Notes for Astrophysical Fluid Dynamics
UC Santa Cruz, Fall 2022

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Lecture 1: Introduction

Lecturer: Ruth Murray-Clay *26 September* *Aditya Sengupta*

Note: *LaTeX format adapted from template for lecture notes from CS 267, Applications of Parallel Computing, UC Berkeley EECS department.*

What is a fluid?

- Moving particles
- Things that aren't held together in a lattice
- Has bulk properties like volume and velocity

In astrophysics, fluids are not generally liquids, but gases. What makes a gas behave as a fluid or not? Wind behaves as a fluid; it's sensible to talk about its velocity even though it's made up of many smaller particles. Why? This is because molecules are held together by collisions.

Fluid dynamics consists of a statistical treatment of a gas. If you consider a gas of particles, each one has a position $\vec{x}$ and a velocity $\vec{v}$. If you wanted to describe this in perfect detail, you'd need six coordinates per particle, so $6N$ coordinates, but we can describe the velocity of the wind in a bulk collection of particles without needing to deal with all N. So it makes sense to talk about a *fluid element* that has its own properties: pressure, temperature, mass, density, volume, bulk velocity, and so on.

What does each of these things refer to, and how do they relate to collisions? Pressure is the force per area exerted by molecules bouncing off a solid surface and imparting momentum to it. Temperature is a measure of microscopic kinetic energy. Specifically, if the molecules have some average velocity $v_{thermal}$, each molecule has a kinetic energy $\frac{1}{2}mv_{th}^2$. We can relate this to the thermal energy k_BT, where k_B is the Boltzmann constant. We'll be using cgs units, so k_B has units of ergs per Kelvin.

Setting these equal (with an extra prefactor of $\frac{3}{2}$ for a monatomic gas),

$$\frac{3}{2}k_BT = \frac{1}{2}mv_{th}^2, \tag{1.1}$$

and we can think of this as what defines temperature. For this class, we won't care so much about the $\frac{3}{2}$ prefactor; it's also fine to say $k_BT \approx \frac{1}{2}mv_{th}^2$.

So far, we've been assuming that there is a well-defined average velocity for these molecules. In fact, it's possible for there to be a whole range of velocities. If a collection of molecules with varying velocities undergo a few collisions per molecule, their velocities average out to all be similar. It'll follow a Boltzmann distribution, which we'll talk about later.

If we take the fluid approximation, we can talk about continuous fields $\rho(\vec{x}, t), \vec{v}(\vec{x}, t), T(\vec{x}, t)$, and so on. How big does our system have to be to use these?

We want a system size L that is much greater than the distance before a collision for a molecule. We call this expected distance the *mean free path*, λ_{mfp}. We want many collisions within the system so that everything is well-mixed. Similarly for time, we want to evolve the system much longer than the average time for a collision to happen, which we can say is λ_{mfp}/v_{th}.

Next, we'll start thinking about how we calculate collision rates, the mean free path, and if we can estimate these things for the air in a room and decide if it's a fluid.

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Lecture 2: The collision formula

Lecturer: Ruth Murray-Clay *28 September* *Aditya Sengupta*

Collision rates depend on density (number of particles per volume), the velocity of the moving particles, and the size of the particle.

Consider a tube. On the front of it is the cross-section for collisions, σ. Within the tube, we have n representing the number of targets per volume, and we say the cross-section sweeps through at a velocity v. In time t, this cross-section sweeps out a volume $V = \sigma l = \sigma v t$. In this time, it collides with $nV = n\sigma v t$ particles. So the number of collisions per unit time is $n\sigma v$.

Let's try to apply this to get an expression for the mean free path λ_{mfp}, which is the distance a particle goes before one collision is expected. We can say the collision rate is the inverse of the collision time, $t_{coll} = \frac{1}{n\sigma v}$, so the mean free path is $\lambda_{mfp} = v \cdot t_{coll} = \frac{1}{n\sigma}$. Another way of thinking about this is: if we set a tube with a length λ_{mfp}, we should expect only one particle, so there's one collision. $\underbrace{\sigma\lambda_{mfp}}_{V} n = 1$, so $\lambda_{mfp} = \frac{1}{n\sigma}$.

The last piece of terminology we'll need is the impact parameter b. When we have two things that are colliding, the distance of closest approach (when the relative motion is at a right angle to the line between the centers) is b. We denote by b_{coll} the impact parameter required for a collision. If particles 1 and 2 have radii $r_{1,2}$, then we'd have $b_{coll} = r_1 + r_2$. In 3D, if we have particles approaching a circular cross-section, we could say $\sigma = \pi b_{coll}^2$.

An additional comment: a common phrase Ruth uses is "order unity change" in something. In the homework this week, we have a mass M and a point mass m. (Also if she says "force" and it looks like it should be "force per mass", it probably is). We've been asked to find b such that the change in v due to the collision is order unity. This means $\Delta v \sim v$.

Let's figure out if the air in the room is a fluid! (It is, but let's show it!) What we need to know is:

- The volume of the room/the density of air
- The temperature
- The composition of the air

and we'd like to check that the mean free path is much, much smaller than the system size (any linear dimension of the room), and that the collision time $t_{coll} = \frac{\lambda_{mfp}}{v_{th}}$ is much, much less than the time we care about.

The temperature is about 300K, the pressure is about 1 bar, and air is mostly N_2 which weighs $28m_H$. To get the density of air, we can use the ideal gas law $PV = NRT$. In this class, we'll use $PV = Nk_BT$. The difference is R is what we use if N is in moles, but k_B is what we use if N is just a number of particles. k_B for us carries units of ergs/K, so it must be the case that pressure has units of energy per volume, which is equivalent to the traditional force per distance definition.

If we let n be the specific number of particles, i.e. $n = N/V$, we get $P = nk_BT$. We know pressure and temperature, so we can get our number density:

$$n = \frac{P}{k_B T} = \frac{1 \text{ bar}}{k_B \cdot 300\text{K}} = \frac{10^6 \text{ Ba}}{k_B \cdot 300\text{K}} \tag{2.1}$$

where Ba is defined as above and represents the "barye", the cgs unit of pressure.

Therefore

$$n = \frac{10^6}{1.4 \times 10^{-16} \times 300} \sim 3 \times 10^{19}\text{cm}^{-3}. \tag{2.2}$$

For comparison, the mass density of water is $\rho \sim \frac{1 \text{ g}}{\text{cm}^3}$, and most solids are a few times that. Solid metal is denser; gold is at about 20 grams per cc. The average density of the Sun is also about a gram per cc (by coincidence).

Finally, we can get the density of air: $\rho = n \cdot \frac{\text{mass}}{\text{particle}}$, and we know each number in there if we know that $m_H = 1.7 \times 10^{-24}$g. $\rho = 3 \times 10^{19} \times 1.7 \times 10^{-24} \times 28$ which comes to about 10^{-3}. In fact, ρ_{air} is about a kilogram per cubic meter, or 10^{-3}g/cc.

We're missing a cross-section σ, which we'll fill in next time.

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Lecture 3: Air, hydrostatic equilibrium

Lecturer: Ruth Murray-Clay *30 September* *Aditya Sengupta*

3.1 The air in the room

Last time, we derived the collision rate formula $n\sigma v$. It's noteworthy that σ is not necessarily the physical cross-section of any object in the problem. For example, we talk about cross-sections for strong gravitational scattering or Coulomb scattering "collisions" for ions.

All else equal, an ionized gas is more collisional than neutral because σ is larger. We need interactions to effectively exchange energy and angular momentum for a fluid.

Last time, we almost derived the fact that the air in the room is a fluid. We used $P = nk_BT$ and $nm = \rho$, and we found that ρ for air is about 10^{-3}g/cm^3.

We have n, and now we need σ and v. To get velocity, we use the relationship between kinetic energy and thermal energy, because all the particles have the same average velocity and temperature:

$$\frac{1}{2}mv_{th}^2 \approx k_BT \implies v_{th} \approx \sqrt{\frac{k_BT}{m}} = \left(\frac{1.4 \times 10^{-16} \times 300}{28 \times 1.7 \times 10^{-24}}\right)^{1/2} \approx 3 \times 10^4\text{cm/s}. \tag{3.1}$$

These are really fast, so the only way for this to make sense is for the mean free path to be really small.

For the cross-section, let's go to the Bohr atom for an estimate of the size of the hydrogen atom, and assume anything else is in the same order of magnitude. Force balance in the Bohr atom gives us an electrostatic force of e^2/r^2 equal to a centrifugal force of $\frac{m_ev^2}{r}$. The key thing to remember from quantum mechanics is that the angular momentum is quantized to the value $\hbar$, whch is equal to m_erv. We can get rid of v:

$$v = \frac{\hbar}{m_er} \implies \frac{e^2}{r^2} = \frac{m_e}{r}\left(\frac{\hbar}{m_er}\right)^2 \tag{3.2}$$

Therefore

$$r = \frac{m_e}{e^2}\left(\frac{\hbar}{m_e}\right)^2 \implies a_0 = \frac{\hbar^2}{m_ee^2}. \tag{3.3}$$

If we plug numbers in, we find $a_0 \approx 0.5\text{Å}$, and so we can get $\sigma = \pi a_0^2 \approx 10^{-16}\text{cm}^2$. Most things are a bit bigger, so a good rule of thumb is to take $\sigma \sim (3\text{Å})^2 \sim 10^{-15}\text{cm}^2$.

Finally, let's calculate the MFP of air in the room and the collision time. The MFP is

$$\lambda_{mfp} = \frac{1}{n\sigma} = \frac{1}{3\times 10^{19}\text{cm}^{-3}\cdot 10^{-15}\text{cm}^2} \approx 3\times 10^{-5}\text{cm}. \tag{3.4}$$

This is about one micron. As we would hope, this is much less than the size of the room. The collision time is

$$t_{coll} = \frac{\lambda_{mfp}}{v_{th}} \approx \frac{3\times 10^{-5}\text{cm}}{3\times 10^{4}\text{cm/s}} = 10^{-9}\text{s}. \tag{3.5}$$

This is a nanosecond, which is much less than human timescales.

Therefore the air in the room is a fluid.

3.2 Hydrostatic equilibrium

Assume a plane parallel atmosphere. The "hydro" means we're treating it as a fluid, the "static" means the bulk fluid is not moving, and the "equilibrium" means the system overall is not changing.

Consider a small fluid element, i.e. a parcel of size Δz with area A, where $\Delta z \gg \lambda_{mfp}$ but $\Delta z \ll L$ where L is the length scale of the system. We do this to be able to say the properties of the fluid within the fluid element do not vary.

Let z be the 'up' direction, and consider gravitational acceleration g in the downward direction. The total mass of the flud element is $m = \rho V = \rho A \delta z$. The force on the fluid element from gravity is $\vec{F} = -mg\hat{k} = -\rho A \Delta z g \hat{k}$.

The total force upward due to pressure can be described as

$$F_{\text{pressure}} = (P_{\text{bottom}} - P_{\text{top}})A \tag{3.6}$$

and we can say

$$P_{\text{top}} \approx P_{\text{bottom}} + \frac{\mathrm{d}P}{\mathrm{d}z}\Delta z. \tag{3.7}$$

This gives us

$$F_{\text{pressure}} = -A\Delta z\frac{\mathrm{d}P}{\mathrm{d}z} = -V\frac{\mathrm{d}P}{\mathrm{d}z}. \tag{3.8}$$

Therefore, the force per volume on our fluid element is $-\frac{\mathrm{d}P}{\mathrm{d}z}$.

For force balance, we require that $F_{\text{grav}} + F_{\text{pressure}} = 0$, so

$$-\underbrace{m}_{\rho V} g - V\frac{\mathrm{d}P}{\mathrm{d}z} = 0 \tag{3.9}$$

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = -g. \tag{3.10}$$

This is the *equation of hydrostatic balance*. Further, let's use the ideal gas law, $P = nk_BT = \rho c_s^2$ (where c_s is the speed of sound; we can take this as a definition for now). This lets us rewrite hydrostatic balance as

$$c_s^2 \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}z} = -g \tag{3.11}$$

$$\frac{1}{P}\mathrm{d}P = -\frac{g}{c_s^2}\mathrm{d}z \tag{3.12}$$

$$\ln P = -\frac{g}{c_s^2}z + C \tag{3.13}$$

$$P = P_0 \exp\left(-z/(\underbrace{c_s^2/g}_{H})\right), \tag{3.14}$$

where we refer to H as the "scale height" of the atmosphere. Every time you go up in z by H, you divide your pressure by a factor of e.

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Lecture 4: Scale height and the milk demo

Lecturer: Ruth Murray-Clay *3 October* *Aditya Sengupta*

Last time, we saw the equation of hydrostatic balance, $\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = -g$, which we integrated in the isothermal case and simplified further using the ideal gas law $P = \rho \underbrace{\frac{kT}{\mu}}_{c_s^2}$. We found that the pressure falls off exponentially, where the rate was set by the *scale height* H of the atmosphere. Since P and ρ are proportional, ρ also falls off exponentially with that rate.

What do we think the scale height of the Earth is? Lick Observatory is at about 1.2 km, and we noticed a significant reduction in pressure but not nearly a full scale height. Mount Everest is at about 10 km, and that definitely has significant reduction in pressure that might be a full scale height (37% of the surface's). Another benchmark is that airplanes fly at 30,000 ft.

Let's actually calculate it now. We know $H = \frac{c_s^2}{g}$, and $c_s^2 = \frac{kT}{m}$. We know that $\frac{1}{2}mv_{th}^2 \approx \frac{3}{2}kT$ and so $v_{th}^2 \approx \frac{kT}{m}$. So up to some numerical prefactor, $c_s \sim v_{th}$. This is interesting: the speed of sound is about the thermal velocity of its constituent molecules. We already calculated this to be about 0.3 km/s, so

$$H = \frac{c_s^2}{g} \approx \frac{(0.3\text{km/s})^2}{9.8\text{m/s}^2} \approx \frac{0.1}{10}\frac{\text{km}}{10^{-3}} \sim 10\text{km}. \tag{4.1}$$

Let's derive H again a different way. We know that $\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = -g$, and we know that H is the distance over which P changes by a factor of e.

$$\frac{1}{\rho}\left(\frac{-P}{H}\right) \sim -g \tag{4.2}$$

$$H \sim \frac{P}{\rho}\frac{1}{g} = \frac{\rho c_s^2}{\rho}\frac{1}{g} = \frac{c_s^2}{g}, \tag{4.3}$$

as we expected. But what happened here? We took an "order-of-magnitude derivative". Consider a function $f(x)$ that varies smoothly with the input x. If you want to consider an order unity change in f, you can compute a characteristic length scale over which this happens, and the original quantity over that length represents a derivative at that point.

