}\int_{-\infty}^{t} \frac{x(\tau) - y_c(\tau)}{R}d\tau \tag{7.24}$$

$$\tag{7.25}$$

Differentiating, we get

$$y_c(t) = \frac{1}{RC}\left(x(t) - y_c(t)\right)\dot{y}_c(t) + \frac{1}{RC}y_c(t) = \frac{1}{RC}x(t) \tag{7.26}$$

Let $x(t) = e^{i\omega t} \implies y_c(t) = H_c(\omega)e^{i\omega t}$. Then

$$i\omega + \frac{1}{RC}H_c(\omega) = \frac{1}{RC} \tag{7.27}$$

$$1 + RCi\omega = H_c(\omega) \tag{7.28}$$

The magnitude of the frequency response can be plotted:

$$|H_c(\omega)|\,\frac{1/RC}{|| \to ||} \tag{7.29}$$

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Lecture 8: More Than You Ever Wanted To Know About RC Circuits

Lecturer: Babak Ayazifar *18 September* *Aditya Sengupta*

8.1 RC Circuit Frequency Response contd.

As covered previously, the DE describing the output voltage over a capacitor in an RC circuit is

$$RC\dot{y}_c(t) + y_c(t) = x(t) \tag{8.1}$$

Let $x(t) = e^{i\omega t}$ and take derivatives, then

$$y_c(t) = H_c(\omega)e^{i\omega t} \implies \dot{y}(t) = (i\omega)H(\omega)e^{i\omega t} \tag{8.2}$$
$$RC(i\omega)H_c(\omega)e^{i\omega t} + H_c(\omega)e^{i\omega t} = e^{i\omega t} \tag{8.3}$$
$$(i\omega RC + 1)H_c(\omega) = 1 \tag{8.4}$$
$$H_c(\omega) = \frac{1/RC}{i\omega - \left(-\frac{1}{RC}\right)} \tag{8.5}$$

which can be graphically analyzed with coloured vectors. The magnitude response is 1 at 0, and going in either direction it curves downwards (with a flatter slope at greater ω due to marginal changes in the length of the vector from $\frac{-1}{RC}$ to $i\omega$ with greater ω) to reach 0 eventually.

The phase response:

$$\sphericalangle H_c(\omega) = \sphericalangle\frac{1}{RC} - \sphericalangle\left(i\omega - \frac{-1}{RC}\right) = -\sphericalangle\text{red vector} \tag{8.6}$$

This is roughly a negative sine wave with magnitude $\frac{\pi}{2}$.

We have found that zero frequency corresponds to a gain of unity. Send in an input signal of $e^{i0t} = 1$, and we get $H_c(0)e^{i0t} = 1$. This passes the "smell test"; physically, the capacitor opens up, so all of the voltage drop is across the capacitor.

When $\omega \to \infty$, the system becomes a short circuit, so the voltage drop goes to 0. Since $H_c(\infty) = 0$, this is consistent.

8.2 Side Note

We know that $y(t) = H(\omega)e^{i\omega t}$ for an input of $e^{i\omega t}$. We can rewrite this in polar form to get

$$|H(\omega)|e^{i\omega\left(t+\frac{\sphericalangle H(\omega)}{\omega}\right)} \tag{8.7}$$

We can set a cutoff frequency $\omega_c = \frac{1}{RC}$. The input $e^{i\omega_c t}$ yields the output

$$y(t)_{\omega_c} = \frac{1}{\sqrt{2}}e^{i\frac{1}{RC}\left(t-\frac{\pi/4}{1/RC}\right)} \tag{8.8}$$

(significance?)

Example

Consider a system with impulse response $g(t) = \beta e^{-\alpha t}u(t)$, with positive α and β. We want to find the frequency response:

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_0^{\infty} \beta e^{-\alpha t}e^{-i\omega t} \tag{8.9}$$

$$G(\omega) = \frac{\beta}{\alpha + i\omega}e^{-(\alpha+i\omega)t}|_0^{\infty} \tag{8.10}$$

$e^{-i\omega t}$ oscillates about the unit circle as $t \to \infty$, so there is no limit, but since there is a negative real component $e^{-\alpha t}$, the quantity goes to 0 as $t \to \infty$. Therefore,

$$G(\omega) = \frac{\beta}{\alpha + i\omega} \tag{8.11}$$

We can use this to convert a frequency response to an impulse response. Consider an RC circuit characterized by

$$RC\dot{h}_c(t) + h_c(t) = \delta(t) \tag{8.12}$$

whose frequency response we have already derived as

$$H_c(\omega) = \frac{1/RC}{i\omega + 1/RC} \tag{8.13}$$

This looks similar to the expression for $G(\omega)$, with $\alpha = \beta = \frac{1}{RC}$. Therefore, $h_c(t) = \frac{1}{RC}e^{-t/RC}u(t)$.

This can also be found by solving the differential equation:

$$RC\dot{h}_c(t) + h_c(t) = \delta(t), h_c(0^-) = 0RC\dot{h}_c(t)e^{t/RC} + h_c(t)e^{t/RC} = e^{t/RC}\delta(t) = \delta(t) \tag{8.14}$$

$$RC\left(\dot{h}_c(t)e^{t/RC} + h_c(t)\frac{1}{RC}e^{t/RC}\right) = \delta(t) \tag{8.15}$$

$$\frac{d}{dt}\left(h_c(t)e^{t/RC}\right) = \frac{1}{RC}\delta(t) \tag{8.16}$$

$$\int_{-\infty}^{\tau}\frac{d}{dt}\left(h_c(t)e^{t/RC}\right)dt = \frac{1}{RC}\int_{-\infty}^{\tau}\delta(t)dt = \frac{1}{RC}u(\tau) \tag{8.17}$$

(We evaluate the last step by recognizing that the Dirac delta is the derivative of the unit step function)

$$h_c(\tau)e^{\tau/RC} = \frac{u(\tau)}{RC} \tag{8.18}$$

$$h_c(t) = \frac{1}{RC}e^{-t/RC}u(t) \tag{8.19}$$

8.3 RC Circuit Step Response

Let $x(t) = u(t)$, then

$$y(t) = \int_{-\infty}^{\infty} h_c(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h_c(\tau)u(t-\tau)d\tau \tag{8.20}$$

$$0 \le t-\tau \iff \tau \le t \implies y(t) = \int_{-\infty}^{t} h_c(\tau)d\tau \tag{8.21}$$

We derived the impulse response previously, therefore this integral (only for positive t; with negative t, everything is 0) becomes

$$\frac{1}{RC}\int_0^t e^{-\tau/RC}d\tau = 1 - e^{-t/RC} \tag{8.22}$$

We can remove the caveat that this only works for positive t by multiplying by $u(t)$. Wait long enough, and the circuit's response becomes close to 1; the capacitor is charging before then, and after a certain point, it opens.

8.4 RC Circuit Finite-Duration Pulse

Consider an input signal that is $\frac{1}{T}$ for all $t \in [0, T]$, and 0 elsewhere. We can express this signal as a combination of unit-steps:

$$x(t) = \frac{1}{T}u(t) - \frac{1}{T}u(t-T) \tag{8.23}$$

Since the system is LTI, we can change this to the output of each input expression:

$$y_c(t) = \frac{1}{T}y(t) - \frac{1}{T}y(t-T) \tag{8.24}$$

where $y(t)$ is the step-response.

$$y_c(t) = \frac{1}{T}\left((1-e^{-t/RC})u(t) - (1-e^{-(t-T)/RC})u(t-T)\right) \tag{8.25}$$

This can be analyzed based on where t is:

$$y_c(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{T}(1-e^{-t/RC})u(t) & 0 < t < T \\ \frac{e^{T/RC}-1}{T}e^{-t/RC} & t \geq T \end{cases} \tag{8.26}$$

For t between 0 and T, the response increases; when the decaying exponential term is added in the $t > T$ case, that dominates to make the graph go back down to 0 starting at T. This is consistent with what we would expect from a finite-duration pulse.

In the limit $T \to 0$, this becomes the Dirac delta, which can be verified with the height of the peak going to ∞, and the area going to 1.

8.5 Integrator-Adder-Gain Block Diagrams

These are the counterparts to DAG block diagrams in discrete time. Using the RC circuit as an example,

$$RC\dot{y} + y = x \tag{8.27}$$

$$\dot{y} = -\frac{1}{RC}y + \frac{1}{RC}x \tag{8.28}$$

$$y = -\frac{1}{RC}\int y + \frac{1}{RC}\int x \tag{8.29}$$

This can be translated into a diagram that looks exactly like a DAG diagram, except with the integral sign instead of a delay block.

EE 120: Signals and Systems **Fall 2018**

Lecture 9: LCCDEs, State-Space Representations

Lecturer: Babak Ayazifar *20 September* *Aditya Sengupta*

9.1 Characterizing a System from a DE

A system is characterized by the generalized differential equation

$$a_N \frac{d^N y(t)}{dt^N} + \cdots + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + \cdots + b_0 x(t) \tag{9.1}$$

whose order is max(M, N). Assume this system is causal and BIBO stable; what is its frequency response?

We can give the system an input of $e^{i\omega t}$ and assume an output of $H(\omega)e^{i\omega t}$, which gives us this generalized form:

$$a_N \frac{d^N H(\omega)e^{i\omega t}}{dt^N} + \cdots + a_0 H(\omega)e^{i\omega t} = b_M \frac{d^M e^{i\omega t}}{dt^M} + \cdots + b_0 e^{i\omega t} \tag{9.2}$$

$$H(\omega) \sum_{k=0}^{N} a_k (i\omega)^k = \sum_{k=0}^{M} b_k (i\omega)^k \tag{9.3}$$

$$H(\omega) = \frac{\sum_{k=0}^{M} b_k (i\omega)^k}{\sum_{k=0}^{N} a_k (i\omega)^k} \tag{9.4}$$

This lets us simplify our analysis. Consider the DE for the RC circuit from before:

$$RC\dot{y}(t) + y(t) = x(t) \tag{9.5}$$

This has $a_1 = RC, a_0 = 1, b_0 = 1$. Therefore the frequency response is

$$H(\omega) = \frac{1}{1 + RC(i\omega)} \tag{9.6}$$

Similarly, for the mass-spring-damper system,

$$M\ddot{y}(t) + D\dot{y}(t) + Ky(t) = x(t) \tag{9.7}$$

$$G(\omega) = \frac{1}{M(i\omega)^2 + D(i\omega) + K} \tag{9.8}$$

9.2 State-Space Representation

We can provide a state-space representation of a system by taking the output of each integrator in the IAG block diagram as a state variable. This gives us

$$y(t) = q_1(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + 0 \cdot x(t) \tag{9.9}$$

$$\tag{9.10}$$

where $q_2(t)$ is the output of the first integrator. We can make a state-evolution equation as well. The input to the integrator yielding $q_1(t)$ is $q_2(t) - \frac{D}{M} q_1(t)$. Therefore

$$\dot{q}_1(t) = -\frac{D}{M} q_1(t) + q_2(t) \tag{9.11}$$

$$\dot{q}_2(t) = -\frac{K}{M} q_1(t) + 0 \cdot q_2(t) + \frac{1}{M} x(t) \tag{9.12}$$

$$\tag{9.13}$$

which can be represented in matrix form:

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{D}{M} & 1 \\ -\frac{K}{M} & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} x(t) \tag{9.14}$$

This is now in the general form $\dot{q}(t) = Aq(t) + Bx(t)$.