This only works this way in this case because the exponential function is its own derivative (you can pick whatever; this is all order of magnitude, so there might be a numerical prefactor. We just know in this case that there wasn't because we did it exactly before).

In general, we'll always have a relationship of the form $\frac{\mathrm{d}f}{\mathrm{d}x} \sim \frac{f}{L}$. For instance, $f(x) = ax^b$ has

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = bax^{b-1} = b\frac{f}{x} \tag{4.4}$$

and so for polynomials, $L \sim x$. Polynomials are often referred to as "scale-free" for this reason: a zoomed-in version looks just like the zoomed-out one.

(demo!)

How much milk do we have to pour into a fish tank full of water? We have a tank filled with water to about 25cm / 50cm / 15cm, or a volume of about 20,000 cm^3. We know from lived experience that whole milk has about 3% fat, and we know from Ruth telling us that fat globules are about the size of the wavelength of light, so they're about 500 nm.

Let's think about the mean free path we want. We want it to be within an order-1 factor of the length of the fish tank: not so short that light just gets blocked, but not so long that no scattering happens. So we want an MFP of about 50 cm.

$$\lambda_{mfp} = \frac{1}{n\sigma}. \tag{4.5}$$

We can calculate both of these. $\sigma = \pi r^2$ with $r = 500$ nm. The cross-section from here is $\sigma = 10^{-8}$ cm^2. We can back this out to a desired number density.

The volume of fat globules we put in is 3% of the volume of milk we put in, and it consists entirely of spherical fat globules with a radius of 500 nm. In class we finished this too fast for me to take notes on, but we found that we needed about 0.3 cm^3 of milk, or about one cap, which worked!

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Lecture 5: Conservation of mass for fluids

Lecturer: Ruth Murray-Clay *5 October* *Aditya Sengupta*

On the module, we learned that the column density n (the number of particles per area) is given by $N = nL$. If we have variable density, the optical depth is defined as $\tau = \int n\sigma \mathrm{d}l$; if σ doesn't change with l, we can also say $\tau = \sigma N$ (exactly) because $N = \int n\mathrm{d}l$.

Today, we'll start deriving the fluid equations. We consider an ideal fluid, which has no viscosity, friction, conductivity, magnetic fields, and so on.

Our governing equations will be conservation equations on mass, momentum, energy.

5.1 Eulerian view

We take an Eulerian view, in which we consider a fixed region in space. Consider an arbitrary region of space whose size is much bigger than λ_{mfp}, but much smaller than the length scale of the system. Let its volume be $\mathrm{d}V$ and its outward-pointing area vector be $\mathrm{d}\vec{A}$. We want the change in mass over time overall to be 0. Start with the mass the region has already:

$$M = \int \rho \mathrm{d}V \implies \frac{\mathrm{d}M}{\mathrm{d}t} = \int \frac{\partial \rho}{\partial t} \mathrm{d}V. \tag{5.1}$$

To this we add something depending on a bulk velocity. We define a mass flux by transporting the density ρ through a volume Al in a time t, which gives us a rate ρAv. The general idea is that $\rho\vec{v}$ is equivalent to the mass over area moving with the fluid. This is the mass flux. The net gain or loss of mass overall is therefore

$$\left.\frac{\mathrm{d}m}{\mathrm{d}t}\right|_{flow} = -\int_{\text{area}} \rho\vec{v} \cdot \mathrm{d}\vec{A} \tag{5.2}$$

where we put a negative sign because the area vector is pointing outwards. Using Stokes' theorem (all of the equations that relate surface integrals to volume integrals are Stokes' theorem, just in different dimensions) we get

$$\left.\frac{\mathrm{d}m}{\mathrm{d}t}\right|_{flow} = -\int_V \vec{\nabla} \cdot (\rho\vec{v}) \mathrm{d}V. \tag{5.3}$$

Now we need $\frac{\mathrm{d}M}{\mathrm{d}t}$ to equal $\left.\frac{\mathrm{d}m}{\mathrm{d}t}\right|_{flow}$, so we get

$$\int_V \frac{\mathrm{d}\rho}{\mathrm{d}t} \mathrm{d}V = -\int_V \vec{\nabla} \cdot (\rho\vec{v}) \mathrm{d}V. \tag{5.4}$$

Since this must be true at every point in space, we get

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\rho \vec{v}) \tag{5.5}$$

or

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0. \tag{5.6}$$

When we're dealing with incompressible fluids, we know ρ won't vary much under the divergence or under the time derivative, so we just get $\vec{\nabla} \cdot \vec{v} = 0$. In astro, this is only good when motions are very subsonic, leading to sound waves bouncing back and forth maintaining pressure.

5.2 Lagrangian view

In the Lagrangian view, we consider the motion of a fluid element and "move along with it". The fluid element moves with both time and position, so we use the chain rule to let time be our only independent variable:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\vec{v} \cdot \vec{\nabla}). \tag{5.7}$$

We can justify this in more depth by taking v in components using the time derivatives of each spatial direction, and then combining them back into a dot product. Sometimes, we denote the total derivative in t by $\frac{\mathrm{D}}{\mathrm{D}t}$.

We can relate this to the Eulerian continuity equation.

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) &= 0 \\ \underbrace{\frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho}_{\frac{\mathrm{d}\rho}{\mathrm{d}t}} + \rho \vec{\nabla} \cdot \vec{v} &= 0 \\ \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \vec{\nabla} \cdot \vec{v} &= 0 \end{aligned} \tag{5.8}$$

and this last equation is the Lagrangian form of the continuity equation.

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Lecture 6: The momentum equation

Lecturer: Ruth Murray-Clay *7 October* *Aditya Sengupta*

Last time, we saw the two forms of the mass conservation equation, or the continuity equation:

- Eulerian: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$
- Lagrangian: $\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \vec{\nabla} \cdot \vec{v} = 0$.

Now, we consider conservation of momentum. This is different in that we have the possibility of external sources or sinks due to forces that can change the momentum. We start with the Eulerian perspective, where we take a volume element fixed in space with a volume $\mathrm{d}V$ and an outward-pointing area vector $\mathrm{d}\vec{A}$.

Before we do anything else, we can guess what a conservation of momentum equation should look like; it's based on $F = ma$, so we might guess that it'll look something like $\frac{\vec{F}}{V} = \frac{\mathrm{d}(\rho \vec{v})}{\mathrm{d}t} = \boldsymbol{\nabla} P$.

Consider pressure (an internal force) and gravity (an external force). We could add more forces, but this is sufficient for now. As we said, we're looking for something like

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{total momentum in volume}) = \text{flow through boundaries} + \text{forces}. \tag{6.1}$$

The momentum in $\mathrm{d}V$ is given by $\int \rho \vec{v} \mathrm{d}V$, as $\rho \vec{v}$ represents the momentum per volume. We're going to start using tensor notation, so we write $\int \rho v_i \mathrm{d}V$. The momentum change due to flow through the boundaries can be found analogously to how we did this for mass. There, we did a surface integral of something like $n\sigma v$ times a mass per particle. Here, we do the same thing for momentum per particle, which corresponds to mv_i. This gives us $-\int_S \rho v_i \vec{v} \cdot \mathrm{d}\vec{A}$. This is three separate equations, for $i = x, y, z$.

From Gauss's law, this is also $-\int_V \vec{\nabla} \cdot (\rho v_i \vec{v}) \mathrm{d}V$.

If we expand the dot product, we can get that the divergence term is $\partial_j(\rho v_i v_j)$. Here, $\rho v_i v_j$ is the "momentum flux tensor", which represents bulk momentum transport due to bulk motions. We call this kind of motion "advection".

When v isn't bulk but instead represents turbulent or thermal motions, it's called the Reynolds tensor or the stress tensor.

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Lecture 7: The momentum equation

Lecturer: Ruth Murray-Clay *10 October* *Aditya Sengupta*

Last time, we saw that there is a conservation equation for each component of linear momentum, and motion in all directions affects each component. We use tensors to describe this interrelationship.

Addition or loss of momentum through boundaries of a fluid element can be written as "$-\int \rho\vec{v}\vec{v}\cdot \mathrm{d}\vec{A}$" (this is shorthand notation).

Next, we convert this surface integral to a volume integral:

$$-\oiint \rho v_i v_j \mathrm{d}A_j = -\iiint \partial_j(\rho v_i v_j)\mathrm{d}V. \tag{7.1}$$

We need to add in force terms. We start with pressure, which is a surface force $-\int p\mathrm{d}\vec{A} = -\int \nabla p \mathrm{d}V$. Next, gravity: $\vec{F}$ is the force per unit mass, and for gravity $\vec{F} = \vec{g} = -\nabla\Phi$. Force is the rate of change in time of momentum, i.e.

$$\vec{F} = \frac{\partial}{\partial t}\int \rho\vec{v}\mathrm{d}V = \int \frac{\partial}{\partial t}(\rho\vec{v})\mathrm{d}V. \tag{7.2}$$

Here we've taken an Eulerian view, so we have a volume that's fixed in time.

Putting everything together, we get

$$\int \frac{\partial}{\partial t}(\rho v_i)\mathrm{d}V = -\int \partial_j(\rho v_i v_j)\mathrm{d}V + \int \rho F_i \mathrm{d}V - \partial_i P\mathrm{d}V. \tag{7.3}$$

Since this is true for all volume elements, we drop the integral and find *Euler's equation*,

$$\frac{\partial(\rho v_i)}{\partial t} + \partial_j(\rho v_i v_j) = -\partial_i P + \rho F_i. \tag{7.4}$$

This is also called the "conservation form" or the "standard form" of the conservation law. The reason why this is standard is because the right-hand side has all the source and sink terms.

The pressure term $\partial_i P$ is sometimes written $\partial_j(P\delta_{ij})$ so that it can be pulled over to the left-hand side, leaving only truly external forces to the right-hand side.

$$\frac{\partial}{\partial t}(\rho v_i) + \partial_j(\rho v_i v_j + P\delta_{ij}) = \rho F_i. \tag{7.5}$$

Let's turn this into a Lagrangian equation, because that's the kind of thing $F = ma$ might apply to. You can't have an acceleration of a fixed volume element, but you can with something moving as a fluid element.

Rewrite:

$$\frac{\partial(\rho v_i)}{\partial t} = v_i \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial t} \tag{7.6}$$

and note that mass continuity says $\frac{\partial \rho}{\partial t} + \underbrace{\boldsymbol{\nabla} \cdot (\rho \vec{v})}_{\partial_j(\rho v_j)} = 0$, so

$$\frac{\partial(\rho v_i)}{\partial t} = -v_i \partial_j(\rho v_j) + \rho \frac{\mathrm{d}v_i}{\mathrm{d}t}. \tag{7.7}$$

Note also that by the chain rule,

$$\partial_j(\rho v_i v_j) = v_i \partial_j(\rho v_j) + \rho v_j \partial_j v_i, \tag{7.8}$$

so noting the common term $v_i \partial_j(\rho v_j)$, we can write

$$\frac{\partial(\rho v_i)}{\partial t} = \rho v_j \partial_j v_i - \partial_j(\rho v_i v_j) + \rho \frac{\mathrm{d}v_i}{\mathrm{d}t}. \tag{7.9}$$

Substituting this into Equation 7.3 and cancelling the $\partial_j(\rho v_i v_j)$ term, we get

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \partial_j v_i = -\partial_i P + \rho F_i. \tag{7.10}$$

Note that we've got the Lagrangian derivative!

$$\rho \left(\frac{\partial}{\partial t} + v_j \partial_j \right) v_i = \rho \frac{\mathrm{D}v_i}{\mathrm{D}t}. \tag{7.11}$$

So we can just rewrite this as

$$\rho \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = -\boldsymbol{\nabla} P + \rho \vec{F}. \tag{7.12}$$

An aside: when we say something is "static", it is not moving. When we say something is "steady-state", it means the Eulerian derivative $\frac{\partial}{\partial t}$ is 0. In an inertial frame, the quantity is not moving with the fluid and the flow is not changing, but it doesn't necessarily mean it's static. The Lagrangian derivative $\frac{\mathrm{D}}{\mathrm{D}t}$ may be nonzero; it's equal to $\vec{v} \cdot \vec{\nabla}$.

Now, we're going to move on to the conservation of energy equation. This one gets even more nasty to figure out the different forms of, so we're leaving it to the homework! To start with here, we'll be working with entropy. We start with the first? law of thermodynamics,

$$\mathrm{d}U = -P\mathrm{d}V + T\mathrm{d}S \tag{7.13}$$

and we divide throughout by M,

$$\frac{\mathrm{d}U}{M} = \frac{-P\mathrm{d}V}{M} + \frac{T\mathrm{d}S}{M}. \tag{7.14}$$

We rewrite $\mathrm{d}V$ in terms of $\mathrm{d}\rho$:

$$\frac{\mathrm{d}V}{M} = \mathrm{d}\frac{V}{M} = \mathrm{d}\frac{1}{\rho} = -\frac{1}{\rho^2}P\mathrm{d}\rho. \tag{7.15}$$

Therefore, rewriting using this and $\epsilon = U/M$, we get

$$\mathrm{d}\epsilon = -\frac{P}{\rho^2}\mathrm{d}\rho + T\mathrm{d}s \tag{7.16}$$

where we've introduced $s = S/M$.

We can mathematically describe a lack of heating/cooling effects as $\frac{\mathrm{D}s}{\mathrm{D}t} = 0$.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 8: Conservation of energy

Lecturer: Ruth Murray-Clay *12 October* *Aditya Sengupta*

The Eulerian view of conservation of energy requires us to consider the total energy in the gas, including the bulk kinetic energy of the gas and the internal energy due to thermal motion. The bulk KE of the gas is $\frac{1}{2}\rho v^2$ per volume, and we said the internal energy was $\rho\epsilon$ where ϵ is the energy per mass. So the total energy is given by

$$E = \frac{1}{2}\rho v^2 + \rho\epsilon. \tag{8.1}$$

If we go to conservation law form,

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \vec{F}_E = \underbrace{\rho\vec{F} \cdot \vec{v}}_{\text{sources/sinks}} \tag{8.2}$$

where $\vec{F}_E$ is the *energy flux*, which we add in for continuity. We'll see what's in there in a minute. We typically lump pressure work into the $\boldsymbol{\nabla} \cdot \vec{F}_E$ term.

The energy flux is given by $E\vec{v} + P\vec{v}$. This lets us rewrite the equation as

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot (E\vec{v}) = \rho\vec{F} \cdot \vec{v} - \boldsymbol{\nabla} \cdot (P\vec{v}) \tag{8.3}$$

Intuitively, we're using $\boldsymbol{\nabla} \cdot (P\vec{v})$ because we're considering the energy contribution due to a pressure surface force, which we transform into a volume integral using Stokes' theorem:

$$\int_A P\vec{v}\mathrm{d}A = \int_V \boldsymbol{\nabla} \cdot (P\vec{v})\mathrm{d}V. \tag{8.4}$$

Continuing to rewrite,

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \vec{F}_E = \rho\vec{F} \cdot \vec{v}. \tag{8.5}$$

We substitute in

$$\vec{F}_E = (E + P)\vec{v} = \left(\frac{1}{2}\rho v^2 + \rho\epsilon + P\right)\vec{v} \tag{8.6}$$

$$= \frac{1}{2}\rho v^2\vec{v} + \rho\vec{v}\underbrace{\left(\epsilon + \frac{P}{\rho}\right)}_{\text{enthalpy}} \tag{8.7}$$

The enthalpy (not per mass) is given by $H = U + PV$, so

$$\frac{H}{M} = \frac{U}{M} + \frac{PV}{M} \implies h = \epsilon + \frac{P}{\rho}. \tag{8.8}$$

This represents "the energy inside the gas that's available" (to do $P\mathrm{d}V$ work).

Overall we have

$$\frac{\partial}{\partial t}E + \boldsymbol{\nabla}\cdot\left(\left(\frac{1}{2}\rho v^2 + h\right)\vec{v}\right) = \rho\vec{F}\cdot\vec{v} + \Gamma + \Lambda. \tag{8.9}$$

Here, Γ and Λ represent heating gain and cooling loss, respectively. How these are set depends on the case.

Let's go back to the Lagrangian form. We claim there is no heating and cooling, so $\vec{F}, \Gamma, \Lambda$ are all zero. Then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0 = \frac{\partial s}{\partial t} + (\vec{v}\cdot\vec{\nabla})S \tag{8.10}$$

For an ideal gas with adiabatic index γ, we have

$$s = \frac{1}{\gamma - 1}\frac{k}{m}\ln\frac{P}{\rho^\gamma}. \tag{8.11}$$

We can take derivatives of this to derive the fundamental law of thermodynamics.

Let's take a time derivative of this; we get

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P}{\rho^\gamma}\right) = 0. \tag{8.12}$$

We use the ideal gas law, $P = \frac{\rho kT}{m}$, to get

$$\frac{P}{\rho^\gamma} = \frac{\rho kT}{m\rho^\gamma} = \frac{\rho^{1-\gamma}kT}{m} = \frac{kT}{n^{\gamma-1}m^\gamma}. \tag{8.13}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P}{\rho^\gamma}\right) = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{T}{n^{\gamma-1}}\right) = 0. \tag{8.14}$$

Applying the chain rule, we get

$$\frac{1}{n^{\gamma-1}}\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{(\gamma-1)T}{n^{\gamma}}\frac{\mathrm{d}n}{\mathrm{d}t} = 0 \tag{8.15}$$

$$n\frac{\mathrm{d}T}{\mathrm{d}t} - (\gamma-1)T\frac{\mathrm{d}n}{\mathrm{d}t} = 0 \tag{8.16}$$

$$\frac{1}{\gamma-1}nk\frac{\mathrm{d}T}{\mathrm{d}t} - kT\frac{\mathrm{d}n}{\mathrm{d}t} = 0. \tag{8.17}$$

This is the energy equation of the gas in the absence of any sources or sinks.

We can relate this to the fundamental law of thermodynamics in the per-mass form:

$$\mathrm{d}\epsilon = T\mathrm{d}s + \frac{P}{\rho^2}\mathrm{d}P. \tag{8.18}$$

We know that $\epsilon = \frac{3}{2}kT$ for a monatomic ideal gas (and in general, $\frac{l}{2}kT$ for a gas with l degrees of freedom). We also often use the specific heats $C_P = \left(\frac{đQ}{đT}\right)\Big|_P$ and $C_V = \left(\frac{đQ}{đT}\right)\Big|_V$.

We use the differentials $\mathrm{d}E = đQ + \frac{P}{\rho^2}\mathrm{d}\rho$.

Here,

$$C_V = \left(\frac{\mathrm{d}Q}{\mathrm{d}T}\right)_V \underbrace{=}_{\mathrm{d}\rho=0} \left(\frac{\mathrm{d}\epsilon}{\mathrm{d}T}\right)_V = \frac{l}{2}k \tag{8.19}$$

$$C_P = \left(\frac{\mathrm{d}Q}{\mathrm{d}T}\right)_P = \left(\frac{\mathrm{d}\epsilon}{\mathrm{d}T}\right)_P - \frac{P}{\rho^2}\left(\frac{\rho}{T}\right)_P \tag{8.20}$$

By the ideal gas law,

$$P = \frac{\rho kT}{m} \implies \rho = \frac{mP}{kT} \implies \left(\frac{\mathrm{d}\rho}{\mathrm{d}T}\right)_P = \frac{-mP}{kT^2}. \tag{8.21}$$

Therefore

$$C_P = \frac{l}{2}k + \frac{P}{\rho^2}\frac{m^2P}{kT^2} \tag{8.22}$$

Substituting in $P^2 = \frac{\rho^2k^2T^2}{m^2}$, we get

$$C_P = \frac{l}{2}k + k \tag{8.23}$$

Therefore, we get

$$\gamma = \frac{C_P}{C_V} = \frac{\frac{l}{2} + 1}{\frac{l}{2}} = \frac{l + 2}{l}. \tag{8.24}$$

For example, if $l = 3$ we get $\gamma = \frac{5}{3}$, and if $l = 5$ we get $\gamma = \frac{7}{5}$.

There's a missing m somewhere that I have to fill in so that we get the right dimensions.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 9: Lagrangian energy, motion on streamlines

Lecturer: Ruth Murray-Clay *14 October* *Aditya Sengupta*

Last time, we were trying to show that

$$\frac{1}{\gamma-1}nk\frac{\mathrm{d}T}{\mathrm{d}t} - kT\frac{\mathrm{d}n}{\mathrm{d}t} = 0 \tag{9.1}$$

is equivalent to the fundamental theorem of thermodynamics, which we can put in the form (for the case with no heating and cooling)

$$T\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}\epsilon}{\mathrm{d}t} - \frac{P}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0. \tag{9.2}$$

We know that

$$\epsilon = \frac{\frac{l}{2}kT}{m} = \frac{1}{\gamma-1}\frac{kT}{m}. \tag{9.3}$$

Starting with equation 9.1, dividing through by n and rearranging, we get

$$\frac{1}{\gamma-1}\frac{k}{m}\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{kT}{mn}\frac{\mathrm{d}n}{\mathrm{d}t} = 0 \tag{9.4}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} - \frac{kT}{m^2n}\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0. \tag{9.5}$$

Using the ideal gas law, $P = \frac{\rho kT}{m}$, we get

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} - \frac{Pm}{\rho m^2 n}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{d}\epsilon}{\mathrm{d}t} - \frac{P}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0, \tag{9.6}$$

which was what we wanted.

We can interpret this as saying the Lagrangian energy equation just is the energy conservation equation. $\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V$, and we're assuming no change in entropy (no heating/cooling), so in this case, the only change in internal energy is from $P\mathrm{d}V$ work.

Overall, we now have a system of five equations: continuity, three momentum equations, and the energy equation. Our unknowns are $\rho, T, v_{x,y,z}, P$, so we're one equation short. This comes in with the *equation of state*, $P = P(\rho, T)$. In this case, we're using the ideal gas law $P = \frac{\rho kT}{m}$.

Let's start applying this system of equations! To begin with, we'll talk about streamlines. Bernoulli's equation is defined along streamlines, which define the path of the fluid. It is most useful in steady state, i.e. $\frac{\mathrm{d}}{\mathrm{d}t} = 0$ and the tangent to the path of the fluid in space doesn't change with time. For example, the streamlines for uniform motion and rotation (diagrams here).

Assume steady flow $\frac{\mathrm{d}}{\mathrm{d}t} = 0$ and an external gravitational force $\vec{F} = -\boldsymbol{\nabla}\Phi$. Consider the energy conservation law,

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \boldsymbol{\nabla} \cdot ((E+P)\vec{v}) = \rho\vec{F} \cdot \vec{v}. \tag{9.7}$$

In this case,

$$\rho\vec{F} \cdot \vec{v} = -\rho\vec{\nabla}\Phi \cdot \vec{v} \tag{9.8}$$

and by the chain rule,

$$\boldsymbol{\nabla} \cdot (\rho\Phi\vec{v}) = \rho\vec{v} \cdot \boldsymbol{\nabla}\Phi + \Phi\boldsymbol{\nabla} \cdot (\rho\vec{v}) \tag{9.9}$$

and so

$$\rho\vec{F} \cdot \vec{v} = \Phi \underbrace{\boldsymbol{\nabla} \cdot (\rho\vec{v})}_{\text{continuity: } \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\vec{v}) = 0} - \boldsymbol{\nabla} \cdot (\rho\Phi\vec{v}). \tag{9.10}$$

so

$$\rho\vec{F} \cdot \vec{v} = -\boldsymbol{\nabla} \cdot (\rho\Phi\vec{v}). \tag{9.11}$$

Plugging this back in, we get

$$\boldsymbol{\nabla} \cdot (E\vec{v} + P\vec{v}) = \boldsymbol{\nabla} \cdot (-\rho\Phi\vec{v}) \tag{9.12}$$

$$\boldsymbol{\nabla} \cdot (\vec{v}(E + P + \rho\Phi)) = 0 \tag{9.13}$$

Recall that $E = \rho\epsilon + \frac{1}{2}\rho v^2$, so

$$\boldsymbol{\nabla} \cdot \left[\rho\vec{v}\left(\epsilon + \frac{1}{2}v^2 + \frac{P}{\rho} + \Phi\right)\right] \tag{9.14}$$

and we can combine two of these terms into the enthalpy $h = \epsilon + \frac{P}{\rho}$, to get

$$\boldsymbol{\nabla}\cdot\left[\left(\frac{1}{2}v^2+h+\Phi\right)\rho\vec{v}\right]=0. \tag{9.15}$$

Recall that

$$\frac{\mathrm{D}}{\mathrm{D}t}=\frac{\partial}{\partial t}+\vec{v}\cdot\vec{\nabla}. \tag{9.16}$$

What we have here is almost that, but not quite. To proceed, let's expand out Equation 9.15 using the chain rule,

$$\vec{v}\cdot\vec{\nabla}\left(\frac{1}{2}v^2+h+\Phi\right)+\left(\frac{1}{2}v^2+h+\Phi\right)\underbrace{\boldsymbol{\nabla}\cdot(\rho\vec{v})}_{=0}=0 \tag{9.17}$$

so

$$\rho\vec{v}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}v^2+h+\Phi\right)=0. \tag{9.18}$$

This is the equation for energy conservation in a steady state fluid. Since $\vec{v}\cdot\vec{\nabla}$ represents motion along streamlines, we have

$$B\triangleq\frac{1}{2}v^2+h+\Phi=\text{constant along streamlines.} \tag{9.19}$$

This looks like an energy conservation equation, with the addition of enthalpy encapsulating $P\mathrm{d}V$ work along the flow.

This equation holds for lift on airplane wings. If we have Φ roughly constant, we get that $\frac{1}{2}v^2+h$ is also roughly constant. That is, along a streamline, regions of high v have low h and so low pressure. Similarly, regions of low v have high h and high pressure.

This explains lift: different sides of a wing have different vs, creating a pressure differential, which causes lift. Why this lift goes up depends on the details of wings.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 10: Accretion and outflow

Lecturer: Ruth Murray-Clay *17 October* *Aditya Sengupta*

We've seen how an isothermal hydrostatic atmosphere has a nonzero pressure and density at infinity, which is nonphysical. We resolve this either by dropping the "isothermal" assumption (i.e. the temperature changes through the atmosphere) or the "hydrostatic" assumption (i.e. there's some outgassing). The model for the latter case is called a *Parker wind*.

The Sun has a hot corona and wind. The corona radiates as a blackbody at about 10^6K, which puts it in the UV and soft x-ray range. The wind has a mass loss rate of about $10^{-14}\frac{M_\odot}{\text{year}}$ and energy loss rate of about $10^{-6}L_\odot$, so we can neglect both.

As we've done before, we can compute the density as $r \to \infty$; this comes out to $\rho_0 \exp\left(-\frac{GM}{r_0 c_s^2}\right) \sim \rho_0 e^{-12}$ and similarly $P \to P_0 e^{-12}$. This is pretty small but it's much greater than the pressure of the interstellar medium. We need a dynamic/time-dependent model. It can be steady-state, but there need to be velocities involve; it can't be static.

A full treatment would involve dropping both assumptions and treat all the equations, but this would be hard to do analytically and people who have actually solved this have found a class of isothermal solutions. We'll assume *a priori* that such a solution set exists and try to find it.

We treat wind as spherically symmetric and steady state, but not static. This means $\frac{\partial}{\partial t} = 0$ but $\frac{\mathrm{D}}{\mathrm{D}t} \neq 0$. Let v be the outflow velocity. We use the momentum equation with $\vec{F} = \frac{GM}{r^2}\hat{r}$:

$$\rho\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = -\nabla P - \frac{GM\rho}{r^2}\hat{r}. \tag{10.1}$$

Note that if we had $\vec{v} = 0$ and $\frac{\mathrm{D}}{\mathrm{D}t} = 0$, we'd have

$$-\nabla P - \rho\frac{GM}{r^2}\hat{r} = 0$$

which is a rearranged version of the hydrostatic equilibrium equation, as we'd expect.