9.3 Discrete-Time State Space Equation

Consider the following difference equation:

$$y(n) - \alpha^2 y(n-2) = x(n) - x(n-2) \tag{9.15}$$

$$\tag{9.16}$$

Using established methods, we can find the frequency response:

$$H(\omega) = \frac{1 - e^{-2i\omega}}{1 - \alpha^2 e^{2i\omega}} \tag{9.17}$$

In general, if we have a difference equation

$$sum_{k=0}^{N} a_N y(n-N) = \sum_{k=0}^{M} b_M x(n-M) \tag{9.18}$$

then we can use basically the same method as in continuous time to solve for the frequency response and get

$$H(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-ik\omega}}{\sum_{k=0}^{N} a_k e^{-ik\omega}} \tag{9.19}$$

and substituting the above coefficients into this equation will give the same response.

We can also draw a DAG block diagram, and select state-space variables to be after each delay element. Let $q_1(n)$ be after the last delay block, then $q_2(n) = q_1(n+1)$, and the input to the first delay block is $q_2(n+1)$. Then

$$q_1(n+1) = 1 \cdot q_2(n) + 0 \cdot q_1(n) + 0 \cdot x(n) \tag{9.20}$$

$$q_2(n+1) = \alpha^2 \left(q_1(n) + x(n)\right) - x(n) = 0 \cdot q_2(n) + \alpha^2 q_1(n) + (\alpha^2 - 1)x(n) \tag{9.21}$$

which can be represented in matrix form,

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha^2 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha^2 - 1 \end{bmatrix} x(n) \tag{9.22}$$

Also, we can express the output $y(t)$ as a function of state variables and input,

$$y(n) = \begin{bmatrix} 1 & 0 \end{bmatrix} q(n) + 1 \cdot x(n) \tag{9.23}$$

We can find the eigenvalues of this system. Take the determinant of $|\lambda I - \alpha| = \lambda^2 - \alpha^2 \implies \lambda = \pm\alpha$. We previously found the frequency response, which we can rewrite as

$$H(\omega) = \frac{(e^{i\omega} + 1)(e^{i\omega} - 1)}{(e^{i\omega} + \alpha)(e^{i\omega - \alpha})} \tag{9.24}$$

Let $\alpha = 0.99$, very close to 1 inside the unit circle. We can draw coloured vectors to show that the vectors to the $+$ terms on the numerator and denominator are roughly equal for ω reasonably far from the real axis, and the same for the $-$ terms. Therefore the frequency response has magnitude 1 except around integer multiples of π.

$q(n)$ can be written as a linear combination of Av_1 and Av_2, the matrix multiplied by the eigenvectors. So

$$q(0) = \beta_1 \vec{v_1} + \beta_2 \vec{v_2} \tag{9.25}$$

$$q(1) = \beta_1 A\vec{v_1} + \beta_2 A\vec{v_2} + Bx(0) = \beta_1 \alpha \vec{v_1} - \beta_2 \alpha \vec{v_2} + Bx(0) \tag{9.26}$$

$$q(2) = \beta_1 \alpha^2 \vec{v_1} - \beta_2 \alpha^2 \vec{v_2} + ABx(0) + Ax(1) \tag{9.27}$$

9.4 System with Irrational Frequency Response

Consider a system whose impulse response is $\frac{1}{2T}$ for $t \in [-T, T]$ and 0 elsewhere. We can find the frequency response:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt = \int_{-T}^{T} \frac{1}{2T}e^{-i\omega t} \tag{9.28}$$

$$\tag{9.29}$$

using which we can find the DC gain of the system $H(0) = 1$. Anyway, some integrals get you

$$H(\omega) = \frac{1}{2T}\frac{e^{i\omega T} - e^{-i\omega T}}{i\omega} = \frac{\sin(\omega T)}{\omega T} \tag{9.30}$$

Sinc looks weird. As T increases, frequency becomes narrower, and as T decreases, frequency becomes broader.

EE 120: Signals and Systems **Fall 2018**

Lecture 10: Fourier Analysis

Lecturer: Babak Ayazifar *27 September* *Aditya Sengupta*

10.1 Discrete-Time Fourier Series

Consider the set of 2-periodic signals, with the property $x(n+2) = x(n) \forall n$. We can represent these signals as a vector:

$$\vec{x} = \begin{bmatrix} x(0) \\ x(1) \end{bmatrix} \tag{10.1}$$

This allows us to represent signals in this space as a linear combination of basis elements, say $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A signal that is 4 when n is even and 2 when n is odd can be written as $x(n) = 4\phi_0(n) + 2\phi_1(n)$, for appropriate definitions of ϕ_0 and ϕ_1.

The concept of Fourier analysis is to change the basis for periodic signals so as to make it easier to interpret in the frequency domain. Consider signal $\psi_0(n) = 1 \forall n$, with corresponding vector $\psi_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and signal $\psi_1(n) = (-1)^n$, with corresponding vector $\psi_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Note that both of these vectors have a squared magnitude of 2, which is the period of the signal. We also observe that these signals are orthogonal, using the following standard inner product:

$$\langle \psi_0, \psi_1 \rangle = \psi_0^T \psi_1^* \tag{10.2}$$

In this case, the signals are real-valued, so the complex conjugate is the signal itself. We can write any signal in terms of this new basis:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = X_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + X_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{10.3}$$

We find that $X_0 = 3$ and $X_1 = 1$. In general, we can find this by projecting the signal onto each of the basis elements:

$$\vec{x} = \sum_{i=0}^{p-1} \frac{\langle \vec{x}, \psi_i \rangle}{\langle \psi_i, \psi_i \rangle} \psi_i(n) \tag{10.4}$$

By rewriting the input signal in these terms, we know

$$x(n) = X_0 e^{i0\omega_0 n} + X_1 e^{i1\omega_0 n} \tag{10.5}$$

where $\omega_0 = \frac{2\pi}{2} = \pi$.

We see that the frequencies are harmonically related; they are integer multiples of the fundamental frequency. The linear combination of $p = 2$ harmonically related exponentials can therefore construct x. In general, the coefficient X_i tells us how much frequency $i \cdot \omega_0$ contributes to x. Note that no frequency other than $0, \omega_0, \ldots, (p-1)\omega_0$ contributes to x. We can construct a vector of new basis elements, as follows:

$$\psi_k = \begin{bmatrix} \psi_k(0) \\ \psi_k(1) \end{bmatrix} = \begin{bmatrix} 1 \\ e^{ik\omega_0} \end{bmatrix} \tag{10.6}$$

For $p = 3$, we find that $\omega_0 = \frac{2\pi}{3}$, therefore any signal that is 3-periodic can be represented as

$$x(n) = X_0 e^{i0\omega_0 n} + X_1 e^{i1\omega_0 n} + X_2 e^{i2\omega_0 n} \tag{10.7}$$

More generally,

$$x(n) = \sum_{k=0}^{p-1} X_k \psi_k(n) = \sum_{k=0}^{p-1} X_k e^{ik\omega_0 n} \tag{10.8}$$

This is the *synthesis equation*.

"I feel like a Sunday morning preacher. I'm going to ask you to BELIEVE that the ψ_ks are all orthogonal. BELIEVE, my flock."

Like in the $p = 2$ case, orthogonality allows us to find the coefficients just by projecting the vector onto each of the basis elements:

$$X_k = \frac{\langle \vec{x}, \psi_k \rangle}{\langle \psi_k, \psi_k \rangle} \tag{10.9}$$

This linear combination allows us to express signals as vectors, consisting of the coefficients in the Fourier expansion. This lets us define an inner product on signals,

$$\langle f(n), g(n) \rangle = \vec{f}^T \vec{g}^* = \sum_{n=0}^{p-1} f(n) g^*(n) \tag{10.10}$$

Here, the complex conjugate matters, as the basis elements are in general complex-valued. So the coefficient expression becomes

$$X_k = \frac{\sum_{n=0}^{p-1} x(n)\psi_k^*(n)}{\sum_{n=0}^{p-1} \psi_k(n)\psi_k*(n)} = \frac{1}{p}\sum x_n e^{-ik\omega_0 n} \tag{10.11}$$

The last expression is arrived at using the fact that the complex conjugate of e^{ix} is e^{-ix} in the numerator. In the denominator, we have

$$\langle \psi_k, \psi_k \rangle = \psi_k^T \psi_k^* = \sum_{n=0}^{p-1} \psi_k(n)\psi_k^*(n) = \sum_{n=0}^{p-1} e^{ik\omega_0 n} e^{-ik\omega_0 n} = p \tag{10.12}$$

We can use this to reverse the synthesis equation to the *analysis equation*, which gives us the coefficients of the basis element associated with each frequency:

$$X_k = \frac{1}{p}\sum_{n=0}^{p-1} x(n) e^{-ik\omega_0 n} \tag{10.13}$$

This will allow us to show why $\psi_k \psi_l$.

$$\langle \psi_k, \psi_l \rangle = \sum_{n=0}^{p-1} \psi_k(n)\psi_l^*(n) = \sum_{n=0}^{p-1} e^{i(k-l)\omega_0 n} \tag{10.14}$$

Some things with geometric series will tell us

$$\langle \psi_k, \psi_l \rangle = \frac{e^{i(k-l)\omega_0 P} - 1}{e^{i(k-l)\omega_0} - 1} \tag{10.15}$$

The numerator is something on the unit circle minus 1, therefore the inner product is 0. The inner product is only nonzero when $k = l$. Therefore

$$\langle \psi_k, \psi_l \rangle = p\delta(k-l) \tag{10.16}$$

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Lecture 11: The Discrete-Time Fourier Series and the DFT

Lecturer: Babak Ayazifar *2 October* *Aditya Sengupta*

11.1 DTFS

Consider the set of p-periodic discrete time signals, i.e. signals such that

$$x(n+p) = x(n)\ \forall n \in \mathbb{Z} \exists p \in \{1, 2, \dots\} \tag{11.1}$$

We know that we can decompose such signals into

$$x(n) = \sum_{k=0}^{p-1} X_k e^{ik\omega_0 n} = \sum_{k=0}^{p-1} X_k \psi_k(n) \tag{11.2}$$

which is a linear combination of complex exponentials, where $\omega_0 = \frac{2\pi}{p}$. We know that the coefficients can be found by

$$X_k = \frac{1}{p} \sum_{n=0}^{p-1} x(n) e^{-ik\omega_0 n} = \sum_{n=0}^{p-1} x(n) \psi_k^*(n) \tag{11.3}$$

and so

$$x(n) = X_0 e^{i \cdot 0 \cdot \omega_0 n} + X_1 e^{i\omega_0 n} + \dots +_k e^{ik\omega_0 n} + \dots + X_{p-1} e^{i(p-1)\omega_0 n} \tag{11.4}$$

Any p-periodic signals has at most p harmonically related frequencies $k\omega_0$, $0 \leq k \leq p-1$.

What about the frequencies not in this range that are still integral multiples of ω_0? Why do these not count as harmonically related frequencies in the DTFS?

Consider the case $k = -1$.

$$e^{-i\omega_0 n} = e^{-i\omega_0 n} \cdot e^{i \cdot 2\pi n} = e^{-i\omega_0 n} \cdot e^{ip\omega_0 n} = e^{i(p-1)\omega_0 n} \tag{11.5}$$

because $p\omega_0 = 2\pi$ by definition.