Expanding out the Lagrangian derivative, we get

$$\rho \underbrace{\cancel{\frac{\partial \vec{v}}{\partial t}}}_{\text{steady state}} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla P - \rho\frac{GM}{r^2}\hat{r}. \tag{10.2}$$

We assume the system is spherically symmetric, so we can write this as an equation where r is the only variable and where $v = v_r\hat{r}$:

$$\rho(\vec{v}\cdot\nabla)\vec{v} = -\frac{\partial P}{\partial r}\hat{r} - \rho\frac{GM}{r^2}\hat{r}. \tag{10.3}$$

This gives us the momentum equation we're going to solve:

$$\rho v \frac{\partial v}{\partial r} = -\frac{\partial P}{\partial r} - \rho \frac{GM}{r^2}. \tag{10.4}$$

Since the atmosphere is isothermal, we take $P = \rho c_s^2$ and so

$$\rho v \frac{\partial v}{\partial r} = -c_s^2 \frac{\partial \rho}{\partial r} - \rho \frac{GM}{r^2}. \tag{10.5}$$

We use this as our replacement for the energy equation. Next, we bring in mass continuity:

$$\underbrace{\frac{\partial \not{\rho}}{\not{\partial} t}}_{0} + \nabla \cdot (\rho \vec{v}) = 0 \tag{10.6}$$

and due to spherical coordinates, this reduces to

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \rho v \right) = 0. \tag{10.7}$$

Therefore $r^2 \rho v$ is constant with r.

$\dot{M}$ is the rate of mass moving through a spherical shell of radius r and therefore surface area $4\pi r^2$. We can get this by computing $n\sigma v$:

$$\dot{M} = \rho 4\pi r^2 v. \tag{10.8}$$

By our continuity equation, this is constant with r. This gives us our full system:

$$\begin{aligned} \dot{M} &= 4\pi r^2 \rho v = \text{const} \\ \rho v \frac{\partial v}{\partial r} &= -c_s^2 \frac{\partial \rho}{\partial r} - \rho \frac{GM}{r^2} \\ T &= \text{const} \end{aligned} \tag{10.9}$$

Note that there is symmetry in the equations: we could replace v with $-v$ and it would also be a solution. Physically, this would mean spherical wind changes to spherical accretion.

Let's expand out the first equation. $\dot{M}$ being constant means its r derivative is 0, so

$$\frac{\partial}{\partial r} \left(\rho v r^2 \right) = 0 \tag{10.10}$$

$$2r\rho v + r^2 v \frac{\partial \rho}{\partial r} + r^2 \rho \frac{\partial v}{\partial r} = 0. \tag{10.11}$$

Isolating $\frac{\partial \rho}{\partial r}$ and plugging it into the momentum equation, we get

$$\rho v \frac{\partial v}{\partial r} = -c_s^2 \underbrace{\rho \left[-\frac{2}{r} - \frac{1}{v}\frac{\partial v}{\partial r} \right]}_{\frac{\partial \rho}{\partial r}} - \rho \frac{GM}{r^2} \tag{10.12}$$

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = c_s^2 \left[\frac{2}{r} + \frac{1}{v}\frac{\partial v}{\partial r} \right] - \frac{GM}{r^2}. \tag{10.13}$$

We get the *isothermal wind/accretion equation*,

$$\frac{1}{v}\frac{\partial v}{\partial r}\left[v^2 - c_s^2\right] = \frac{2c_s^2}{r} - \frac{GM}{r^2}. \tag{10.14}$$

We can rearrange this to get

$$\boxed{\frac{1}{v}\frac{\partial v}{\partial r} = \frac{\frac{2c_s^2}{r} - \frac{GM}{r^2}}{v^2 - c_s^2}}. \tag{10.15}$$

This leads to a singularity at $v = c_s$ unless $\frac{2c_s^2}{r} - \frac{GM}{r^2} = 0$, or

$$\boxed{r_{\text{sonic}} = \frac{GM}{2c_s^2}}. \tag{10.16}$$

This is called the *sonic point radius*.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 11: Parker wind, the sonic point

Lecturer: Ruth Murray-Clay *19 October* *Aditya Sengupta*

Last time, we derived the expression for accretion/outflow:

$$\frac{1}{v}\frac{\partial v}{\partial r} = \frac{\frac{2c_s^2}{r} - \frac{GM}{r^2}}{v^2 - c_s^2}. \tag{11.1}$$

To avoid $\frac{\partial v}{\partial r} \to \infty$ when $v^2 - c_s^2 = 0$, we had three options:

1. the flow is always supersonic
2. the flow is always subsonic
3. when $v = \pm c_s$, the numerator also has to be 0, $\frac{2c_s^2}{r} - \frac{GM}{r^2} = 0$.

The third point is the *Parker wind solution*, and it's specified by $v = \pm c_s$ and $r = r_s = \frac{GM}{2c_s}$. We can plot $v(r)$ to visualize what this means. There are two solutions that pass through $v(r_s) = c_s$, one going up and one going down. The family of subsonic solutions is always below the minimum of both solutions, and the family of supersonic solutions is always above the maximum of both solutions. These are steady-state solutions, and in practice subsonic time-dependent solutions can "push up" to the Parker wind solution and supersonic ones can "push down" to the Bondi accretion solution.

This is the principle behind a de Laval nozzle, which expects $v < c_s$ at the inlet and $v > c_s$ at the outlet. It narrows towards the center to a critical point that controls the flow, and then widens again. If we want $\dot{M}$ to be constant, we should get increasing ρ and v at the center where it narrows.

The expression for r_s might remind us of that of the scale height: $H = \frac{c_s^2 r^2}{GM} = \frac{r^2}{2r_s}$. This tells us that $v_{esc} \sim c_s$ at r_s.

Let's formally resolve the 0/0 in the Parker/Bondi solution using L'Hôpital's rule evaluated at the sonic point s.p.,

$$\left.\frac{1}{v}\frac{\partial v}{\partial r}\right|_{s.p.} = \frac{\left.\frac{2c_s^2}{r} - \frac{GM}{r^2}\right|_{s.p.}}{\left.v^2 - c_s^2\right|_{s.p.}} = \frac{\left.\frac{\partial}{\partial r}\left(\frac{2c_s^2}{r} - \frac{GM}{r^2}\right)\right|_{s.p.}}{\left.\frac{\partial}{\partial r}(v^2 - c_s^2)\right|_{s.p.}}. \tag{11.2}$$

Taking derivatives and plugging everything in at the sonic point $r = r_s, v = c_s$,

$$\left.\frac{\partial v}{\partial r}\right|_{s.p.} = \pm\frac{2c_s^3}{GM}. \tag{11.3}$$

This lets us drop a boundary condition. Instead of our boundary conditions being ρ, v, T at the base, we use ρ, T, and the fact that the above must be satisfied for a solution that passes through the sonic point.

Let's try and order-of-magnitude each term in the energy equation. In the case $r \ll r_s$, we can say $\rho v \frac{\partial v}{\partial r} \sim 0$ because it OOMs to something with a v^2, so it's negligible compared to v_{esc}^2 or c_s^2. This lets us say

$$-c_s^2 \frac{\partial \rho}{\partial r} \sim \rho \frac{GM}{r^2}.$$

This is about the hydrostatic equilibrium equation, so the density profile is about the hydrostatic one.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 12: Accretion conditions

Lecturer: Ruth Murray-Clay *21 October* *Aditya Sengupta*

Consider a gas of density ρ and temperature T around a body with radius r and mass M. At what temperature will the gas accrete onto the body, and what is the accretion rate $\dot{M}$?

We start with hydrostatic equilibrium,

$$-\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} < \frac{GM}{r^2} \tag{12.1}$$

where we take a negative sign because we want $\frac{\mathrm{d}P}{\mathrm{d}r}$ to be negative. Taking an order-of-magnitude derivative with a scale length L,

$$\frac{1}{\rho}\frac{P}{L} < \frac{GM}{r^2}. \tag{12.2}$$

We use the ideal gas law $P = \rho\frac{kT}{\mu} = \rho c_s^2$ to get

$$\frac{c_s^2}{L} < \frac{GM}{r^2}. \tag{12.3}$$

To finish this condition, we need to specify L. The natural answer is $L = r$, as it's the only length in the problem. This gives us

$$\frac{c_s^2}{r} < \frac{GM}{r^2} \implies c_s^2 < \frac{GM}{r} \tag{12.4}$$

or $c_s^2 \lesssim v_{esc}^2$. This makes sense: motion of the gas is dominated by the gravitational escape speed rather than the pressure/sound waves. This gives us the condition

$$\frac{kT}{\mu} < \frac{GM}{r} \implies kT < \frac{GM\mu}{r}. \tag{12.5}$$

We could also take $L = H_{hydro} = \frac{c_s^2 r^2}{GM}$, but plugging this in would give us $0 = 0$. This implies we have the freedom to pick a scale length $H \neq H_{hydro}$ and see what it tells us. If we take $L > H_{hydro}$, the pressure would be too small. If $H_{hydro} < r$ we'd get

$$\frac{c_s^2 r^2}{GM} < r \implies c_s^2 < \frac{GM}{r} \tag{12.6}$$

which is the same condition. We could also take $L = r_{sonic}$ and get an equivalent condition to what's above.

For the accretion rate, we use $\dot{M} = 4\pi r^2 \rho v$. We take this at the sonic point, so $r_s = \frac{GM}{2c_s^2}$ and $v = c_s$. We plug in the density at infinity of the gas for ρ.

The luminosity is $L \sim \frac{GM\dot{M}}{r} \sim \frac{1}{2}v_{esc}^2\dot{M}$.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 13: Accretion recap, sound waves

Lecturer: Ruth Murray-Clay *24 October* *Aditya Sengupta*

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13.1 Accretion recap

Last time, we talked about Bondi accretion. We chose the sonic point as where to plug in for the mass accretion rate $\dot{M} = 4\pi r^2 \rho v$. We usually pick the Bondi radius $R_B = \frac{GM}{c_s^2}$, at which $v_{esc} \sim c_s$ and $\rho \sim \rho_g$. A way to justify this is by showing

$$\frac{H}{r} = \frac{c_s^2 r^2}{GM}\frac{1}{r} = \frac{c_s^2 r}{GM} = \frac{r}{R_B}. \tag{13.1}$$

That is, if $r \gtrsim R_B$, we can say $H \gtrsim r$, so ρ hasn't fallen much by R_B. This lets us say

$$\dot{M} \approx 4\pi R_B^2 \rho_g c_s. \tag{13.2}$$

For this to work, we needed $c_s < v_{esc}$ for accretion We can also write thisas $c_s^2 < \frac{2GM}{R}$, or $R \lesssim \frac{GM}{c_s^2} = R_B$. (We're used to dropping factors of 2 indiscriminately by this point). Therefore the Bondi radius is the threshold for accretion to take place.

At the end of class last time, we said the luminosity is given by $L = \frac{1}{2}\dot{M} v_{esc}^2$. This makes sense as it's an energy per time where we assume all the infalling mass gets converted to energy and comes in at the escape velocity. This in turn is the case because it's the opposite of starting at vesc at the surface and going to 0 at infinity. For this to work, we have to assume it's able to freely accelerate due to gravity, i.e. gravity is much greater than pressure.

13.2 Sound waves

We start with mass conservation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \vec{v}) = 0 \tag{13.3}$$

and momentum conservation:

$$\rho \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = -\boldsymbol{\nabla} P + \underbrace{\rho \vec{F}_g}_{\text{drop this}} . \tag{13.4}$$

We're eventually going to find it's okay to neglect gravity, but for now let's drop it using an order of magnitude argument. Sound waves are compressional waves in a gas, and they have a much smaller wavelength than the scale height of the atmosphere. Also, the acceleration from gravity over a wavelength is much much less than c_s.

For energy, we consider two cases:

1. isothermal, $T =$const
2. adiabatic/isentropic, $\frac{\mathrm{D}s}{\mathrm{D}t} = 0$.

We'll also have the ideal gas equation of state $P = \frac{\rho k T}{\mu} = \rho c_s^2$. We haven't yet shown that $c_s^2 = \frac{kT}{\mu}$ represents the isothermal sound speed; that's our goal here.

We know that in case 1, $P \propto \rho$, and in case 2,$s = c_v \ln \frac{P}{\rho^\gamma}$. This implies $\frac{\mathrm{D}s}{\mathrm{D}t} = 0$ implies $\frac{\mathrm{D}}{\mathrm{D}t}\frac{P}{\rho^\gamma} = 0$ implies $P \propto \rho^\gamma$. We can encapsulate both of these with $P = K\rho^\gamma$, where $\gamma = 1$ for the isothermal case and $\gamma =$the adiabatic index for the adiabatic case. This kind of equation of state is called a polytropic equation of state if γ is arbitrary. Polytropes are often used to approximate physical systems because it's useful even if you don't know what's going on with your actual equation of state, so you can get a good approximation.

Our set of equations is now

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\vec{v}) = 0 \tag{13.5}$$

$$\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = -\boldsymbol{\nabla}P \tag{13.6}$$

$$P = K\rho^\gamma. \tag{13.7}$$

We're going to consider perturbations on top of known solutions to these equations, and we hope that they don't grow without bound. We want to solve for $\rho, T, P, \vec{v}$, so let's pick an internally consistent set of these. We can set $\vec{v} = \vec{v}_0 = 0$ and set the others to constants; $\rho = \rho_0, T = T_0, P = P_0$. This kills all the derivatives in space and time because none of these vary as anything changes.

Now we perturb the solution by saying $\rho = \rho_0 + \rho_1$, $T = T0 + T1$, and so on. Assume the perturbation is small, so $\rho_1 \ll \rho_0$, $T_1 \ll T_0$, and so on. Since $\vec{v}_0 = 0$, there's no obvious choice for what it means for $\vec{v}_1$ to be small, because it has to be small with respect to something, so we say $\vec{v}_1 \ll \vec{v}_{thermal}$. We plug these back into the governing equations:

$$\frac{\partial(\rho_0 + \rho_1)}{\partial t} + \boldsymbol{\nabla}\cdot((\rho_0+\rho_1)(\vec{v}_0 + \vec{v_1})) = 0 \tag{13.8}$$

$$\color{blue}{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho_1}{\partial t} + \color{blue}{\boldsymbol{\nabla}\cdot(\rho_0\vec{v}_0)} + \boldsymbol{\nabla}\cdot(\rho_0\vec{v}_1) + \boldsymbol{\nabla}\cdot(\rho_1\vec{v}_0) + \color{red}{\rho_1\vec{v}_1} = 0. \tag{13.9}$$

We drop the red term because it's second order, and the blue terms because they're constant, and even if they weren't they sum to the left hand side of the unperturbed continuity equation so they have to sum to 0.

$$\frac{\partial}{\partial \rho_1} + \boldsymbol{\nabla}\cdot(\rho_0\vec{v}_1) + \underbrace{\cancel{\boldsymbol{\nabla}\cdot(\rho_0 v_1)}}_{\vec{v}_0} = 0. \tag{13.10}$$

It doesn't change the physics if we had $\vec{v}_0 \neq 0$, it just makes the math more annoying. Expanding out the one surviving divergence term,

$$\frac{\partial \rho_1}{\partial t} + \vec{v}_1 \cdot \boldsymbol{\nabla} \rho_0 + \rho_0 \boldsymbol{\nabla} \cdot \vec{v_1}. \tag{13.11}$$

We drop the blue term because the $\boldsymbol{\nabla}\rho_0$ is a perturbation on a scale around the scale height; it can be approximated as ρ/H, and $H \gg R_B$ so $\rho/H \ll \rho/R_B$, meaning it's much smaller than we're looking for.