Similarly, any frequency that is an integer multiple of $\omega_0 n$ can be mapped to a frequency in the above-mentioned range: $e^{ik'\omega_0 n} = e^{ik\omega_0 n}$, $k \in [0, p-1]$ for $k' \pmod p = k$.

Algebraically k and n are identical, so if a signal is periodic in n with period p, it should also be periodic in k with period p. Therefore

$$\psi_k(n+p) = e^{ik\omega_0(n+p)} = e^{ik\omega_0 n} e^{ik\omega_0 p} = e^{ik\omega_0 n} = \psi_k(n) \tag{11.6}$$

but also

$$\psi_{k+p}(n) = e^{i(k+p)\omega_0 n} = e^{ik\omega_0 n} e^{ip\omega_0 n} = e^{ik\omega_0 n} = \psi_k(n) \tag{11.7}$$

This explicitly shows that the frequency $(k+p)\omega_0$ is the same as the frequency $k\omega_0$. This means any set of p continguous frequencies consist of a valid DTFS expansion of a signal. This can be used to exploit symmetry or antisymmetry in problems, by combining frequencies from $-\frac{p}{2}\omega_0$ to $\frac{p}{2}\omega_0$ to form sines or cosines.

Technically, the frequencies do not have to be contiguous, as long as they directly correspond modulo p to each of the unique frequencies in a contiguous set, but this is usually not helpful.

11.2 Examples

Determine the DTFS coefficients of $x(n) = \cos\left(\frac{\pi}{3}n\right)$.

We first determine p by shifting the signal and requiring that it is the same as the original,

$$\cos\left(\frac{\pi}{3}(n+p)\right) = \cos\left(\frac{\pi}{3}n\right) \tag{11.8}$$

$$\frac{\pi}{3}p = 2\pi l \implies p = 6l \implies p = 6 \tag{11.9}$$

Therefore, six coefficients are required. Ordinarily this would need the previously-derived decomposition equation, but by the inverse Euler's formula, we know that

$$x(n) = \cos\left(\frac{\pi}{3}n\right) = \frac{1}{2}e^{-i\frac{\pi}{3}n} + \frac{1}{2}e^{i\frac{\pi}{3}n} \tag{11.10}$$

and therefore by pattern-matching to the DTFS, we see that $X_{-1} = X_1 = \frac{1}{2}$. Since these two harmonics completely determine the cosine, the other DTFS coefficients are zero. The X_{-1} can be shifted into the range $k \in [0,5]$ by adding p, which is 6. Therefore, we have

$$\begin{gathered} X_0 = X_2 = X_3 = X_4 = 0 \\ X_1 = X_5 = \frac{1}{2} \end{gathered}$$

Determine the DTFS coefficients of $x(n) = \cos^2\left(\frac{\pi}{3}n\right)$.

$$x(n) = \cos^2\left(\frac{\pi}{3}n\right) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{3}n\right) \tag{11.11}$$

We find p based on this,

$$\frac{2\pi}{3}p = 2\pi l \implies p = 3l \implies p = 3 \tag{11.12}$$

Therefore we need 3 coefficients. We rewrite the signal in terms of complex exponentials,

$$x(n) = \frac{1}{2} + \frac{1}{4}e^{-i\frac{2\pi}{3}n} + \frac{1}{4}e^{i\frac{2\pi}{3}n} \tag{11.13}$$

Matching coefficients, we see that $X_0 = \frac{1}{2}$, and $X_{-1} = X_1 = \frac{1}{4}$.

Some functions are not periodic in an integer p; these functions, for example $x(n) = \cos(n)$, do not have a DTFS expansion.

11.3 A DTFS Puzzle

Consider a signal x with the following properties:

$$x(n) \in \mathbb{R}$$
$$x(n+4) = x(n)\ \forall n \in \mathbb{Z}$$
$$\sum_{n=-1}^{2} x(n) = 2$$
$$\sum_{n=-1}^{2} (-1)^n x(n) = 4$$
$$\sum_{n=-1}^{2} x(n)\cos\left(\frac{\pi}{2}n\right) = \sum_{n=-1}^{2} x(n)\sin\left(\frac{\pi}{2}n\right) = 0$$

First, we see that $p = 4$. This is not necessarily the fundamental period, but it is an upper bound on this period. It is required to find 4 DTFS coefficients.

We look at the analysis equation,

$$X_k = \frac{1}{p}\sum_{n\in\langle p\rangle} x(n)e^{-ik\omega_0 n} \tag{11.14}$$

Note that all the facts about summations are over an interval $n = -1$ to $n = 2$, so it is convenient to select this as our interval. Note that the notation $\langle p \rangle$ means any sequence of p contiguous integers.

We use the analysis equation to find X_0:

$$X_0 = \frac{1}{p} \sum_{n=-1}^{2} x(n) = \frac{1}{4} \cdot 2 = \frac{1}{2} \tag{11.15}$$

and X_1:

$$X_1 = \frac{1}{4} \sum_{n=-1}^{2} x(n) e^{-i\frac{\pi}{2}n} = \frac{1}{4} \sum x(n) \left(\cos\left(\frac{\pi}{2}n\right) + i \sin\left(\frac{\pi}{2}n\right)\right) = 0 \tag{11.16}$$

which we know because $\sum_{n=-1}^{2} x(n) \cos\left(\frac{\pi}{2}n\right) = \sum_{n=-1}^{2} x(n) \sin\left(\frac{\pi}{2}n\right) = 0$. Similarly, we know that $X_{-1} = 0$.

Finally, we calculate X_2:

$$X_2 = \frac{1}{4} \sum_{n=-1}^{2} x(n) e^{-i2\frac{\pi}{2}n} = \frac{1}{4} \sum_{n=-1}^{2} x(n) e^{-i\pi n} = \frac{1}{4} \sum_{n=-1}^{2} (-1)^n x(n) = 1 \tag{11.17}$$

Therefore the signal is

$$x(n) = \frac{1}{2} + e^{i\pi n} = \frac{1}{2} + (-1)^n \tag{11.18}$$

which we can plot, if I can ever get around to getting sympy to work. We get

$$x(n) = \begin{cases} \frac{3}{2} & n \text{ even} \\ \frac{-1}{2} & n \text{ odd} \end{cases} \tag{11.19}$$

11.4 The DTFS vs the DFT

The DTFS is characterized by

$$x(n) = \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 n};\ X_k = \frac{1}{p} \sum_{n \in \langle p \rangle} x(n) e^{-ik\omega_0 n} \tag{11.20}$$

and the DFT is the above just with the $\frac{1}{p}$ on the synthesis equation instead, so

$$X_k = \frac{1}{p}\hat{X}_k \tag{11.21}$$

We can use the DTFS to expand any finite-duration signal, by replicating it throughout the time axis and finding that infinite signal's DTFS.

11.5 Periodic DT Signals through DT-LTI Systems

Consider a signal $x(n)$ that is periodic in p, being used as input to a DT-LTI system with impulse response $h(n)$ and frequency response $H(\omega)$. We shift the signal by its period, and send it through the filter:

$$\hat{x}(n) = x(n+p) \tag{11.22}$$

By time-invariance, we see that

$$\hat{y}(n) = y(n+p) \tag{11.23}$$

Also, we know that $y(n+p) = y(n)$; since $x(n+p) = x(n)$, the filter receives the same input in both cases, when x or $\hat{x}$ are fed to it, and therefore produces the same output in both cases as well. This means that y is p-periodic.

Alternatively, by linearity, we know that when $\widetilde{x}(n) = x(n) - \hat{x}(n)$ is input to the system, the output by linearity is $\widetilde{y}(n) = y(n) - \hat{y}(n)$. Therefore, when we know by periodicity that $x(n) - \hat{x}(n) = 0 \; \forall n$, then $y(n) - \hat{y}(n) = 0$, therefore $y(n) = \hat{y}(n) = y(n+p)$. Therefore, a p-periodic signal input to an LTI system must produce a p-periodic output (potentially with a smaller fundamental period such that p is an integer multiple of its fundamental period, but never with a larger one).

Consider the DTFS expansion of an arbitrary signal being input to a system with frequency response $H(\omega)$. The input is

$$x(n) = \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 n} \tag{11.24}$$

and the output can be found using the eigenfunction property of complex exponentials in an LTI system:

$$y(n) = \sum_{k \in \langle p \rangle} X_k H(k\omega_0) e^{ik\omega_0 n} \tag{11.25}$$

EE 120: Signals and Systems **Fall 2018**

Lecture 12: The Continuous-Time Fourier Series

Lecturer: Babak Ayazifar *4 October* *Aditya Sengupta*

In inner products, we should be aware that

$$\langle x, y \rangle = \langle y, x \rangle^* \tag{12.1}$$

12.1 CTFS

In continuous time, we can still have periodicity, as defined by the property

$$x(t+p) = x(t) \forall t \ in \mathbb{R} \exists p > 0 \tag{12.2}$$

$$\omega = \frac{2\pi}{p} \tag{12.3}$$

So the synthesis equation changes from an infinite sum to an integral over a period. This is quite cool because in general there are an uncountably infinite number of input points that we can represent in a countably infinite set of coefficients. If that seems too good to be true, it is. In general, the input signal is not continuous, but the constituent parts are. So we make the input signal kinda wavy around the discontinuities.

$$\hat{x}_N(t) = \sum_{k=-N}^{N} X_k e^{ik\omega_0 t} = \sum_{k=-N}^{N} X_k \psi_k(t) \tag{12.4}$$

The only frequencies contributing to x are $\ldots, -2\omega_0, -\omega_0, 0, \omega_0, 2\omega_0, \ldots$

How do we find the X_ks?

$$x(t) = \sum_k X_k \psi_k(t) \tag{12.5}$$

where $\psi_k(t)$ forms a basis for different values of k. Specifically, this basis is orthogonal. Claiming orthogonality requires that we define an inner product and show that $\langle \psi_k, \psi_l \rangle = 0$.

In discrete time, our inner product was

$$\langle f, g \rangle = \sum_{n \in \langle p \rangle} f(t) g^* t(t) dt \tag{12.6}$$

In continuous time, this is an integral:

$$\langle f, g\rangle = \int_{\langle p\rangle} f(t)g^*(t)dt \tag{12.7}$$

Note that the ψ_ks are periodic with period p, as

$$\psi_k(t+p) = e^{ik\omega_0(t+p)} = e^{ik\omega_0 t}e^{ik(2\pi)} = e^{ik(\omega_0)t} = \psi_k(t) \tag{12.8}$$

so we can be reassured that our signal with p-periodicity is a linear combination of p-periodic elements. Let's look at the inner product of two arbitrary elements, $\langle\psi_k, \psi_l\rangle$.

$$\langle\psi_k, \psi_l\rangle = \int_0^p \psi_k(t)\psi_l^*(t)dt = \int_0^p e^{i(k-l)\omega_0 t}dt \tag{12.9}$$

In the case where $k = l$, this becomes

$$\langle\psi_k, \psi_k\rangle = \int_0^p 1dt = p \implies ||\psi_k||^2 = p \tag{12.10}$$

This is the energy of ψ_k. This can be found more directly and generally using the following expression:

$$\epsilon_f = \langle f, f\rangle = \int_{\langle p\rangle} f(t)f^*(t)dt \tag{12.11}$$

When $k \neq l$, the integral for the inner product becomes

$$\langle\psi_k, \psi_l\rangle = \int_0^p \cos((k-l)\omega_0 t)dt + i\int_0^p \sin((k-l)\omega_0 t) \tag{12.12}$$

These are both 0, as sine and cosine are being integrated over an integer number of periods.

Therefore,

$$\langle\psi_k, \psi_l\rangle = p\delta(k-l) \tag{12.13}$$

where the δ is a Kronecker delta. Therefore, we can determine X_k by projecting x onto ψ_k:

$$\langle x, \psi_k\rangle = \left\langle \sum_{l=-\infty}^{\infty} X_l\psi_l, \psi_k \right\rangle = \sum_{l=-\infty}^{\infty} X_l \langle\psi_l, \psi_k\rangle = X_k \langle\psi_k, \psi_k\rangle \tag{12.14}$$

Therefore

$$X_k = \frac{\langle x, \psi_k \rangle}{\langle \psi_k, \psi_k \rangle} = \frac{1}{p} \langle x, \psi_k \rangle = \frac{1}{p} \int_{\langle p \rangle} x(t) \psi_k * (t) dt \tag{12.15}$$

$$X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-ik\omega_0 t} dt \tag{12.16}$$

We can write down the CTFS equations:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} \tag{12.17}$$

$$X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-ik\omega_0 t} dt \tag{12.18}$$

This is really similar to the DTFS equations, with the sole exception that we act in t and not n, and that the analysis equation takes an integral rather than an infinite sum.

12.2 Examples

12.2.1 Example 1

Consider a Dirac impulse train $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - np)$. Find the CTFS coefficients.

This is a p-periodic signal that is a Dirac delta at one point and zero elsewhere. We use the analysis equation,

$$X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-ik\omega_0 t} dt = \frac{1}{p} \int_{-p/2}^{p/2} \delta(t) e^{-ik\omega_0 t} dt \tag{12.19}$$

By the sampling property, this is just sampled at $t = 0$, so

$$X_k = \frac{1}{p} e^0 = \frac{1}{p} \tag{12.20}$$

That is, the p-periodic Dirac delta has equal CTFS coefficients that sum to 1.

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - np) = \frac{1}{p} \sum_{k=-\infty}^{\infty} e^{ik\omega_0 t} \tag{12.21}$$

12.2.2 Example 2

Consider a pulse train repeating every p seconds, of magnitude $\frac{1}{\Delta}$ going from time $np - \frac{\Delta/2}{}$ to time $np + \frac{\Delta}{2}$. This signal is $z(t)$; find the coefficients Z_k.

$$Z_k = \frac{1}{p}\int_{\langle p\rangle} z(t)e^{-ik\omega_0 t}dt = \frac{1}{p}\int_{-\Delta/2}^{\Delta/2}\frac{1}{\Delta}e^{-ik\omega_0 t} \tag{12.22}$$

$$Z_k = \frac{1}{p\Delta}\frac{1}{-ik\omega_0}\left(e^{-ik\omega_0\frac{\Delta}{2}} - e^{-ik\omega_0\frac{-\Delta}{2}}\right) \tag{12.23}$$

$$Z_k = \frac{\sin(k\omega_0\frac{\Delta}{2})}{pk\omega_0\Delta/2} = \frac{1}{p}(k\omega_0\Delta/2) \tag{12.24}$$

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Lecture 13: CTFS Wrap-Up, DTFT

Lecturer: Babak Ayazifar *9 October* *Aditya Sengupta*

13.1 CTFS

We know the CTFS equations,

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} \tag{13.1}$$

$$X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-ik\omega_0 t} \tag{13.2}$$

Suppose we have a signal that cannot be perfectly expressed as a linear combination of ψ_ks. We want to find the closest approximation to the signal in terms of the ψ_k basis. That is, we want to minimize

$$||e_N||^2 = \langle e_N, e_N \rangle = \int_{\langle p \rangle} e_N(t) e_N^*(t) = \int_{\langle p \rangle} |e_N(t)|^2 dt \tag{13.3}$$

This is the energy of the error in one period.

Since $e_N \perp span\{\psi_k\}$, we can say that $\langle e_N, \psi_k \rangle = 0$ for all k. Let $e_N = x - \hat{x}_N$, then

$$\langle x - \hat{x}_N, \psi_l \rangle = 0 \tag{13.4}$$

$$\langle x, \psi_l \rangle = \langle \hat{x}_N, \psi_l \rangle = \left\langle \sum_{k=-N}^{N} a_k \psi_k \right\rangle \tag{13.5}$$

$$\langle x, \psi_l \rangle = a_l \langle \psi_l, \psi_l \rangle \implies a_l = \frac{\langle x, \psi_l \rangle}{\langle \psi_l, \psi_l \rangle} \tag{13.6}$$

The linear combination that minimizes the energy of the error turns out to be the CTFS coefficients.

13.2 Disappointment

We claim that

$$\lim_{N \to \infty} ||e_N^2|| = 0 \tag{13.7}$$

for every p-periodic signal x with finite energy in one period (finite energy is the case for pretty much every real-world signal). That is, as more terms are added to the expansion, the approximation gets better and better until there is no error.

Let's look at the magnitude of the error:

$$\langle e_N, e_N \rangle = \langle x - \hat{x}_N, e_N \rangle = \langle x, e_N \rangle - \langle \hat{x}_N, e_N \rangle \tag{13.8}$$

We can violently murder the second term due to orthogonality, which leaves us with

$$\langle e_N, e_N \rangle = \langle x, x - \hat{x}_N \rangle = ||x||^2 - \langle x, \hat{x}_N \rangle \geq 0 \tag{13.9}$$

Suppose $x(t) = \sum_{k \in \mathbb{Z}} X_k \psi_k(t)$. Then the inner product becomes

$$\left\langle \sum_{k \in \mathbb{Z}} X_k \psi_k, \sum_{l=-N}^{N} X_l \psi_l \right\rangle = p \sum_{k=-N}^{N} |X_k|^2 \tag{13.10}$$

Therefore, we can say

$$\langle e_N, e_N \rangle = p \sum_{k=-\infty}^{\infty} |X_k|^2 - p \sum_{k=-N}^{N} |X_k|^2 \tag{13.11}$$

Somewhere in this derivation, we derived Parseval's identity:

$$\int_{\langle p \rangle} |x(t)|^2 dt = p \sum_{k=-\infty}^{\infty} |X_k|^2 \tag{13.12}$$

13.3 Aperiodic Discrete-Time Signals: the DTFT

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n} \tag{13.13}$$

This is the analysis equation of the DTFT. We want to use this to make the synthesis equation, which allows us to make an impulse response from the frequency response. We note that $H(\omega)$ is a 2π-periodic continuous function. Recall that in CTFS, we had a function of a continuous variable that could be written as a linear combination of complex exponentials or ψ_ks. The coefficients were given by

$$X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-ik\omega_0 t} dt \tag{13.14}$$

We can now draw a parallel between the frequency-to-time DTFT expression and the time-to-frequency CTFS expression.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{i(-n)\frac{2\pi}{2\pi}\omega} = \sum_{l=-\infty}^{\infty} h(-l) e^{il\omega} \tag{13.15}$$

Also, using the CTFS analysis equation,

$$h(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} H(\omega) e^{i\omega n} d\omega \tag{13.16}$$

This is the DTFT synthesis equation.

We live in the universe of functions of a continuous variable ω, where those functions are 2π-periodic. Let F, G be in this universe.

$$\langle F, G \rangle = \int_{\langle 2\pi \rangle} F(\omega) G^*(\omega) d\omega \tag{13.17}$$

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Lecture 14: Examples in and properties of the DTFT

Lecturer: Babak Ayazifar *11 October* *Aditya Sengupta*

14.1 Examples

1. *Consider $x(n) = e^{i\omega_0 n}$. Find its spectrum $X(\omega)$.*

   We can just substitute this into the DTFT analysis equation,

$$X(\omega) = \sum_{n=-\infty}^{\infty} e^{i\omega_0 n} e^{-i\omega n} = \sum_{n=-\infty}^{\infty} e^{i(\omega_0 - \omega)n} \tag{14.1}$$

   However, this sum does not converge, so we try a different approach. In frequency space, $x(n)$ is represented by a Dirac delta somewhere between $-\pi$ and π: $X(\omega) = \alpha\delta(\omega - \omega_0)$, where α is an unknown magnitude. To find the magnitude, we can use the synthesis equation:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \alpha\delta(\omega - \omega_0)e^{i\omega n} d\omega \tag{14.2}$$

$$x(n) = \frac{\alpha}{2\pi} e^{i\omega_0 n} = e^{i\omega_0 n} \implies \alpha = 2\pi \tag{14.3}$$

   This makes use of the sifting property of the Dirac delta. This is not quite a complete solution, as it does not account for 2π periodicity. Adding in an impulse train (choo choooooo) to account for this, we get

$$X(\omega) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) \tag{14.4}$$

2. *Let $Y(\omega) = e^{-i\omega}$. Find the time-domain representation $y(n)$.*

   We plug this into the synthesis equation,

$$y(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} e^{-i\omega} e^{i\omega n} d\omega = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} e^{i\omega(n-1)} d\omega \tag{14.5}$$

$$\tag{14.6}$$

   This splits into two cases: when $n = 1$, $y(1) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} 1 = 1$. When $n \neq 1$, we get

$$y(n) = \int_{-\pi}^{\pi} \cos(\omega(n-1)) + i\sin(\omega(n-1)) d\omega \tag{14.7}$$

$$y(n) = 0 \tag{14.8}$$

which is essentially a Kronecker delta $\delta(n-1)$.

Alternatively, we can see that the spectrum of a Dirac delta input is

$$\delta(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} e^{i\omega n} d\omega \tag{14.9}$$

which is different from the above only in that it's shifted by 1. Therefore we shift the input by 1 and get $\delta(n-1)$.

3. *Consider an ideal low-pass filter whose frequency response is*

$$H(\omega) = \begin{cases} 1 & |\omega| \leq B \\ 0 & \text{otherwise} \end{cases} \tag{14.10}$$

Find the impulse response.

We use the synthesis equation,

$$h(n) = \frac{1}{2\pi}\int_{-B}^{B} e^{i\omega n} d\omega \tag{14.11}$$

$$\tag{14.12}$$

This splits apart into cases: when $n = 0$, we get $h(0) = \frac{B}{\pi}$, and otherwise,

$$h(n) = \frac{1}{2\pi i n}\left(e^{iBn} - e^{-iBn}\right) = \frac{\sin Bn}{\pi n} \tag{14.13}$$

Thanks to M. Hoˆpital, this reduces to $\frac{B}{\pi}$ at $n = 0$, so we don't even need the special case:

$$\lim_{n\to 0}\frac{\sin Bn}{\pi n} = \lim_{n\to 0}\frac{B\cos Bn}{\pi} = \frac{B}{\pi} \tag{14.14}$$

14.2 Dealing with signals that aren't absolutely summable

How would we get back to the spectrum from the last example? We could try and use the analysis equation, but we'd get something that isn't absolutely summable:

$$\sum_{n=-\infty}^{\infty} h(n)e^{-i\omega n} = \sum_{n=-\infty}^{\infty} \frac{\sin Bn}{\pi n} e^{-i\omega n} \tag{14.15}$$

The square, however, is absolutely summable (which physically corresponds to a signal having finite energy); more formally, $\sum_{n=-\infty}^{\infty}|x(n)|^2 < \infty$. In general, if $h \in l^2$ but $h \notin l^1$ (where $l^k = \{x : \mathbb{Z} \to \mathbb{R}\text{or}\mathbb{Z} \to |\sum_{n=-\infty}^{\infty} x(n)^k < \infty\}$), then we take the Fourier transform as follows:

$$H(\omega) = \lim_{N\to\infty} \sum_{n=-N}^{N} h(n)e^{-i\omega n} \tag{14.16}$$

For signals of slow growth (those that grow at most polynomially with n), the DTFT has a Dirac delta, such as with $x(n) = e^{i\omega_0 n}$. If x grows exponentially, there is no spectrum; this will be handled with the $z-$transform later on.