We do the same procedure on the momentum equation,

$$(\rho_0 + \rho_1)\frac{\mathrm{D}\vec{v}_0 + \vec{v}_1}{\mathrm{D}t} = -\boldsymbol{\nabla}(P_0 + P_1). \tag{13.12}$$

We can expand this, drop all terms that only have background terms (subscripts 0), drop all terms that have more than one 1 subscript as this would be second-order, and use the fact that $\vec{v}_0 = 0$. This gives us the perturbed equation

$$\rho_0 \frac{\mathrm{D}\vec{v}_1}{\mathrm{D}t} = -\boldsymbol{\nabla} P_1. \tag{13.13}$$

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Lecture 14: Sound waves

Lecturer: Ruth Murray-Clay *26 October* *Aditya Sengupta*

Last time, we wrote down a set of equations from which we could solve for sound waves:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0; \rho \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = -\nabla P; P = K\rho^{\gamma}.$$

We further perturbed each variable by some small amount away from an equilibrium solution, and dropped terms that are second order in the perturbations because those are smaller than other terms. We also used the fact that ρ_0, P_0, etc., are a solution to get rid of terms that were only the background solution. Finally, we took a special case to make the math easier: $\vec{v}_0 = 0$. This gave us the equations

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{v}_1 = 0 \tag{14.1}$$

$$\rho_0 \frac{\mathrm{D}\vec{v}_1}{\mathrm{D}t} = -\nabla P_1. \tag{14.2}$$

Our third equation comes from $P = K\rho^{\gamma}$, which implies $\nabla P = \frac{\mathrm{D}P}{\mathrm{D}\rho}\nabla\rho$, and we can say

$$\frac{\mathrm{D}P}{\mathrm{D}\rho} = \gamma K \rho^{\gamma-1} = \frac{\gamma P}{\rho}. \tag{14.3}$$

Define as our sound speed $a_0^2 = \frac{\mathrm{d}P}{\mathrm{d}\rho} = \gamma\frac{P}{\rho}$. Note that in the isothermal case, this just reduces to $a_0^2 = c_s^2$ as we can plug in $P = c_s^2\rho$.

We plug in the perturbation:

$$\nabla P = \gamma \frac{P}{\rho} \nabla \rho \tag{14.4}$$

$$\nabla (P_0 + P_1) = \gamma \frac{(P_0 + P_1)}{(\rho_0 + \rho_1)} \nabla (\rho_0 + \rho_1) \tag{14.5}$$

$$\nabla P_0 + \nabla P_1 = \gamma \frac{P_0}{\rho_0} \nabla \rho_0 + \gamma \frac{P_0}{\rho_0} \nabla \rho_1 + \text{second-order terms} + \text{terms proportional to } \nabla \rho_0, \tag{14.6}$$

where we've assumed $\nabla \rho_0$ is negligible compared to a perturbative term. Cancelling the two blue terms (because they're equal from the unperturbed equation), we get

$$\rho_0 \frac{\mathrm{D}\vec{v}_1}{\mathrm{D}t} = -\nabla P_1 = -\gamma \frac{P_0}{\rho_0} \nabla \rho_1. \tag{14.7}$$

We can also write this as

$$\rho_0 \frac{\mathrm{d}\vec{v}_1}{\mathrm{d}t} = -a_0^2 \boldsymbol{\nabla} \rho_1, \tag{14.8}$$

where $a_0^2 = \frac{\mathrm{D}P}{\mathrm{D}\rho}\Big|_0$.

This gives us a system consisting of Equation 14.1 and Equation 14.8 in ρ_1 and $\vec{v}_1$. Let's try and combine these into a wave equation. A general wave equation has the form

$$\boldsymbol{\nabla}^2 f - C \frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = 0 \tag{14.9}$$

where the speed of the wave is given by $1/\sqrt{C}$. This is solved by expressions like $f(x,t) = A\cos(kx - \omega t)$ or $e^{i(kx-\omega t)}$; by taking derivatives, we can show that these solve the equation and that the speed is $\frac{\omega}{k}$.

How can we show that we have a wave equation? We take a time derivative on the mass conservation equation to get

$$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \boldsymbol{\nabla} \cdot \frac{\partial \vec{v}_1}{\partial t} = 0. \tag{14.10}$$

We get

$$\rho_0 \frac{\mathrm{D}\vec{v}_1}{\mathrm{D}t} = \rho_0 \frac{\partial \vec{v}_1}{\partial t} + \rho_0 (\vec{v}_0 \cdot \boldsymbol{\nabla}) \vec{v}_1, \tag{14.11}$$

where we cancel the $\vec{v}_0$ term by saying $\vec{v}_0 = 0$. Plugging this into momentum conservation, we get

$$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \boldsymbol{\nabla} \cdot \left(-\frac{a_0^2}{\rho_0} \boldsymbol{\nabla} \rho_1 \right) = 0. \tag{14.12}$$

$$\frac{\partial^2 \rho_1}{\partial t^2} - a_0^2 \boldsymbol{\nabla}^2 \rho_1 = 0. \tag{14.13}$$

This is a wave equation! In the isothermal case, we have $\gamma = 1$ and $a_0^2 = c_s^2$. In the adiabatic case, we have $a_0^2 = \gamma \frac{P_0}{\rho_0}$.

Now, we're going to Fourier decompose our perturbation so that we can build up a solution set and write down a *dispersion relation*. We'll require each Fourier component to satisfy the equations.

$$\rho_1 = \delta\rho \exp\Big(i(\vec{k} \cdot \vec{x} - \omega t) \Big) \tag{14.14}$$

$$\vec{v}_1 = \delta\vec{v} \exp\Big(i(\vec{k} \cdot \vec{x} - \omega t) \Big). \tag{14.15}$$

Let's substitute these into the mass equation.

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \vec{v}_1 = 0 \tag{14.16}$$

$$-i\omega\delta\rho \exp\Big(i(\vec{k}\cdot\vec{x} - \omega t)\Big) + \rho_0 ik\delta\vec{v} \exp\Big(i(\vec{k}\cdot\vec{x} - \omega t)\Big) = 0\delta\rho \qquad = \rho_0 \frac{\vec{k}\cdot\delta\vec{v}}{\omega}. \tag{14.17}$$

We do the same for the momentum equation:

$$\rho_0 \frac{\partial \vec{v}_1}{\partial t} + a_0^2 \boldsymbol{\nabla}\rho_1 = 0 \tag{14.18}$$

$$\rho_0(-i\omega\delta\vec{v})e^{i(\dots)} + a_0^2 i\vec{k}\delta\rho e^{i(\dots)} = 0 \tag{14.19}$$

$$\delta\vec{v} = \frac{a_0^2\delta\rho}{\rho_0\omega}\vec{k}. \tag{14.20}$$

We've found that $\delta\vec{v}$ is proportional to $\vec{k}$, meaning that sound waves travel in the same direction as the velocity/density perturbation. In other words, sound waves are compression waves.

Combining the mass and momentum expressions,

$$\vec{k}\cdot\delta\vec{v} = \vec{k}\cdot\left(\frac{a_0^2\delta\rho}{\rho_0\omega}\vec{k}\right) = \frac{\delta\rho}{\rho_0}a_0^2\frac{k^2}{\omega} \tag{14.21}$$

$$\delta\rho = \rho_0\frac{\vec{k}\cdot\delta\vec{v}}{\omega} = \frac{\rho_0}{\omega}\frac{\delta\rho}{\rho_0}a_0^2\frac{k^2}{\omega} = \delta\rho a_0^2\frac{k^2}{\omega^2}. \tag{14.22}$$

Therefore $1 = a_0^2\frac{k^2}{\omega^2} \implies \frac{\omega}{k} = a_0$. (Note here that we're writing $k^2 = \vec{k}\cdot\vec{k}$.) This is the *dispersion relation* (relation between ω and k) for a wave that moves at velocity a_0. This restricts our space of possible solutions from all choices of $(\omega, \vec{k})$ to those that satisfy the dispersion relation.

Lecture 15: Sound physics

Lecturer: Ruth Murray-Clay *28 October* *Aditya Sengupta*

Last time, we found a mathematical expression for sound waves; today, we'll connect that to physics. We found that sound waves are compression waves, so what does the speed of those compressions tell us? First, we'll intuitively reason that the speed of propagation and the speed of compressions are the same. When we transmit sound from one location to another, no molecules move; instead, perturbations move through space, and we're interested in the speed at which, say, an individual peak moves. This is the same as the speed of the overall wave (up to a factor of 2, because we usually measure the time for a peak to move half a wavelength, but that's fine) so whenever we have a v in what follows we can treat it all as the same v.

We have regions of high pressure and low pressure, which implies there is a force due to a pressure gradient. The velocity of compressions can be approximated by $F\Delta t$, where F represents a pressure gradient. Further, we know it's a wave, so it should have a frequency ν, wavelength λ, and period $T = \frac{1}{\nu} = \frac{2\pi}{\omega}$. The velocity of the sound wave is then given by $\frac{\lambda}{T} = \lambda\nu$. We don't know λ or ν, but we do know T should be greater than the collision time and less than the damping time.

Let's try and put some of this together. First, let's see if we have an expression for F. Since we have a direction of propagation, we can look at changes in pressure over that direction, and average this over volumes. We normalize this over mass, defined by the integral of density over volume.

$$F = \frac{-\int_V \frac{\mathrm{d}P}{\mathrm{d}x}\mathrm{d}V}{\int_V \rho \mathrm{d}V}. \tag{15.1}$$

Therefore

$$\Delta v \sim -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x}\Delta t \sim \frac{1}{\rho}\frac{P}{\lambda}T = \frac{P}{\rho v}. \tag{15.2}$$

That is, $v \sim \frac{P}{\rho v} \implies v^2 \sim \frac{P}{\rho} = c_s^2$. So $v \sim c_s$, as we would hope!

Let's look at specific wave solutions. In one dimension, we have a wave $A\cos(kx - \omega t) = A\cos\left(\frac{\omega}{a_0}x - \omega t\right)$. The period of this wave is $\frac{2\pi}{\omega}$. We can also write the wave as $A\cos(k(x - a_0 t))$.

Overall, we have a phase velocity $\frac{\omega}{k} = a_0$ and a group velocity $\frac{\mathrm{d}\omega}{\mathrm{d}k}$ which in this case is also a_0. The fact that the phase and group velocities are the same means these are "non-dispersive" waves, meaning all frequencies travel at the same speed. Regardless of λ, we can say $v = a_0$ due to the fluid equations. We can have any λ as long as $\lambda \ll H$ and $\lambda \gg \lambda_{mfp}$.

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Lecture 16: SooOOOOUUUuund physics, viscosity

Lecturer: Ruth Murray-Clay *31 October* *Aditya Sengupta*

The speed of sound sets a limit on how fast information can travel through the medium. What if you try? We can certainly have things that move at $v > c_s$ through the atmosphere. We know that at $v < c_s$, the source emits a sound wave travelling at the speed of sound, but we run into a problem with sending a wave that's travelling slower than its source. Further, if we want to send sound faster than the thermal velocity, it's hard for the molecules to catch up on the back end of the sound wave.

Physically, we expect that at $v < c_s$, the perturbation will create an overdensity close to where the perturbation happens, which then pushes back on the source and we reach a new equilibrium. What about $v > c_s$? We physically see this with the Parker solar wind interacting with the interstellar medium. For equilibrium to be attained, we need the "ram pressure" (the pressure induced by the solar wind ramming into the ISM, $\sim \rho v_{bulk}^2$) plus the thermal pressure to match the ISM pressure, which happens at about 40 AU, around Neptune. We can estimate the ram pressure using $n\sigma v \sim \rho A v_{bulk}$, which tells us the force is about $\rho A v_{bulk}^2$ and the pressure is about ρv_{bulk}^2.

When we have $v > c_s$, we get a *shock* because the perturbation hits faster than sound can travel. Even though this means the medium can't re-equilibriate itself nicely, we can describe the behaviour on either side of a shock using the mass/momentum equations together with the so-called "jump conditions". To understand this, we'll need to look at viscosity.

Viscosity is opposition to differential flow, i.e. pieces of the fluid moving compared to one another. If we have shear flow that looks something like $x\hat{y}$ (i.e. the y-direction velocity varies in the $\hat{x}$ direction), viscosity wants to even this out so that it doesn't. This is caused by intermolecular forces, but for gases, forces like the Van der Waals force and adhesion that would be important for liquids aren't important. So what does cause gaseous viscosity?

In gases, collisions are what hold the gas together. An individual molecule has a component of velocity along the bulk as well as a random component $v_{th} > v_{bulk}$. These collisions damp out the shear force, but why? This is caused by more average thermalized motion to regions with less bulk velocity, and vice versa. Based on this, we should expect that as temperature goes up, what happens? And as density goes up, what happens?

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Lecture 17: Viscosity

Lecturer: Ruth Murray-Clay *2 November* *Aditya Sengupta*

Last time, we saw that viscosity was caused by collisions within a fluid that try to equilibriate shear flows. Today, we'll look at what impact that has on an individual fluid elements, and find that it is a diffusion process governed by the mathematics of random walks.

Consider a flow $\vec{v} = v_y(x)\hat{y}$. Pick a point x and look at the interval $\left(x - \frac{\lambda_{mfp}}{2}, x + \frac{\lambda_{mfp}}{2}\right)$. If we consider a fluid element of size greater than the MFP but less than the length scale, the flow pushes it in the $\hat{y}$ direction. Collisions in the $\hat{x}$ direction are causing motion in the $\hat{y}$ direction. This is different from the pressure force.

Let our fluid element be a cuboid that has a face at x and extends to a length $\Delta x \gg \lambda_{mfp}$, and let it have a surface area $A \perp \hat{x}$. To see what happens to the fluid, we'll need to consider what happens to each face due to the shear force and add them up.