14.3 Relating input and output in DT-LTI systems with the transform

We know that in a DT-LTI system with impulse response h,

$$y(n) = \sum_m h(m)x(n-m) \tag{14.17}$$

and that

$$Y(\omega) = \sum_n y(n)e^{-i\omega n} = \sum_n \left(\sum_m h(m)x(n-m)\right) e^{-i\omega n} \tag{14.18}$$

With a maniacal laugh, we exchange the order of the sum expressions,

$$Y(\omega) = \sum_m h(m) \sum_n x(n-m)e^{-i\omega n} = \sum_m h(m) \sum_k x(k)e^{-i\omega(k+m)} \tag{14.19}$$

with the change of variable $k = n - m$. We can simplify this to

$$Y(\omega) = \sum_m h(m) \left(\sum_k x(k)e^{-i\omega k}\right) e^{-i\omega m} \tag{14.20}$$

$$Y(\omega) = X(\omega)H(\omega) \tag{14.21}$$

Convolution in time is multiplication in frequency.

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Lecture 15: The Continuous-Time Fourier Transform

Lecturer: Babak Ayazifar *16 October* *Aditya Sengupta*

15.1 Synthesis and Analysis

The synthesis equation in the CTFT is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega \tag{15.1}$$

which is an infinite sum of frequencies with coefficients $\frac{d\omega}{2\pi} X(\omega)$. $|X(\omega)|$ is a measure of the participation of frequency ω in the signal x. By varying the $X(\omega)$s, we can limit a signal to certain frequency bands.

The analysis equation is

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \tag{15.2}$$

If we ***BELIEVE*** that the exponentials $\psi_t(\omega) = e^{-i\omega t}$ are mutually orthogonal, meaning that $\langle \psi_t, \psi_\tau \rangle = 0$ for all $t \neq \tau$. So we can use projections. This is more difficult than before, because $e^{-i\omega t}$ is no longer periodic, because t is a continuous variable. We'l have to slightly change up the definition of the inner product,

$$\langle X(\omega), \psi_\tau(\omega) \rangle = \int_{-\infty}^{\infty} \psi_\tau^*(\omega) d\omega \tag{15.3}$$

$$\langle X(\omega), \psi_\tau(\omega) \rangle = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} dt x(t) \psi_t(\omega) \right) \psi_\tau^*(\omega) d\omega \tag{15.4}$$

$$\langle X(\omega), \psi_\tau(\omega) \rangle = \int_{-\infty}^{\infty} dt x(t) \int_{-\infty}^{\infty} \psi_t(\omega) \psi_\tau^*(\omega) d\omega \tag{15.5}$$

In the case $t \neq \tau$, we get $\langle \psi_t, \psi_\tau \rangle = \int_{-\infty}^{\infty} e^{i\omega(\tau - t)} d\omega$, which is equal to $2\pi\delta(\tau - t)$ (as a given), therefore we can get the synthesis equation:

$$x(\tau) = \frac{1}{2\pi} \langle X(\omega), \psi_\tau(\omega) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \psi_\tau^*(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega\tau} d\omega \tag{15.6}$$

15.2 Rapid-Fire Examples

15.2.1 $x(t) = \delta(t)$

The transform of a Dirac delta is wide in frequency, requiring all frequencies to provide equal contributions.

15.2.2 $x(t) = e^{i\omega_0 t}$

The transform of a single-frequency signal has (surprise) one frequency, $X(\omega) = \alpha\delta(\omega - \omega_0)$. To find the strength,

$$X(\omega) = \frac{\alpha}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{i\omega t}d\omega = \frac{\alpha}{2\pi}e^{i\omega_0 t} \implies \alpha = 2\pi \tag{15.7}$$

Given that $e^{i\omega_0 t}$ transforms to $2\pi\delta(\omega - \omega_0)$, we can say in Hz that $e^{i2\pi f_0 t}$ transforms to $\delta(f - f_0)$.

15.2.3 Box from -Δ to Δ

Let $x(t) = \frac{1}{\Delta}$ from $t = -\Delta/2$ to $\Delta/2$, and 0 elsewhere. Then

$$X(\omega) = \int_{-\Delta}^{\Delta}\frac{1}{\Delta}e^{-i\omega t}dt = \frac{1}{\Delta}\frac{e^{-i\omega t}}{-i\omega}\,|_{-\Delta/2}^{\Delta/2} \tag{15.8}$$

$$X(\omega) = \text{algebra} = \frac{\sin(\omega\Delta)}{\omega\Delta} \tag{15.9}$$

15.2.4 Ideal LPF

Just like the box, but in frequency space: $H(\omega) = 1 for |\omega| < W$ and 0 otherwise.

$$h(t) = \frac{1}{2\pi}\int_{-W}^{W}e^{i\omega t}d\omega = \frac{1}{2\pi i t}e^{i\omega t} = \frac{\sin(Wt)}{\pi t} \tag{15.10}$$

This signal is not BIBO stable (slight abuse of terminology here; BIBO stability only makes sense when we think of this function as the impulse response of a filter). To analyze, we square the function and use the fact that it is square integrable (has finite energy). We can do some stuff to show that $H(0) = \int_{-\infty}^{\infty} h(t)dt = 1$.

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Lecture 16: The CTFT Contd.

Lecturer: Babak Ayazifar *18 October* *Aditya Sengupta*

16.1 CTFT Recap

We know that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{-i\omega t}dt \tag{16.1}$$

based on which we can decompose signals into constituent frequencies. For example:

$$x(t) = \cos(\omega_0 t) = \frac{1}{2}e^{-i\omega_0 t} + \frac{1}{2}e^{i\omega_0 t} \tag{16.2}$$

$$X(\omega) = \pi\left(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right) \tag{16.3}$$

and similarly for sin:

$$x(t) = \cos(\omega_0 t) = -\frac{1}{2i}e^{-i\omega_0 t} + \frac{1}{2i}e^{i\omega_0 t} \tag{16.4}$$

$$X(\omega) = \frac{\pi}{i}\left(-\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right) \tag{16.5}$$

16.2 Time Shifting Property

Suppose we have a signal $x(t)$ with a CTFT $X(\omega)$. We time-shift x by delaying it to make $\hat{x}(t) = x(t-t_0)$, then transform it to frequency space; how does the new $\hat{X}(\omega)$ relate to $X(\omega)$? We can answer this using the CTFT analysis equation,

$$\hat{X}(\omega) = \int_{-\infty}^{\infty} \hat{x}(t)e^{-i\omega t} = \int_{-\infty}^{\infty} x(t-t_0)e^{-i\omega t}dt \tag{16.6}$$

We do a change of variables, letting $\tau = t - t_0$:

$$\hat{X}(\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-i\omega(\tau+t_0)}d\tau = \int_{-\infty}^{\infty} x(\tau)e^{-i\omega\tau}d\tau e^{-i\omega\tau} = X(\omega)e^{-i\omega t_0} \tag{16.7}$$

16.3 Frequency Shifting Property

Suppose we have a signal in frequency space $X(\omega)$ with a CTFT $x(t)$. We frequency-shift ω to make $\hat{X}(\omega) = X(\omega - \omega_0)$ then transform it to time space; how does the new $\hat{x}(t)$ relate to $x(t)$? We can answer this using the CTFT synthesis equation,

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega - \omega_0)e^{i\omega t}d\omega \tag{16.8}$$

Let $\alpha = \omega - \omega_0$ (not λ, that's disgusting), then the integral becomes

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\alpha)e^{i(\alpha+\omega_0)t}d\alpha \tag{16.9}$$

$$\hat{x}(t) = x(t)e^{i\omega_0 t} \tag{16.10}$$

This can be applied in weird ways.

Example

Let $\widetilde{x}(t) = x(t)\cos(\omega_0 t)$. Find $\widetilde{X}(\omega)$ in terms of $X(\omega)$.

We can apply the inverse Euler's formula and reverse the frequency-shifting property:

$$\widetilde{x}(t) = \frac{1}{2}x(t)e^{i\omega_0 t} + \frac{1}{2}x(t)e^{-i\omega_0 t} \tag{16.11}$$

Since these changes are linear, we observe that $e^{i\omega_0 t}$ comes from a frequency delay of ω_0 and $e^{-i\omega_0 t}$ comes from a frequency advancement of ω_0. So, in frequency space, these translate to

$$\widetilde{X}(\omega) = \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0) \tag{16.12}$$

This is the idea behind amplitude modulation. A fast cosine goes between 1 and -1 very rapidly, and it is bounded by the amplitude, which is the value of the signal $x(t)$.

16.4 Time and Frequency Scaling

Suppose there is a signal $x(t)$ with CTFT $X(\omega)$. Let $\hat{x}(t) = x(at)$; for example, at $a = 2$, this is how students watch lectures. How is the frequency spectrum affected? We use the analysis equation

$$\hat{X}(\omega) = \int_{-\infty}^{\infty} \hat{x}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} x(at)e^{-i\omega t}dt \tag{16.13}$$

Let $\tau = at$, then $d\tau = adt$. We take two cases. First, where $a > 0$, we get

$$\hat{X}(\omega) = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{i\frac{\omega}{a}\tau}d\tau = \frac{1}{a}X\left(\frac{\omega}{a}\right) \tag{16.14}$$

Contracting a signal in time is dilation in frequency, and vice versa.

Second, where $a < 0$, we get

$$\hat{X}(\omega) = \frac{1}{a}\int_{\infty}^{-\infty} x(\tau)e^{i\frac{\omega}{a}\tau}d\tau = \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)e^{i\frac{\omega}{a}\tau}d\tau = \frac{1}{|a|}X\left(\frac{\omega}{a}\right) \tag{16.15}$$

16.5 Differentiation in Time

Suppose there is a signal $x(t)$ with CTFT $X(\omega)$. Let $\hat{x}(t) = \frac{dx(t)}{dt}$; how does $\hat{X}(\omega)$ relate to $X(\omega)$? We use the analysis equation and differentiate within the integral,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega \tag{16.16}$$

$$\hat{x}(t) = \frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega X(\omega)e^{i\omega t}d\omega \tag{16.17}$$

Therefore $\hat{X}(\omega) = i\omega X(\omega)$.

We can use this to understand CT-LTI systems described by LCCDEs, which (as covered previously) have the general expression

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m} \tag{16.18}$$

We take the Fourier transform of both sides, which as we derived above involves multiplying by $i\omega$ for each differentation operation:

$$\sum_{n=0}^{N} a_n \mathcal{F}\left(\frac{d^n y(t)}{dt^n}\right) = \sum_{m=0}^{M} b_m \mathcal{F}\left(\frac{d^m x(t)}{dt^m}\right) \tag{16.19}$$

$$\left(\sum_{n=0}^{N} a_n (i\omega)^n\right) Y(\omega) = \left(\sum_{m=0}^{M} b_m (i\omega)^m\right) X(\omega) \tag{16.20}$$

We know that $Y(\omega) = H(\omega)X(\omega)$, as convolution in time is multiplication in frequency, therefore

$$H(\omega) = \frac{\sum_{m=0}^{M} b_m (i\omega)^m}{\sum_{n=0}^{N} a_n (i\omega)^n} \tag{16.21}$$

16.6 Modulation Property

Let there be two functions of time f and g, each of which has a CTFT, and let $h(t) = f(t)g(t)$. What can we say about $H(\omega)$?

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(t)g(t)e^{-i\omega t}dt \tag{16.22}$$

$$\tag{16.23}$$

By the synthesis equation, we know $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha t}d\alpha$, so

$$H(\omega) = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha t}d\alpha\right)g(t)e^{-i\omega t}dt \tag{16.24}$$

$$H(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha F(\alpha)\left(\int_{-\infty}^{\infty} g(t)e^{-i(\omega-\alpha)t}dt\right) \tag{16.25}$$

$$H(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)G(\omega-\alpha)d\alpha = \frac{1}{2\pi}(F\circledast)(\omega) \tag{16.26}$$

Multiplication in time is convolution in frequency, ignoring the $\frac{1}{2\pi}$ prefactor. If this were written in Hz, we could get rid of the 2π: $H(f) = (F \circledast G)(f)$.

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Lecture 17: Amplitude Modulation

Lecturer: Babak Ayazifar *23 October* *Aditya Sengupta*

$$l \geq \frac{\lambda}{4} \tag{17.1}$$

This is the agreed-upon condition on antenna length for an antenna to transmit a signal. If $f = 3.3 \times 10^3$, then wavelength is on the order of 100km, so we need an antenna that's 25km long. Probably not practical. So what do we do? We modulate the amplitude. Consider an information-bearing signal $x(t)$, bounded in frequency by $\omega = -B$ and $\omega = B$, which we modulate by multiplying it by $e^{i\omega_0 t}$. We know that in frequency space, we get

$$y(t) = x(t)c(t) \implies Y(\omega) = \frac{1}{2\pi}(X \circledast C)(\omega) \tag{17.2}$$

Convolution of the delta in frequency (due to the single carrier frequency ω_0) with the arbitrarily-shaped signal X gives us a frequency-shifted signal centered at ω_0 and bounded by $\omega_0 - B$ and $\omega_0 + B$. We can do the same thing in reverse at the other end assuming perfect transmission, multiplying at the receiver by $e^{-i\omega_0 t}$, which shifts the signal back to the original input.

We can try different carriers. Let $c(t) = \cos(\omega_0 t)$, which using the inverse Euler's formula is two frequencies, so the output signal is two copies of the arbitrary shape (call them lobes), one centered at $-\omega_0$ and the other at ω_0 and both with half the height. Then, when we shift it back, let's try just multiplying by $\cos(\omega_0 t)$. The right delta acting on the frequency response Y shifts it to the right, leading to two lobes, one centered at 0 and the other centered at $2\omega_0$, and the left delta acting on the frequency response Y shifts it to the left, leading to two lobes, one centered at 0 and the other centered at $-2\omega_0$. All of these have a height of 1/4; the two at frequency 0 add to make a lobe with height 1/2.

An assumption in this process is that there is no phase difference between the transmitter carrier signal and the reciever's local oscillator. Adding this factor in, we get

$$c(t) = \cos(\omega_0 t) \tag{17.3}$$

$$y(t) = x(t)\cos(\omega_0 t) \tag{17.4}$$

$$q(t) = y(t)\cos(\omega_0 t + \theta) = x(t)\cos(\omega_0 t)\cos(\omega_0 t + \theta) \tag{17.5}$$

Using the Corner of Useless Information,

$$q(t) = \frac{1}{2}x(t)\left(\cos(2\omega_0 t + \theta)\right) + \frac{1}{2}x(t)\cos\theta \tag{17.6}$$

The first term is sitting at $\pm 2\omega_0$, and the second term is sitting at 0.

Suppose we have a low-pass filter that just multiplies by $\cos\theta$, which is basically what this phase offset is. If $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, this is a big problem.

A local oscillator has this equation that helps you translate this phase shift to a time shift,

$$c_L(t) = \cos(2\pi f_0 t + \theta) = \cos\left(2\pi f_0\left(t + \frac{\theta}{2\pi f_0}\right)\right) \tag{17.7}$$

Plugging in numbers, we get the timescale of 250 ns.

17.1 Frequency Drift

Suppose we modulate a signal with a cosine, and on the other end with a frequency shifted sign:

$$c_L(t) = \cos(2\pi(f_0 + \Delta f)t) \tag{17.8}$$

$$q(t) = x(t)\cos(2\pi f_0 t)\cos(2\pi(f_0 + \Delta f)t) = \frac{x(t)}{2}\cos\left((4\pi f_0 + 2\pi\Delta f)t\right) + \frac{x(t)}{2}\cos\left(2\pi\Delta f t\right) \tag{17.9}$$

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Lecture 18: Amplitude Modulation and Other Fun Things

Lecturer: Babak Ayazifar *25 October* *Aditya Sengupta*

18.1 Amplitude Modulation

Suppose we have $y(t) = x(t)\cos(2\pi f_0 t)$, and at the output we split y into two parts; one is multiplied by $\cos(2\pi(f_0 + \Delta f)t + \theta)$, and the other by $\sin(2\pi(f_0 + \Delta f)t + \theta)$. We want to reconstruct x based on these. We use the Corner of Useless Information, and then low-pass filter it:

$$q_1(t) = \frac{x(t)}{2}\left(\cos(2\omega_0 t + \Delta\omega t + \theta) + \cos(\Delta\omega t + \theta)\right) \tag{18.1}$$

$$r_1(t) = \frac{A}{2}x(t)\cos(\Delta\omega t + \theta) \tag{18.2}$$

$$q_2(t) = \frac{x(t)}{2}\left(\sin(2\omega_0 t + \Delta\omega t + \theta) + \sin(\omega_0 t + \theta)\right) \tag{18.3}$$

$$r_2(t) = \frac{A}{2}x(t)\sin(\Delta\omega t + \theta) \tag{18.4}$$

We can then extract $x(t)$ from these by squaring and adding the terms:

$$r_1^2(t) + r_2^2(t) = \frac{A^2}{4}(x(t))^2 \tag{18.5}$$

If we let $A = 2$ (where A is the magnitude of the LPF) then $x(t) = \sqrt{r_1^2(t) + r_2^2(t)}$.

WLOG let $-1 \leq x(t) \leq 1$, then add 1 to it so that $\hat{x}(t) \geq 0$. And we know how to reconstruct positive signals, so we modulate this onto $\cos(\omega_0 t)$ and transmit it.

18.2 Encoding Multiple Signals

Can we encode two signals $x_1(t)$ and $x_2(t)$ onto the same carrier frequency?

Yes! But how, you ask? I will answer even if you didn't ask.

$$y(t) = x_1(t)\cos(\omega_0 t) + x_2(t)\sin(\omega_0 t) \tag{18.6}$$

and then at the receiver,

$$
\begin{aligned}
q_1(t) &= y(t)\cos(\omega_0 t) && (18.7)\\
q_2(t) &= y(t)\sin(\omega_0 t) && (18.8)\\
& && (18.9)
\end{aligned}
$$

LPF these and you'll get the two signals. (Do the algebra later)

18.3 Sampling of CT Signals

We've reached the digital age. Eyyyyy, thaaaaat's greeeeaaaattt.

Given a continuous-time signal $x(t)$, we sample it over some sampling period T_s to create a DT signal $x_d(n)$, we pass it through a DT-LTI filter to create $y_d(n)$ and we reconstruct it with some reconstruction period T_r to make $y_c(t)$.

Suppose we have $x(t)$, and we sample it at discrete intervals separated by time T_s: $x_d(n) = x_c(nT_s)$. This would be the same as hitting the CT signal with an impulse train (CHOOO CHOOO):

$$\hat{x_c}(t) = x_c(t)\sum_n \delta(t - nT_s) \tag{18.10}$$

So there is no information loss in converting this to a DT signal. This is now a completely CT problem; can we recover x_c from $\hat{x}_c$?

We know that

$$
\begin{aligned}
\hat{x}_c(t) &= x_c(t)P(t) && (18.11)\\
\hat{X}_c(\omega) &= \frac{1}{2\pi}\left(X_c \circledast P\right)(\omega) && (18.12)\\
P(t) &= \sum_k P_k e^{ik\omega_s t} && (18.13)
\end{aligned}
$$

where $\omega_s = \frac{2\pi}{T_s}$. Then, the P_ks are given by

$$P_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)e^{-ik\omega_s t}dt = \frac{1}{T_s} \tag{18.14}$$

So all the coefficients are the same. We can therefore transform,

$$p(t) = \frac{1}{T_s}\sum_k e^{ik\omega_s t} \xrightarrow{\mathcal{F}} P(\omega) = \frac{2\pi}{T_s}\sum_k \delta(\omega - k\omega_s) \tag{18.15}$$

That's an impulse train both in time and in frequency. It turns out that if a certain condition holds, you can actually just completely get the signal from samples. This condition is called the Whittaker-Nyquist-Kotelnikov-Shannon sampling theorem: if a function $x(t)$ contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced $1/(2B)$ seconds apart.

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Lecture 19: Aliasing

Lecturer: Babak Ayazifar *1 November* *Aditya Sengupta*

Aliasing is when Tesla wheels seem to spin backwards.

A little more generally, aliasing occurs when higher frequencies are filtered out and only low frequencies can be observed.

Suppose we sample a CT signal with an impulse train (CHOOOO CHOOOO),

$$\hat{x}_C(t) = \sum_{n=-\inf}^{\inf} x_c(nT_s)\delta(t - nT_s) \tag{19.1}$$

The impulse train has a CTFS of

$$P(\omega) = \frac{2\pi}{T_s} \sum_k \delta(\omega - k\omega_s) = \omega_s \sum_k \delta(\omega - ka_s) \tag{19.2}$$

Convolution with the signal (which, say, ranges from -B to B) yields a series of repetitions of that signals, as convolution with a Dirac delta in frequency space shifts the signal in frequency. For the signal repetitions not to overlap, the right end of the frequency band $\omega = B$ must be less than the left end of the next frequency-space repetition $\omega_s - B$. This gives us the condition that $2B \leq \omega_s$; $2B$ is the Nyquist sampling rate.

If ω_s is reduced below the Nyquist rate, the repetitions in frequency start to overlap; in the overlapping space, there is some contribution both due to the right end of one of the repetitions, and due to the left end of the next one. If the sampling rate is sufficient, a low-pass filter is sufficient to reconstruct this signal, but aliasing occurs when the sampling rate is insufficient so overlapping takes place in frequency; high frequencies take on the alias of low frequencies.

To avoid this, we can carry out anti-alias filtering, preprocessing the signal $x_c(t)$ so that it is band-limited enough to avoid aliasing.

Consider a signal being sampled at a rate of $\omega_s = \frac{3\omega_0}{2}$, so that its sampling rate is below the Nyquist criterion. Take a delta of strength 2π at ω_0, and repeat it at frequencies $\omega_0 + n\omega_s$ (this is what happens in frequency space when the signal is sampled with an impulse train). The only impulse that survives the interpolation filter is a delta of strength 2π at $\omega_0 - \omega_s = -\frac{\omega_0}{2}$.