Consider the left face. We want to figure out the momentum exchange due to collisions within a mean free path of the surface. To do this, we'll need to find the number of particles per time passing through the face ($n\sigma v = nAv_{th}$; we don't do any vector summing of velocities because it's all order of magnitude), and the y-momentum transferred by each one is $mv_y\left(x + \frac{\lambda_{mfp}}{2}\right)$ going right to left, and $mv_y\left(x - \frac{v_y}{2}\right)$ going left to right. Since $\frac{\lambda_{mfp}}{2}$ is small with respect to Δx, we can Taylor expand it:

$$mv_y\left(x + \frac{\lambda_{mfp}}{2}\right) = mv_y(x) + m\frac{\mathrm{d}v_y}{\mathrm{d}x}\frac{\lambda_{mfp}}{2} + \mathcal{O}(\lambda_{mfp}^2). \tag{17.1}$$

The difference is

$$\Delta p = m\frac{\mathrm{d}v_y}{\mathrm{d}x}\lambda_{mfp}. \tag{17.2}$$

Threrefore the momentum transfer, i.e. the force on one face of the surface of the fluid element, is $nAv_{th}m\frac{\mathrm{d}v_y}{\mathrm{d}x}\lambda_{mfp}$. Since $\rho = nm$, we can write this as

$$\dot{p}(x) = \rho A v_{th}\lambda_{mfp}\frac{\mathrm{d}v_y}{\mathrm{d}x} \tag{17.3}$$

Therefore, the total force on the fluid element is $F_y = \dot{p}(x + \Delta x) - \dot{p}(x) \approx \frac{\mathrm{d}\dot{p}}{\mathrm{d}x}\Delta x$. We're allowed to do this because $\Delta x \ll L = \frac{v_y}{\left|\frac{\mathrm{d}v_y}{\mathrm{d}x}\right|}$.

$$F_y = \Delta x\frac{\partial}{\partial x}(\dot{p}) = \Delta x\frac{\partial}{\partial x}\left(\rho A v_{th}\lambda_{mfp}\frac{\mathrm{d}v_y}{\mathrm{d}x}\right). \tag{17.4}$$

Therefore, the viscous force per volume is

$$\frac{F_y}{A\Delta x} = \frac{\partial}{\partial x}\left[\underbrace{\rho v_{th}\lambda_{mfp}}_{\mu}\frac{\mathrm{d}v_y}{\mathrm{d}x}\right] \tag{17.5}$$

where μ is called the "dynamic viscosity" and $\nu = \frac{\mu}{\rho} = v_{th}\lambda_{mfp}$ is called the "kinematic viscosity". Depending on the situation, this might or might not be a constant. What if it is?

The units of ν are $[\nu] = \frac{\mathrm{cm}^2}{\mathrm{s}}$, which looks like a diffusion coefficient. If ρ and ν are constant, we can write this as

$$\frac{\partial v_y}{\partial t} = \nu\frac{\partial^2 v_y}{\partial x^2}. \tag{17.6}$$

This is a diffusion equation. Recall that we were talking about the wave equation $\partial_t^2 f = c\partial_x^2 f$, and we could order-of-magnitude it to find that the constant represented the wave's velocity squared. Similarly for the diffusion equation, we get that $\frac{f}{T} = D\frac{f}{L^2} \implies D = \frac{L^2}{T} = LV = V^2 T$.

This is where we come back to random walks. Collisions are causing momentum to random walk, and viscosity is a process by which this is made to happen.

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Lecture 18: Viscosity length scales

Lecturer: Ruth Murray-Clay *4 November* *Aditya Sengupta*

Last time, we calculated that our viscous force per volume was

$$f_y = \frac{\partial}{\partial x}\left(\rho\nu\frac{\partial v_y}{\partial x}\right). \tag{18.1}$$

In the special case that ρ, ν were constant, we got

$$\frac{\partial v_y}{\partial t} = \nu\frac{\partial^2 v_y}{\partial x^2} \tag{18.2}$$

To order of magnitude, this gave us ν in terms of the length and velocity scales of the system.

We can define $\sigma_{yx} = \rho\nu\frac{\partial v_y}{\partial x}$. σ is called the "viscous stress tensor". Here, $\frac{\partial v_y}{\partial x}$ describes the 'shear' or 'rate of strain'.

To order of magnitude, we have

$$f_y \sim \frac{\rho\nu v_y}{L^2} \tag{18.3}$$

where L is a system length scale that we define to be $\frac{|v_y|}{\left|\frac{\mathrm{d}v_y}{\mathrm{d}x}\right|}$.

Then $\rho\frac{\partial v_y}{\partial t} \sim \frac{\rho\nu v_y}{L^2}$ gives us

$$\lambda_{mfp} v_{th} = LV \tag{18.4}$$

This tells us that our small-scale collision parameters are related to the macroscopic length and velocity scales at which the overall momentum is changing.

But this isn't an exact analogy! We can say $V = \frac{L}{T}$ but that isn't equivalent to $v_{th} = \frac{\lambda_{mfp}}{t_{col}}$, because that would require that every collision moved the particle in the same direction. In reality, this is a diffusive process, so

$$V = v_{th}\frac{\lambda_{mfp}}{L} < v_{th}. \tag{18.5}$$

Specifically, these are related by $L = \sqrt{N}\lambda_{mfp}$ where N is the number of collisions. What's N in terms of stuff in our problem? We know the total time and the average time between collisions, so we can say $N = \frac{T}{t_{col}}$. This tells us

$$L \sim \left(\frac{T}{t_{col}}\right)^{1/2} \lambda_{mfp} \tag{18.6}$$

or

$$\frac{L^2}{T} \sim \frac{\lambda_{mfp}^2}{t_{col}} \tag{18.7}$$

Let's put this back into the fluid equations. Consider the special case where ρ, ν are constant:

$$\rho \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = -\boldsymbol{\nabla} P + \rho \vec{F} + \rho \nu \nabla^2 \vec{v} \tag{18.8}$$

where we've now added in a viscous force term. Note that ∇^2 here refers to $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \vec{v})$. We distribute out the Lagrangian derivative to get

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} v_i = -\boldsymbol{\nabla} P + \rho \vec{F} + \rho \nu \nabla^2 \vec{v}. \tag{18.9}$$

When does viscosity matter? We can order-of-magnitude the viscous force ($\sim \frac{\rho \nu v}{L^2}$) and the inertial force $\rho \vec{v} \cdot \vec{\nabla} v_i \sim \frac{\rho v^2}{L}$, and ask which one is bigger. We can do this using the *Reynolds number*:

$$\mathrm{Re} = \frac{\text{inertial}}{\text{viscous}} = \frac{\rho v^2/L}{\rho \nu v/L^2} = \frac{vL}{\nu}. \tag{18.10}$$

We can also relate this to microscale parameters by saying $\mathrm{Re} = \frac{vL}{v_{th} \lambda_{mfp}}$.

If we have $\mathrm{Re} \gg 1$, then inertia dominates and we can drop viscosity from the equation.

We can write the Reynolds number as a ratio of timescales: $T_{\text{diff}} = \frac{L^2}{\nu}$ and $T_{\text{adv}} = \frac{L}{v}$, then $\mathrm{Re} = \frac{T_{diff}}{T_{adv}}$. Advection refers to motion due to bulk velocity.

In astrophysical fluids, we only need viscosity in specific situations. It usually refers to boundary layers like pipes on Earth, but in our case it might be layers between solid bodies and gases. The other situation in which it usually appears is if you would otherwise not be moving, so long-timescale changes can add up. The next example in class is accretion disks.

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Lecture 19: Accretion

Lecturer: Ruth Murray-Clay *7 November* *Aditya Sengupta*

When we were dealing with Bondi accretion, we neglected angular momentum. We had an angular momentum vector $\vec{\Omega}$ that was perpendicular to $\hat{z}$, so infall in the $x - y$ plane is eventually stalled by centrifugal force. To neglect rotation, we need the centrifugal force to be less than the gravitational force at all distances R during the collapse. If we have so little angular momentum that this balance will never be achieved, then Bondi accretion holds. But if at any point during the infall, we have centrifugal force greater than gravity, accretion is stalled.

For example, consider gas accreting onto a star. At some particular R, we have $v_\theta = \Omega R$, where $\Omega = \dot{\theta}$. In a time $\mathrm{d}t$, we go a distance $R\mathrm{d}\theta$, so $v_\theta = R\frac{\mathrm{d}\theta}{\mathrm{d}t} = R\Omega$. This means our centrifugal force per mass will be

$$F_{cent} = \frac{v_\theta^2}{R} = R\Omega^2 \tag{19.1}$$

which we balance with

$$F_{grav} = \frac{GM}{R^2}. \tag{19.2}$$

Let's try and find the R at which there is the most centrifugal support.

$$\frac{F_{cent}}{F_{grav}} = \frac{R\Omega^2}{GM/R^2} = \frac{R^3\Omega^2}{GM}. \tag{19.3}$$

Recall that Kepler's third law tells us that $P^2 = a^3$, and $P = \frac{2\pi}{\Omega_k}$, so we can rewrite it as $R^3\Omega_k^2 = GM = \text{const}$. Therefore, we know $F_{cent} < F_{grav}$ iff $\Omega < \Omega_k$.

How does Ω change as R changes during infall? We can solve this using conservation of angular momentum. We know that angular momentum per mass is $\vec{r} \times \vec{v} = Rv_\theta$ (as we have a circular orbit) which is $R^2\Omega$. We know this is a constant, so as R goes down, Ω goes up, but so does Ω_k.

$$\Omega = \frac{L}{R^2}; \Omega_k = \frac{GM}{R^3}. \tag{19.4}$$

Therefore

$$\frac{F_{cent}}{F_{grav}} = \frac{\Omega^2}{\Omega_k^2} = \frac{(L/R^2)^2}{(GM/R^3)^2} = \frac{L^2}{GM}\frac{1}{R}. \tag{19.5}$$

As R goes down, this ratio goes up and centrifugal support becomes more important. The smaller you collapse, the more likely you are to have centrifugal support. The criterion for whether you get a disk is at the surface of our star. To avoid creating a disk, we need $\Omega(R_\star) < \Omega_k(R_\star)$.

Let's turn this into something we can evaluate at the outer boundary. Usually our initial conditions operate before the collapse starts, so we want something in those terms. We have $R_{out}^2\Omega_{out} = R_\star^2\Omega_\star$, and we have Kepler's third law which says $\Omega_k(R_\star)^2R_\star^3 = \Omega_k(R_{out})^2R_{out}^3$.

Substituting everything to be in terms of R_{out}, we have

$$\Omega(R_{out})\left(\frac{R_{out}}{R_\star}\right)^2 < \Omega_k(R_{out})\left(\frac{R_{out}}{R_\star}\right)^{3/2} \tag{19.6}$$

$$\left.\frac{\Omega}{\Omega_k}\right|_{R_{out}} = sqrt\frac{R_\star}{R_{out}}. \tag{19.7}$$

Notice that $\left.\frac{v_\theta}{v_k}\right|_{R_{out}} = \left.\frac{\Omega}{\Omega_k}\right|_{R_{out}}$, so it's not enough for v_θ to exceed the Keplerian orbit; it has to exceed it by a factor of the radius ratio above.

A convenient R_{out} to consider is the Bondi radius $R_B = \frac{GM}{c_{s,\infty}^2} = \frac{GM\mu}{kT_\infty}$. Let's take the example of a black hole, where $R_\star = R_{sch} = \frac{2GM}{c^2}$. To avoid an accretion disk, we need

$$\frac{\Omega(R_{out})}{\Omega_k(R_{out})} < \left(\frac{R_{sch}}{R_{out}}\right)^2 = \frac{c_{s,\infty}}{c}. \tag{19.8}$$

So $\frac{R_{sch}}{R_B} \sim \frac{c_{s,\infty}^2}{c^2}$. This is a really small number, so it's hard to believe we'd ever get this. This comes out in most black holes actually having accretion disks.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 20: Accretion disk physics

Lecturer: Ruth Murray-Clay *9 November* *Aditya Sengupta*

Let's basically redo a homework problem. Consider a point a height z above and out at a radius r in the plane of an accretion disk. We want to balance pressure with gravity, $F_{grav} + F_{pressure} = 0$. The gravitational force is $F_g = \frac{GM}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}} \approx \frac{GM}{r^2}\frac{z}{r}$. The pressure force is $F_{pressure} = \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z}$. This gives us something we can OOM derive:

$$\frac{GM}{r^2}\frac{z}{r} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = 0 \tag{20.1}$$

$$\frac{\mathrm{d}P}{\mathrm{d}z} \simeq \frac{P}{H} \tag{20.2}$$

We might be tempted to say this means $\frac{GMz}{r^3} = \frac{1}{\rho}\frac{P}{H}$, but this is a mistake: the left-hand side is a general function of position, whereas the right-hand side is at a particular value. We can only make sense of this at $z = H$.

We did the non OOM version of this for homework and found that $P(z) \propto e^{-z^2/H^2}$. The OOM version isn't quite compatible with this exact solution, because if we plugged in $z = 0$, we'd get $\frac{\mathrm{d}P}{\mathrm{d}z} = 0$ (a Gaussian is flat at its peak). So we need to pick some characteristic point away from the peak to get any physics out.

Solving for H and substituting in $\Omega_k^2 = \frac{GM}{r^3}$, we get

$$\Omega_k^2 H^2 = c_s^2 \implies H = \frac{c_s}{\Omega_k}. \tag{20.3}$$

Note that when we're calculating the scale height, we don't care about how fast the disk is spinning (although usually it's about Ω_k anyway).

If $c_s \sim v_k$, we get that $H \sim r$; this is fat and not very disk-like. So we need $\frac{H}{r} \ll 1$ for a 'cold' situation or a 'thin disk'.

Next, let's apply mass conservation.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \tag{20.4}$$

Since we're in steady state, we can drop the time derivative term. Further, we'll assume azimuthal symmetry, so $\frac{\partial}{\partial \theta} = 0$ and $v_z = 0$.

$$\nabla \cdot (\rho \vec{v}) = 0 \implies \rho \nabla \cdot \vec{v} + \vec{v} \cdot \nabla \rho = 0. \tag{20.5}$$

Replace $\nabla \rho$ with $\frac{\partial \rho}{\partial r}$ and $\nabla \cdot \vec{v}$ with $\frac{1}{r}\frac{\partial}{\partial r}(r v_r)$ to get

$$\frac{\rho}{r}\frac{\partial}{\partial r}(rv_r) + v_r\frac{\partial \rho}{\partial r} = 0 \tag{20.6}$$

or, recombining with the product rule (why did I do this expansion), we get $r\rho v_r = \text{const}$. This is the 2D version of $\dot{M} = \text{const}$.

We can show this by relating this to the mass outflow using

$$\dot{M} = \int_{-H}^{+H} \rho v_r 2\pi r \mathrm{d}z = -2\pi r v_r \underbrace{\int_{-\infty}^{\infty} \rho \mathrm{d}z}_{\Sigma} \tag{20.7}$$

i.e. $\dot{M}$ is constant.

For radial momentum, our equation is

$$\rho\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = \text{forces}. \tag{20.8}$$

In the radial component, the Lagrangian derivative comes out to

$$\rho\frac{\partial v_r}{\partial t} + \rho\vec{v}\cdot\boldsymbol{\nabla} v_r = -\frac{GM}{r^2} - \frac{1}{r}\frac{\mathrm{d}P}{\mathrm{d}r} + \Omega^2 r. \tag{20.9}$$

We're in steady state, so we drop the time derivative. For the $\vec{v}\cdot\boldsymbol{\nabla} v_r$ term, we use $v_z = 0$ and $\frac{\partial}{\partial\theta} = 0$ to drop all terms except the radial one.

$$v_r\frac{\partial}{\partial r}v_r = -\frac{GM}{r^2} - \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \Omega^2 r. \tag{20.10}$$

Let's OOM these terms! We can say $v_r\frac{\partial v_r}{\partial r} \simeq \frac{v_r^2}{r}$; we're using r as the scale length in this case, and in so doing we're assuming that v_r is a power-law in r. Accretion disks generally are power-laws in the interior that exponentially fall off later, so this expression will only be valid in the power-law part of the disk.

$$\frac{v_r^2}{r} \sim \frac{c_s^2}{r} - \frac{v_k^2}{r} + \frac{v_\theta^2}{r}. \tag{20.11}$$

We assume no significant radial outflow, $v_r \sim 0$. We know that because we've got a thin disk, $c_s < v_k$, so we can drop that too. Therefore, $v_\theta \approx v_k$.

We still don't understand the key thing: how quickly do you accrete? This comes from conserving angular momentum, i.e., the θ component of the momentum equation.

$$\rho\vec{v}\cdot\nabla v_\theta = -\nabla P + \rho\vec{F}_g + \Omega^2 r\hat{r} + \vec{F}_\theta. \tag{20.12}$$

We wouldn't get any change if we only considered the first three forces, as they both act in the $\hat{r}$ direction and we want to induce change in the $\hat{\theta}$ direction. We need something else: this is a *shearing force* or viscosity. We know that $v_\theta \sim v_k$, but v_k varies with r (specifically, $v_k \propto r^{-1/2}$) so there's a differential v_θ as we go out. This gives us an azimuthal force that transports angular momentum.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 21:

Lecturer: Ruth Murray-Clay *14 November* *Aditya Sengupta*

Last time, we found that momentum in the $\hat{\theta}$ direction created an azimuthal velocity shear in the radial direction that transported angular momentum. For no shearing, we need Ω not to change with r, because this would effectively give us solid body rotation: no material is 'falling behind' or 'moving ahead' any other.

Working in cylindrical coordinates, we can write out the $\hat{\theta}$ component in terms of the following:

- The 'material derivative' $(A \cdot \boldsymbol{\nabla})\vec{B}$ is

$$A_r \frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial \theta} + A_z \frac{\partial B_\theta}{\partial z} + \frac{A_\theta B_r}{r}. \tag{21.1}$$

- The divergence of a tensor $\boldsymbol{\nabla} \cdot \bar{\bar{T}}$ is

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{1}{r}(T_{\theta r} + T_{r\theta}). \tag{21.2}$$

Assuming axisymmetry we can take $\frac{\partial}{\partial \theta} = 0$, and considering the disk midplane we can take $\frac{\partial}{\partial z} = 0$. This leaves us with a simpler equation:

$$\rho(\vec{v} \cdot \boldsymbol{\nabla})\vec{v} = \rho\left[v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r}\right] = \rho v_r \left[\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\right]. \tag{21.3}$$

Using $v_\theta = \Omega r$, we recombine these two terms using the chain rule to get

$$\rho(\vec{v} \cdot \boldsymbol{\nabla})\vec{v} = \frac{\rho v_r}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r v_\theta) = \rho \frac{v_r}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\underbrace{r^2\Omega}_{\text{angular momentum}}\right) \tag{21.4}$$

On the right-hand side, the velocity in the $\hat{\theta}$ direction is changing as a function of r, so we want the radial derivative of the azimuthal velocity. The viscous stress tensor in cylindrical coordinates is

$$\boldsymbol{\nabla} \cdot \bar{\bar{\sigma}} = \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \sigma_{r\theta}\right). \tag{21.5}$$

Here, $\sigma_{r\theta} = \rho \nu r \frac{\mathrm{d}\Omega}{\mathrm{d}r}$ where $\Omega(r)$ is from Kepler's third law.

Therefore our equation is

$$\rho \frac{v_r}{r} \frac{\mathrm{d}}{\mathrm{d}r}\big(r^2\Omega\big) = \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r}\big(r^2\sigma_{r\theta}\big). \tag{21.6}$$

If we have no viscosity, then $\sigma_{r\theta} = 0$ and we just have a constant angular momentum. Viscosity is what drives changes in angular momentum.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 22: Accretion through diffusion, thermal conductivity

Lecturer: Ruth Murray-Clay *16 November* *Aditya Sengupta*

22.1 Accretion through diffusion

Last time, we integrated the $\hat{\theta}$ component of the momentum equation

$$\dot{M}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\Omega\right) = -\frac{\mathrm{d}}{\mathrm{d}r}\left(2\pi\Sigma\nu r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right) \tag{22.1}$$

to get

$$\dot{M}\left[1-\left(\frac{R_\star}{r}\right)^{1/2}\right] = 3\pi\Sigma\nu \tag{22.2}$$

which for $r \gg R_\star$ became $M = 3\pi\Sigma\nu$.

We can order-of-magnitude the original equation (for what purpose?) To do this, we'll need to set a length scale for the $\frac{\mathrm{d}}{\mathrm{d}r}$ derivatives. We can get this length scale from estimating $r^2\Omega$ using Kepler's third law, and we'll find that $r^3\Omega^2 = GM \implies r^2\Omega \propto r^{2-2/3}$. So we have a power-law in r, which is scale-free, so we can just replace $\frac{\mathrm{d}}{\mathrm{d}r} \mapsto \frac{1}{r}$. On the right-hand side, we can do something similar (using r as the scale length for the $\frac{\mathrm{d}\Omega}{\mathrm{d}r}$, but being agnostic for now about the length scale for the outer derivative):

$$rv_r\frac{\Sigma r^2\Omega}{r} = \frac{\Sigma\nu r^2\Omega}{L} \tag{22.3}$$

We find that we need $L \ll r$ for these two terms to be competitive, which physically corresponds to a Reynolds number much greater than 1: we need significant viscosity for accretion to take place.

Note that we've gotten two estimates of $\dot{M}$, we can compare them:

$$\begin{aligned}\dot{M} \approx 3\pi\Sigma\nu &= -2\pi r v_r \\ v_r &= -\frac{3}{2}\frac{\nu}{r}\end{aligned}$$

and so the accretion timescale is

$$\left|\frac{r}{v_r}\right| \sim \frac{r}{\nu/r} \sim \frac{r^2}{\nu}. \tag{22.4}$$

This tells us physically what we would expect: a stronger viscous force (larger ν) gives us a shorter accretion timescale. Further, this is a diffusion coefficient, and we know we can relate large- and small-scale motion:

$$\frac{\lambda_{mfp}^2}{t_{coll}} = \frac{r^2}{t_{accr}}. \tag{22.5}$$

Accretion is driven by diffusion of angular momentum, and the accretion time is the time it takes to diffuse particles over the length of the disk.

An alpha disk is something we treat as evolving due to viscosity but with an efficiency parameter $0 < \alpha \leq 1$ modifying ν.

We often refer to this process as 'diffusion of angular momentum', which is really 'diffusion of stuff that carries angular momentum'.

It's also worth discussing time-dependent transport. If we reran the derivation above without cancelling $\frac{\partial}{\partial t}$ terms, we'd get

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(\Sigma \nu r^{1/2}\right)\right]. \tag{22.6}$$

This is a diffusion equation that 'flattens out' strongly peaked $\Sigma(r)$ profiles and will eventually reach a steady-state solution.

It's worth talking qualitatively about what happens to angular momentum here. Viscosity in the bulk part of the accretion disk wants to get rid of shear (higher velocity closer in, lower farther out) so that drives a decrease in velocity nearer the center and an increase farther away from it. This reduces the angular momentum per particle further in, so mass will start to accrete inwards and a little bit of mass carries the angular momentum out.

22.2 Thermal conductivity

Thermal conductivity is the transport of heat due to the collision of particles.The transport of thermal energy is determined by the microscopic kinetic energy, $\frac{1}{2}mv_{th}^2$. We know this should be a diffusive process, so there's some diffusion coefficient $D \sim \nu$ and a timescale $t \sim \frac{L^2}{\nu}$.

We can describe conduction as energy transport being proportional to a temperature gradient, where thermal conductivity is the proportionality constant. Let $\vec{F}_c$ be the energy per area per time and let κ be the thermal conductivity. Then

$$\vec{F}_c = -\kappa \boldsymbol{\nabla} T. \tag{22.7}$$

We also have energy continuity,

$$\frac{\mathrm{d}\epsilon\rho}{\mathrm{d}t} = -\boldsymbol{\nabla} \cdot \vec{F}_c. \tag{22.8}$$

For gases, we usually have $\kappa \sim 0.01 - 0.03$ watts/meter/Kelvin, except for hydrogen it's 0.18 and helium it's 0.15. Let's try and figure out the coefficient for hydrogen.

$$\vec{F}_c = -\kappa \boldsymbol{\nabla} T \sim -\frac{\kappa T}{L}. \tag{22.9}$$

Let's estimate $\vec{F}_c$. Consider a surface of area A through which diffusing particles are passing. Per particle, $\frac{1}{2}\mu v_{th}^2 \sim kT$ is the amount of energy being transported, so we need to multiply this by a particle rate nAv_{th}. Therefore, the energy per area is about $kTnv_{th}$, so we can say

$$\frac{\kappa T}{L} \sim kTnv \implies \kappa \sim kn \underbrace{vL}_{D\sim vL\sim v_{th}\lambda_{mfp}}. \tag{22.10}$$

Therefore

$$\kappa \sim \frac{knv_{th}}{\sigma_{coll}} = \frac{kv_{th}}{\sigma_{coll}}. \tag{22.11}$$

If our diffusive picture is correct, this should be (to OOM) the value we said above, 0.18 watts/meter/Kelvin.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 23: Two-body problem

Lecturer: Ruth Murray-Clay *18 November* *Aditya Sengupta*

Previously, we were computing the thermal conduction coefficient. We can OOM it to be equal to $\kappa = knD$ where $D \sim \frac{L^2}{t_{diff}} \sim \frac{\lambda_{mfp}^2}{t_{coll}}$, or equivalently $\lambda_{mfp} v_{th}$.

To get the actual number for hydrogen, we can plug in $v_{th} \sim 1$ km/s ($\sqrt{kT/\mu}$ for $T = 300$K) and $\sigma_{coll} \sim$ Bohr radius^2. These get you the correct number to order of magnitude.

Finally, let's recap what we predicted about viscosity and see what actually happens. As ν goes up, T can either go up or down and ρ can either go up or down, and there are possibly compelling arguments for both. But with the connection to microscopic scales, we can answer this, but I don't knoww the answer because I got a call about my visa and I couldn't pay attention.

Now we can do gravitational dynamics!

Consider masses m_1, m_2 with displacement vectors $\vec{r}_1, \vec{r}_2$ and $\vec{r} = \vec{r}_2 - \vec{r}_1$. The forces acting on both are $\vec{F}_{1,2} = m_{1,2}\ddot{\vec{r}}_{1,2}$. Newton's third law tells us $\vec{F}_1 + \vec{F}_2 = 0$. If the center of mass has a displacement vector $\vec{R}$, then by definition $\vec{R}$ has to satisfy

$$m_1(\vec{r}_1 - \vec{R}) = -m_2(\vec{r}_2 - \vec{R}). \tag{23.1}$$

This gives us

$$(m_1 + m_2)\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2. \tag{23.2}$$

Take two time derivatives:

$$(m_1 + m_2)\ddot{\vec{R}} = m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = \vec{F}_1 + \vec{F}_2 = 0. \tag{23.3}$$

Therefore $\ddot{\vec{R}} = 0$, so $\vec{R} = \vec{A} + \vec{B}t$ where $\vec{A}, \vec{B}$ are constants. So the frame centered on the center of mass is also inertial.

Therefore, we can choose $\vec{R} = 0$ without loss of generality, which is called choosing the *barycentric frame*. In the barycentric frame, we have $m_1\vec{r}_1 = -m_2\vec{r}_2 = -m_2(\vec{r} + \vec{r}_1)$. By doing algebra on this, we can get

$$\vec{r}_1 = -\frac{m_2}{m_1 + m_2}\vec{r}; \vec{r}_2 = \frac{m_1}{m_1 + m_2}\vec{r}. \tag{23.4}$$

Now let's bring in gravity:

$$\vec{F}_1 = \frac{Gm_1m_2}{r^2}\hat{r}; \vec{F}_2 = -\frac{Gm_1m_2}{r^2}\hat{r}. \tag{23.5}$$

Therefore

$$\ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = \frac{\vec{F}_2}{m_2} - \frac{\vec{F}_1}{m_1} \tag{23.6}$$

$$= -\frac{Gm_1}{r}\hat{r} - \frac{Gm_2}{r}\hat{r} = -\frac{G(m_1+m_2)}{r}\hat{r}. \tag{23.7}$$

Note that some sources use $\mu = G(m_1+m_2)$, which is *not* the reduced mass! In this class we're going to refer to the reduced mass as $\tilde{\mu} = \frac{m_1m_2}{m_1+m_2}$.

Our equation of motion in terms of this fake μ is $\ddot{\vec{r}} + \frac{\mu}{r^2} = 0$.

...