Hey, look at that. That's a negative frequency - it looks like it's going slower, and in the opposite direction. Wonder where we've seen that before.

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Lecture 21: Z Transforms I

Lecturer: Babak Ayazifar *8 November* *Aditya Sengupta*

Consider a signal of the form z^n, that is fed into a DT-LTI system with impulse response $y(n)$. Previously, we studied these systems with $z = e^{i\omega}$. Now, consider $z = Re^{i\omega}$, where R is any real number. We can get the output by convolution,

$$y(n) = \sum_k h(k)x(n-k) = \sum_k h(k)z^n z^{-k} = z^n \sum_k h(k)z^{-k} = z^n \hat{H}(z) \tag{21.1}$$

$\hat{H}(z)$ is the transfer function or the system function.

$$h(n) \to \hat{H}(z) = \sum_{n=-\inf}^{\inf} h(n)z^{-n} \tag{21.2}$$

This is referred to as the Z transform of h. Convolution can still be defined in cases where one or perhaps both functions involved are a bereft of a DTFT.

Consider a case in which one of the functions has a finite region of support (region in the time axis over which all the nonzero function values are found). For example, let $h(n) = \delta(n)+2\delta(n-1)+3\delta(n-2)$ and let $x(n) = 2^n u(n)$. Then, convolution is defined; using the Z-transform, $y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2)$.

in the case where both signals are right-sided, i.e. $x(n) = 0 \forall n < N < \infty$. A special subset of right-sided signals is the set of causal signals, for which $N = 0$. Consider $x(m)$ right-sided with $N = N_1$, and $h(m)$ right-sided with $N = N_2$. Then, they can be convolved graphically, by first flipping and shifting h to form $h(n-m)$, and then graphically convolving between N_1 and N_2.

Example

$$x(n) = \frac{1}{2^n}u(n) \to \hat{X}(z) \tag{21.3}$$

The Z-transform is

$$\hat{X}(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1-\frac{1}{2}z^{-1}}, \frac{1}{2} < |z| \tag{21.4}$$

The condition on z is called the region of convergence (ROC). The ROC can be graphed on the complex plane. If the ROC and the unit circle overlap at all, the signal is stable.

Example

$$q(n) = -\left(\frac{1}{2}\right)^n u(-n-1) \tag{21.5}$$

This signal has no DTFT, so we can find the Z transform instead.

$$\hat{Q}(z) = -\sum_{n=-\inf}^{-1} \frac{1}{2^n} z^{-n} = -\sum_{l=1}^{\inf} (2z)^l = -\left(\sum_{l=0}^{\inf (2z)^p - 1}\right) = 1 - \frac{1}{1-2z} = \frac{z}{z-\frac{1}{2}} \tag{21.6}$$

This has an ROC of $|z| < \frac{1}{2}$, which is the only difference between this and the previous example. This carries information about signal causality and BIBO stability.

Consider $r(n) = x(n+1) = \left(\frac{1}{2}\right)^{n+1} u(n+1)$. Then

$$\hat{R}(z) = \sum_n r(n) z^{-n} = \sum_n x(n+1) z^{-n} \tag{21.7}$$

Let $l = n+1$, then

$$\hat{R}(z) = \sum_l x(l) z^{-(l-1)} = z\hat{X}(z) \tag{21.8}$$

Praise the Sun. In general, a time-shifted signal $x(n-N)$ has a Z-transform $\hat{X}(z)z^{-N}$.

In general, the Z transform of $\alpha^n u(n)$ is $\frac{1}{1-\alpha z^{-1}}$ with $|\alpha| < |z|$, and the Z transform of $-\alpha^n u(-n-1)$ is the same but with $|z| < |\alpha|$. These can sum together if and only if the ROCs have a non-trivial overlap.

For example, the Z transform of

$$x(n) = \frac{1}{2^n} u(n) - 2^n u(-n-1) \tag{21.9}$$

is

$$\frac{z}{z-\frac{1}{2}} + \frac{z}{z-2} \tag{21.10}$$

which has the ROC $\frac{1}{2} < |z| < 2$. This can also be written as

$$\frac{2z(z-\frac{5}{4})}{(z-\frac{1}{2})(z-2)} \tag{21.11}$$

So the graph is a donut with poles at $\frac{1}{2}$ and 2, and a zero point at $\frac{5}{4}$.

21.1 Relating the Z-Transform and the DTFT

$$\hat{X}(z) = \sum_n x(n) z^{-n} = \sum_n x(n) R^{-n} e^{-i\omega n} \tag{21.12}$$

This looks like the DTFT. This allows us to conclude that $\hat{H}(Re^{i\omega}) = \mathcal{F}\left\{x(n)R^{-n}\right\}|_\omega$

If the ROC includes the unit circle, then $X(\omega) = \hat{X}(e^{i\omega})$.

For example, $x(n) = u(n)$ has the Z transform $\hat{X}(z) = \frac{1}{1-z^{-1}}$, $1 < |z|$. The problem occurs due to the pole at $z = 1$, which requires the addition of an impulse train (CHOO CHOOOO):

$$X(\omega) = \frac{1}{1 - e^{-i\omega}} + \pi \sum_k \delta(\omega - 2\pi k) \neq \hat{X}(z)|_{z=e^{i\omega}} \tag{21.13}$$

21.2 Z-Transforms of DT-LTI Systems

Consider two cascaded systems with impulse responses $f(n)$ and $g(n)$, with overall impulse response $h(n)$ and transfer functions $\hat{F}$, $\hat{G}$, and $\hat{H}$. The output of the filter F is $\hat{F}(z)z^n$, which is fed into filter G to get overall output $y(n) = \hat{F}(z)\hat{G}(z)z^n$. Convolution in time is multiplication in frequency here as well.

Let a DT-LTI system be described by an LCCDE:

$$a_0 y(n) + a_1 y(n-1) + \cdots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) \tag{21.14}$$

The Z-transform can be found by summing the Z-transforms of individual components,

$$y(n-k) \to \hat{Y}(z) z^{-k} \tag{21.15}$$

and

$$x(n-m) \to \hat{X}(z) z^{-m} \tag{21.16}$$

which gives us a linear combination of Z-transform prefactors,

$$\left(\sum_{k=0}^{N} a_k z^k\right) \hat{Y}(z) = \left(\sum_{m=0}^{M} b_m z^m\right) \hat{X}(z) \tag{21.17}$$

which gives the overall transfer function of the system

$$\hat{H}(z) = \frac{\hat{Y}(z)}{\hat{X}(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^k} \tag{21.18}$$

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Lecture 22: Z-Transforms II

Lecturer: Babak Ayazifar *13 November* *Aditya Sengupta*

22.1 Z-Transform Properties

22.1.1 Time-Reversal

Consider a signal $x_{TR}(n) = x(-n)$. (Recommendation for MT3: derive for yourself a lot of Z transform properties).

$$x_{TR}(n) \leftrightarrow \hat{X}_{TR}(z) \tag{22.1}$$

$$\hat{X}_{TR}(z) = \sum_n x(-n)z^{-n} = \sum_m x(m)z^m = \hat{X}(z^{-1}) \tag{22.2}$$

Let the ROC be $R_1 < |z| < R_2$ originally; then, the ROC of the new transform is $\frac{1}{R_2} < |z| < \frac{1}{R_1}$.

22.1.2 Time-Shift

Consider a signal $x_{TS}(n) = x(n-N) \leftrightarrow \hat{X}_{TS}(z)$. Then, as derived last time, $\hat{X}_{TS}(z) = z^{-N}\hat{X}(z)$. For $N > 0$, we get $\frac{\hat{X}(z)}{z^N}$, which potentially introduces poles at $z = 0$. So the region of convergence is at most $R_x - 0$. For $N < 0$, we get $z^M\hat{X}(z)$. This potentially adds poles at infinity. For example, consider any causal signal that is time-advanced by 1. We have

$$\hat{X}(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots \tag{22.3}$$

$$\hat{X}_{TS}(z) = x(0)z + x(1) + x(2)z^{-1} + \dots \tag{22.4}$$

which has a pole at infinity.

22.1.3 Multiplication by a Complex Exponential

Let $q(n) = z_0^n x(n)$ have a Z transform $\hat{Q}(z)$. How is this related to $\mathcal{Z}\{x(n)\} = \hat{X}(z)$?

$$\hat{Q}(z) = \sum_n x(n)\left(\frac{z}{z_0}\right)^{-n} = \hat{X}\left(\frac{z}{z_0}\right) \tag{22.5}$$

Consider the case where $R_0 = 1$ and $z_0 = e^{i\omega_0}$. Then consider the case $x(n) = u(n)$; we get

$$\hat{X}(z) = \sum_{n=0}^{\inf z^{-n} = \frac{z}{z-1}} \tag{22.6}$$

This has a pole at $z = 1$, and has an ROC outside the unit circle. Then, consider $q(n) = e^{i\frac{\pi}{4}n}u(n)$. We find that $\hat{Q}(z) = \frac{z}{z-e^{i\omega_0}}$. This corresponds to a rotation of ω_0 on the unit circle.

$$q(n) = \frac{1}{2^n} e^{i\frac{\pi}{4}} u(n) \tag{22.7}$$

The ROC grows here. Praise the Sun.

Consider

$$r(n) = \cos(\omega_0 n)u(n) = \frac{1}{2} e^{i\omega_0 n} u(n) + \frac{1}{2} e^{-i\omega_0 n} u(n) \tag{22.8}$$

The Z-transform is therefore

$$\hat{R}(z) = \frac{1}{2}\frac{z}{z - e^{i\omega_0}} + \frac{1}{2}\frac{z}{z - e^{-i\omega_0}} = \frac{z(z - \cos\omega_0)}{(z - e^{i\omega_0})(z - e^{-i\omega_0})} \tag{22.9}$$

22.2

Consider a DT-LTI system H that is causal and rational, with $x(n) = 1 \implies y(n) = -\frac{4}{3}$. It has a pole-zero diagram with poles at $-\frac{1}{2}$ and $\frac{3}{4}$, and a zero at $\frac{3}{2}$. Is H BIBO stable? Determine $\hat{H}(z)$ and $h(n)$.

We know that H is BIBO-stable due to its pole-zero diagram indicating that its ROC is partially inside the unit circle (Praise the Sun). From the diagram, we get

$$\hat{H}(z) = H_0 \frac{z - 3/2}{(z + 1/2)(z - 3/4)} \tag{22.10}$$

where H_0 is a scaling factor. We can find it using the specific point given. $x(n) = 1 = z_0^n$ for $z_0 = 1$, therefore $\hat{H}(1) = -\frac{4}{3}$. Substituting in $z = 1$ to the above expression gives us $H_0 = 1$.

$h(n)$ can be found by carrying out a partial fraction decomposition of $\hat{H}(z)$.

$$\hat{H}(z) = \frac{z - 3/2}{(z + 1/2)(z - 3/4)} = \frac{A}{z + 1/2} + \frac{B}{z - 3/4} \tag{22.11}$$

The numerators are

$$z - 3/2 = A(z - 3/4) + B(z + 1/2) \tag{22.12}$$

By setting $z = 3/4$, we get $B = -3/5$ and $A = 8/5$. Therefore,

$$\hat{H}(z) = \frac{8}{5}\frac{z^{-1}}{1 + \frac{1}{2}z^{-1}} - \frac{3}{5}\frac{z^{-1}}{1 + \frac{3}{4}z^{-1}} \tag{22.13}$$

which has an inverse transform

$$\hat{H}(z) = \frac{8}{5}\left(\frac{1}{2}\right)^{n-1} u(n-1) - \frac{3}{5}\left(\frac{3}{4}\right)^{n-1} u(n-1) \tag{22.14}$$

We see that $h(0) = 0$. We make the bold claim that $h(0) = \lim_{z\to\inf} \hat{H}(z)$ for causal signals. This is the initial value theorem. The initial value theorem states that if $h(n) = 0 \forall n < 0$ then $h(0) = \lim_{z\to\inf} \hat{H}(z)$. We get this from the definition of the transform,

$$\hat{H}(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \ldots \tag{22.15}$$

$$\lim_{z\to\infty} \hat{H}(z) = h(0) \tag{22.16}$$

as required.

Consider a BIBO stable system having the transfer function

$$\hat{G}(z) = \frac{z(z-1)(z-2)}{(z-1/2)(z+1/4)} \tag{22.17}$$

Could this system be causal? The ROC is greater than the circle centered at the origin with radius 1/2 (Praise the Sun).

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Lecture 23: Praise the Sun

Lecturer: Babak Ayazifar *15 November* *Aditya Sengupta*

23.1 Steady-State and Transient Responses

Consider the case

$$y(n) = \alpha y(n-1) + x(n) \tag{23.1}$$

and let $x(n) = u(n)$. This has the transfer function

$$\hat{X}(z) = \frac{1}{1-z^{-1}} \tag{23.2}$$

for $1 < |z|$. Then, we have

$$\hat{Y}(z) = \hat{H}(z)\hat{X}(z) \tag{23.3}$$

so

$$\hat{Y}(z) = \alpha z^{-1}\hat{Y}(z) + \hat{X}(z) \tag{23.4}$$

$$\hat{H}(z) = \frac{1}{1-\alpha z^{-1}} \tag{23.5}$$

Since this is BIBO stable, we can say

$$H(\omega) = \hat{H}(z)|_{z=e^{i\omega}} \tag{23.6}$$

Based on this, we can find $\hat{Y}(z)$:

$$\hat{Y}(z) = \hat{X}(z)/(1-\alpha z^{-1}) = \frac{1}{1-\alpha z^{-1}}\frac{1}{1-z^{-1}} = \frac{z^2}{(z-\alpha)(z-1)} \tag{23.7}$$

Using partial fractions, we can simplify this into a form from which we can carry out an inverse transform,

$$\hat{Y}(z) = z\left(\frac{A}{z-\alpha} + \frac{B}{z-1}\right) \tag{23.8}$$

$$A = \frac{\alpha}{\alpha-1}, B = \frac{1}{1-\alpha} \tag{23.9}$$

Therefore

$$\hat{Y}(z) = \frac{\alpha}{\alpha-1}\frac{z}{z-\alpha} + \frac{1}{1-\alpha}\frac{z}{z-1} \tag{23.10}$$

This has a region of convergence of Praise the Sun, outside the outermost pole at 1. There are poles at α and 1, and two zeros at 0. This corresponds to a right-sided sequence,

$$y(n) = \frac{\alpha}{\alpha-1}\alpha^n u(n) + \frac{1}{1-\alpha}u(n) \tag{23.11}$$

The first term decays, and is called the transient response $y_{TR}(n)$. The second term does not, and it is called the steady-state response $y_{SS}(n)$. We can see this by the position of the poles; the term corresponding to the transient response has a pole inside the unit circle, and the term corresponding to the steady-state response has a pole on the unit circle. If $\alpha > 1$, the transient term would grow unbounded, which would correspond to a pole outside the unit circle.

If the input to this were, instead of the unit step, the constant function 1, the response would be just the steady-state response. $y(n) = \frac{1}{1-\alpha}$.

23.2 Zero-Input Response and Zero-State Response

Consider the case $x(n) = 0$ and $y(-1) \neq 0$, for the above LCCDE. What is the system response? We can easily derive

$$y_{ZIR}(n) = \alpha^{n+1}y(-1) \tag{23.12}$$

This is called the zero-input response (ZIR) of the system.

Now, let $y(-1) = 0$ and let $x(n) = u(n)$. This is what was derived above, and is called the zero-state response (the system has no response prior to time 0). If $y(-1) \neq 0$ and $x(n) = u(n)$, then y is the sum of these two because the LTI system being studied has superposition:

$$y(n) = y_{ZIR}(n) + y_{ZSR}(n) \tag{23.13}$$

Each of these has a transient and a steady-state component; the transient/steady-state decomposition is separate from the ZIR/ZSR decomposition.

The ZSR/ZIR decomposition can be carried out in one go in a few different ways. Start with the difference equation and multiply both sides by the unit step,

$$y(n)u(n) = \alpha y(n-1)u(n) + x(n)u(n) \tag{23.14}$$

Take Z transforms,

$$\widetilde{(\hat{Y}(z))} = \alpha\left(y(-1) + z^{-1}\widetilde{\hat{Y}(z)}\right) + \widetilde{\hat{X}(z)} \tag{23.15}$$

where the middle term can be calculated from the definition of the transform,

$$\mathcal{Z}\{y(n-1)u(n)\} = \sum_{n=0}^{\infty} y(n-1)z^{-n} \tag{23.16}$$

Let $m = n - 1$. Then we get

$$\sum_{m=-1}^{\infty} y(m)z^{-m-1} = y(-1) + z^{-1}\sum_{m=0}^{\infty} y(m)z^{-m} = y(-1) + z^{-1}\widetilde{\hat{Y}(z)} \tag{23.17}$$

Then, by shuffling things around, we get

$$\widetilde{\hat{Y}(z)} = \frac{\alpha y(-1)}{1-\alpha z^{-1}} + \frac{1}{1-\alpha z^{-1}}\widetilde{\hat{X}(z)} \tag{23.18}$$

which, for $n \geq 0$,

$$\alpha y(-1)\hat{H}(z) + \hat{H}(z)\widetilde{\hat{X}(z)} \tag{23.19}$$

and therefore, by the inverse Z transform,

$$y(n) = \alpha y(-1)\alpha^n u(n) + \left(\frac{\alpha}{\alpha-1}\alpha^n u(n) + \frac{1}{1-\alpha}u(n)\right) \tag{23.20}$$

23.3 Feedback

Consider a filter whose Z-transform is $\hat{F}(z) = \frac{1}{1-2z^{-1}}$, placed in a feedback loop as the forward component. The loop component is $\hat{G}(z) = \alpha z^{-1}$; a multiplication and delay by 1. The overall transfer function can be found using Black's formula,

$$\hat{H}(z) = \frac{\hat{F}(z)}{1 + \hat{F}(z)\hat{G}(z)} = \frac{z}{z - (2 - \alpha)} = \frac{1}{1 - (2 - \alpha)z^{-1}} \tag{23.21}$$

The corresponding impulse response is $h(n) = (2 - \alpha)^n u(n)$. This is stable for $\alpha > 1$ and $\alpha < 3$.

BONSAI (I wish). Quiz day.

EE 120: Signals and Systems **Fall 2018**

Lecture 24: Laplace November

Lecturer: Babak Ayazifar *27 Transform* *Aditya Sengupta*

24.1 Definition of the Laplace transform

This is a highly abbreviated version of the Laplace transform.

The Laplace transform is a continuous-time version of the Z transform. Consider a complex exponential of the form

$$x(t) = e^{st} = e^{(\sigma + i\omega)t} \tag{24.1}$$

fed into an LTI system with impulse response $h(t)$. The result can be found by convolution,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \tag{24.2}$$

which is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = \left(\int_{-\infty}^{\infty} h\tau e^{-s\tau}d\tau\right)e^{st} = \hat{H}(s)e^{st} \tag{24.3}$$

$\hat{H}(s)$ is the transfer function of the system. We can say $\mathcal{L}(h(t)) = \hat{H}(s)$.

If we isolate the imaginary component, this is just the CTFT:

$$\hat{H}(s) = \int_{-\infty}^{\infty} h(\tau)e^{-(\sigma+i\omega)\tau}d\tau = \int_{-\infty}^{\infty} \left(h(\tau)e^{-\sigma\tau}\right)e^{-i\omega\tau}d\tau = \mathcal{F}\left(h(\tau)e^{-\sigma\tau}\right) \tag{24.4}$$

24.2 Regions of Convergence

There are three types of ROCs: to the right of some σ, to the left of some σ, or between some σ_1 and σ_2.

Right-sided signals have right-sided ROCs of the form $s > \sigma$, left-sided signals have left-sided ROCs of the form $s < \sigma$, and two-sided signals have ROCs bordered on two sides, of the form $\sigma_1 < s < \sigma_2$.

Consider a right-sided signal x, that is, $x(t) = 0\ \forall t < T$ for some T. Let σ_0 be in the ROC, and let $\sigma_0 < \sigma_1$. Then, we know that $-\sigma_0 > \sigma_1$.

Consider a case where $T < 0$, so that the integral for the Laplace transform can be split into an interval T to 0, and another from 0 to ∞, that is,

$$\hat{X}(s) = \int_{T}^{0} x(t)e^{-st}dt + \int_{0}^{\infty} x(t)e^{-st}dt \tag{24.5}$$

Only a value $\sigma = +\infty$ would cause the first part to blow up (note that t is negative). Therefore, in general for positive times, we can say

$$-\sigma_0 t > -\sigma_1 t \tag{24.6}$$

and be sure that there is no divergence due to negative times.

This inequality can be extended to

$$e^{-\sigma_0 t} > e^{-\sigma_1 t} \tag{24.7}$$

which shows that if σ_0 is in the ROC, then so is σ_1. Therefore, the region of convergence of a right-sided signal extends from some lower threshold to $+\infty$ (and it may or may not include $+\infty$).

Now, consider the case $T > 0$ (the signal is still right-sided). Then the transfer function expression is

$$\hat{X}(s) = \int_{T}^{\infty} x(t)e^{-st}dt \tag{24.8}$$

Here, the case $\sigma = \infty$ is always included in the convergent region, i.e. the ROC in this case is $\sigma_0 < \Re(s)$.

24.3 Examples

24.3.1 $x(t) = u(t)$

$$\hat{X}(s) = \int_{0}^{\inf e^{-st}dt = \frac{e^{-st}}{s}\Big|_0^{\infty} = \frac{1}{s}} \tag{24.9}$$

with the assumption that $\sigma = \Re(s) > 0$. That is, the ROC is the right-hand side of the complex plane, with $\sigma > 0$.

24.3.2 $x(t) = e^{-at}u(t)$

$$\hat{X}(s) = \int_{0}^{\inf e^{-(a+s)t} = \lim_{h\to\infty} \frac{1-e^{-(s+a)h}}{s+a}} \tag{24.10}$$

This diverges for negative values of the real part of $s + a$, that is, the ROC is $-\Re(a) < \Re(s)$. For other values, it becomes $\frac{1}{s+a}$.

For the case $a = 2$, the signal $e^{-2t}u(t) \leftrightarrow \frac{1}{s+2}$ with an ROC $-2 < Re(s)$ and for the case $a = -2$, the signal $e^{2t}u(t) \leftrightarrow \frac{1}{s-2}$ with an ROC $2 < \Re(s)$. The first has an ROC containing the $i\omega$ axis and the second does not. This tells us that the first signal is stable and the second is not.

If the ROC includes the $i\omega$ axis, then $H(\omega) = \hat{H}(s)\Big|_{s=i\omega}$.