We can interpret $\vec{h}$ as an angular momentum per some mass, which turns out to be the reduced mass. In the barycentric frame, we have angular momentum

$$L = \vec{r}_1 \times m_1\dot{\vec{r}}_1 + \vec{r}_2 \times m_2\dot{\vec{r}}_2 \tag{23.8}$$

which in terms of the barycentric frame is

$$\vec{L} = m_1\left(-\frac{\tilde{\mu}}{m_1}\vec{r}\right) \times \left(-\frac{\tilde{\mu}}{m_1}\dot{\vec{r}}\right) + m_2\left(-\frac{\tilde{\mu}}{m_2}\vec{r}\right) \times \left(-\frac{\tilde{\mu}}{m_2}\dot{\vec{r}}\right) \tag{23.9}$$

$$= \left(\frac{\tilde{\mu}^2}{m_1} + \frac{\tilde{\mu}^2}{m_2}\right)\left(\vec{r} \times \dot{\vec{r}}\right) = \tilde{\mu}\vec{r} \times \dot{\vec{r}}. \tag{23.10}$$

Therefore $\vec{L} = \tilde{\mu}\vec{h}$, so $\vec{h}$ is angular momentum per reduced mass.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 24: Solving the two-body problem

Lecturer: Ruth Murray-Clay *21 November* *Aditya Sengupta*

Let's compute $\dot{\vec{r}}\cdot$ the equation of motion:

$$\dot{\vec{r}} \cdot \left[\ddot{\vec{r}} + \frac{\mu}{r}\hat{r}\right] = \dot{\vec{r}} \cdot 0 = 0 \tag{24.1}$$

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} + \frac{\mu}{r^3}\dot{\vec{r}} \cdot \vec{r} = 0. \tag{24.2}$$

We want to integrate this. We do this by taking $\frac{\mathrm{d}}{\mathrm{d}t}v^2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}v^2 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vec{r}} \cdot \dot{\vec{r}}\right) = 2\dot{\vec{r}} \cdot \ddot{\vec{r}} \tag{24.3}$$

and similarly

$$\frac{\mathrm{d}}{\mathrm{d}t}r^2 = 2r\frac{\mathrm{d}r}{\mathrm{d}t} \implies \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\vec{r} \cdot \vec{r}}{r} \tag{24.4}$$

so we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\vec{r} \cdot \dot{\vec{r}}}{r^3}. \tag{24.5}$$

The integral of 24.2 is therefore $\frac{1}{2}v^2 - \frac{\mu}{r} = C$ where C is a constant of integration. This is reminiscent of a kinetic plus a potential energy, so let's look at what the actual total energy is and see if we're close.

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{Gm_1 m_2}{r} \tag{24.6}$$

in the barycentric frame. We can relate v_1, v_2 to the $\vec{r}$s to get

$$v_{1,2}^2 = \frac{\tilde{\mu}^2}{m_{1,2}^2}v^2 \tag{24.7}$$

i.e.

$$E = \frac{1}{2}\left(\frac{\tilde{\mu}^2}{m_1} + \frac{\tilde{\mu}^2}{m_2}\right)v^2 - \frac{G(m_1 + m_2)}{r}\tilde{\mu} \tag{24.8}$$

$$= \frac{1}{2}\tilde{\mu}v^2 - \frac{\tilde{\mu}\mu}{r} = \tilde{\mu}C. \tag{24.9}$$

Therefore C is the energy per reduced mass.

We have four constants of motion: C and the three components of $\vec{h}$.

Let's work in polar coordinates now:

$$\begin{aligned} \hat{r} &= \cos\theta\hat{x} + \sin\theta\hat{y} \\ \hat{\theta} &= -\sin\theta\hat{x} + \cos\theta\hat{y} \\ \vec{r} &= r\cos\theta\hat{x} + r\sin\theta\hat{y} = r\hat{r}. \end{aligned} \tag{24.10}$$

In Cartesian coordinates, it's uninteresting to take time derivatives of these, because they're independent component by component. In polar coordinates, we get

$$\begin{aligned} \vec{r} &= r\hat{r} \\ \dot{\vec{r}} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\ \ddot{\vec{r}} &= (\ddot{r} - r\dot{\theta}^2)\hat{r} + \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right)\right)\hat{\theta}. \end{aligned} \tag{24.11}$$

The $\hat{r}$ term in $\ddot{\vec{r}}$ looks like the centrifugal force $\frac{v_\theta^2}{r} = \frac{r^2\dot{\theta}^2}{r} = r\dot{\theta}^2$.

Similarly, the inner $r^2\dot{\theta}$ component in the $\hat{\theta}$ term looks like angular momentum $rv_\theta = r(r\dot{\theta}) = r^2\dot{\theta} = h$. If you have a force in the $\hat{\theta}$ direction, you change h. Therefore, we've put acceleration in polar coordinates and that's given us terms that will be impacted by force in ways we can clearly see.

Let's restate the constants of motion in polar coordinates:

$$h = \left|\vec{r} \times \dot{\vec{r}}\right| = r^2\dot{\theta} \tag{24.12}$$

$$C = \frac{1}{2}\dot{\vec{r}} \cdot \dot{\vec{r}} - \frac{\mu}{r} = \frac{1}{2}\left(\dot{r}^2 + \frac{h^2}{r^2}\right) - \frac{\mu}{r}. \tag{24.13}$$

Therefore

$$\begin{aligned} \dot{r} &= \pm\left[2\left(C + \frac{\mu}{r} - \frac{h^2}{r^2}\right)\right]^{1/2} \\ \dot{\theta} &= \frac{h}{r^2}. \end{aligned} \tag{24.14}$$

This is our solution to the 2-body problem: we take an initial condition and values for these constants and numerically integrate. But can we get any qualitative information about the shape of our orbits without numerical integration? We want $r(\theta)$.

Coming back to the EOM and putting it in polar coordinates, we get

$$\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + \frac{1}{r}\frac{\mathrm{d}h}{\mathrm{d}t}\hat{\theta}. \tag{24.15}$$

For this to hold component by component, we need

$$\begin{aligned} \hat{r} &: -\frac{\mu}{r^2} = \ddot{r} - r\dot{\theta}^2 \\ \hat{\theta} &: \frac{\mathrm{d}h}{\mathrm{d}t} = 0. \end{aligned} \tag{24.16}$$

So we see h is conserved. Just looking at the $\hat{r}$ equation, we want to remove the time dependence. On a circular orbit, we get $\ddot{r} = 0$ and this is just force balance between centrifugal and gravitational force. Otherwise, substitute $u = \frac{1}{r}$ and $\mathrm{d}u = -\frac{1}{r^2}\mathrm{d}r$.

$$\dot{r} = \frac{\mathrm{d}}{\mathrm{d}t}[r(\theta)] = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = -r^2\frac{\mathrm{d}u}{\mathrm{d}\theta}\dot{\theta} = -h\frac{\mathrm{d}u}{\mathrm{d}\theta}. \tag{24.17}$$

Therefore

$$\ddot{r} = -h\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\dot{\theta} = -h\frac{r^2\dot{\theta}}{r^2}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}. \tag{24.18}$$

And we know $\ddot{r}$ is also equal to the other side of the EOM!

$$-\mu u^2 = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - u^3h^2. \tag{24.19}$$

Dividing throughout by $-h^2u^2$, we get

$$\frac{\mu}{h^2} = \frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u. \tag{24.20}$$

This is a second-order differential equation, so we need two solutions and two integration constants.

For convenience, let $P = \frac{h^2}{\mu}$. Let $w = u - P^{-1}$. Then our differential equation is

$$w = -\frac{\mathrm{d}^2w}{\mathrm{d}\theta^2}. \tag{24.21}$$

The solutions to this are sines and cosines. Let's do the version where we have a phase shift, and suggestively name the constants:

$$w = eP^{-1}\cos(\theta - \varpi) \tag{24.22}$$
$$\frac{\mathrm{d}w}{\mathrm{d}\theta} = -eP^{-1}\sin(\theta - \varpi) \tag{24.23}$$
$$\frac{\mathrm{d}^2 w}{\mathrm{d}\theta^2} = -eP^{-1}\cos(\theta - \varpi) = -w \tag{24.24}$$

which is what we want! Let's take this back to u:

$$u - P^{-1} = eP^{-1}\cos(\theta - \varpi) \tag{24.25}$$
$$u = P^{-1}[1 + e\cos(\theta - \varpi)] \tag{24.26}$$
$$r(\theta) = \frac{P}{1 + e\cos(\theta - \varpi)}. \tag{24.27}$$

This is a generic conic section where P is called the "semi-latus rectum". We sometimes use $f \equiv \theta - \varpi$ and write this as $r = \frac{P}{1+e\cos f}$. Based on e, we have different kinds of shapes. Define a and/or q such that

$$\begin{cases} e = 0 & \text{circle}, P = a \\ 0 < e < 1 & \text{ellipse}, P = a(1 - e^2) \\ e = 1 & \text{parabola}, P = 2q \\ e > 1 & \text{hyperbola}, P = a(e^2 - 1). \end{cases} \tag{24.28}$$

The circle and ellipse are bound orbits, the parabola is a marginally unbound orbit, and the hyperbola is an unbound orbit. Let's look at the ellipse, as we're interested in bound orbits, and the circle is a special case of that. Next time, we're going to find there's one degree of freedom after we've specified E, e, h, and that's where you are on the orbit.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 25: Kepler's laws

Lecturer: Ruth Murray-Clay *23 November* *Aditya Sengupta*

Last time, we found out that the stable solutions to the two-body problem were ellipses. Note that in terms of the ellipse geometry, the eccentricity e is defined such that the distance between a focus and the center is ae, where a is the semi-major axis.

We derived a solution that gave us the shape of the orbits, but not time dependence. We found that angular momentum was given by h where $P = h^2/\mu = a(1-e^2)$, which implies $h = \sqrt{\mu a(1-e^2)}$ Similarly, we can express energy in terms of physical parameters:

$$C = \frac{1}{2}v^2 - \frac{\mu}{r} = \frac{1}{2}\left(\dot{r}^2 + \frac{h^2}{r^2}\right) - \frac{\mu}{r} \tag{25.1}$$

and plugging in $(r\dot{\theta})^2 = (r\dot{f})^2$, we get

$$\begin{aligned}
C &= \frac{1}{2}\left[\left(\frac{h}{r}\frac{e\sin f}{1+e\cos f}\right)^2 + \frac{h^2}{r^2}\right] - \frac{\mu}{r} && (25.2)\\
&= \frac{1}{2}\frac{h^2}{r^2}\left[\frac{e^2\sin^2 f}{(1+e\cos f)^2} + \frac{(1+e\cos f)^2}{(1+e\cos f)^2}\right] - \frac{\mu}{r} && (25.3)\\
&= \text{algebra} = \frac{1}{2}\frac{\mu}{a(1-e^2)}\left[e^2\sin^2 f + 1 + 2e\cos f + e^2\cos^2 f\right] - \frac{\mu}{r} && (25.4)\\
&= \frac{\mu}{2a(1-e^2)}(1 + e^2 + e\cos f) - \frac{\mu}{r} && (25.5)\\
&= \frac{\mu}{a(1-e^2)}\left[\frac{1}{2}(1 + \cancel{2e\cos f} + e^2) - (1 + \cancel{e\cos f})\right] && (25.6)\\
&= \frac{\mu}{a(1-e^2)}\left[-\frac{1}{2}(1-e^2)\right] = -\frac{\mu}{2a} = -\frac{G(m_1+m_2)}{2a}. && (25.7)
\end{aligned}$$

So we've found that only a determines the energy of a Keplerian orbit!

Let's reduce this to a circular orbit case, where $r = a$. We would have

$$C = -\frac{\mu}{r} + \frac{1}{2}v^2 \tag{25.8}$$

and we can further say that balancing centrifugal and gravitational forces gives us

$$\frac{v^2}{r} = \frac{\mu}{r^2} \tag{25.9}$$

i.e.

$$C = -\frac{\mu}{r} + \frac{1}{2}\frac{\mu}{r} = -\frac{\mu}{2r}. \tag{25.10}$$

This is similar to the virial theorem in statistical self-gravitating systems, which has PE $PE = -2KE$, i.e. $E = PE + KE = \frac{PE}{2} = -KE$.

In general, we have

$$C = -\frac{\mu}{2a} = -\frac{\mu}{r} + \frac{1}{2}v^2 \tag{25.11}$$

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right). \tag{25.12}$$

We can also compute the orbital period using the area swept out:

$$\mathrm{d}A = \int_0^r r'\mathrm{d}r'\mathrm{d}\theta = \frac{1}{2}r^2\mathrm{d}\theta. \tag{25.13}$$

Therefore

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h. \tag{25.14}$$

Therefore orbits sweep equal areas in equal times (Kepler's second law). The total area of the ellipse is therefore $\frac{1}{2}hT$ where T is the orbital period. The area is also πab by ellipse geometry, and here $b^2 = a^2(1-e^2)$. Therefore (I missed the time period calculation, but it gives us)

$$\mu = \Omega^2 a^3 \tag{25.15}$$

which is Kepler's third law! Note that $\Omega = \frac{2\pi}{T}$ is sometimes referred to as n by dynamicists, because Ω is an angle describing an orbit in an arbitrary reference frame.

$$h = \sqrt{\mu a(1-e^2)} = \sqrt{\Omega^2 a^3 a(1-e^2)} = \Omega a^2\sqrt{1-e^2} = av_k\sqrt{1-e^2}. \tag{25.16}$$

This says that if your eccentricity is high, your angular momentum is lower than it would be on a circular orbit.

If we dissipate energy without being able to dissipate angular momentum, we reach a circular orbit. If we have C go down, a also goes down. If h is fixed as a goes down, e decreases because $1 - e^2$ has to increase. e can decrease to 0 and can't go down further, so we have a circular orbit.

We actually can't solve $f(t)$ analytically, and instead we need to numerically solve Kepler's equation.

Astrophysical Fluid Dynamics **Fall 2022**

Lecture 26: Tidal gravity, Jeans mass

Lecturer: Ruth Murray-Clay *28 November* *Aditya Sengupta*

26.1 Tidal gravity

Tides on Earth are caused by the moon's differential gravitational pull on the ocean. The side of the Earth that's closest to the moon is pulled more, so it's higher, and the opposite side to that is pulled less than the bulk Earth and so it's "left behind" by the Earth's rotation, so it creates a symmetric bulge.

The Moon and the Earth feel almost the same gravitational force from the Sun, but since the Moon orbits the Earth, there's a slight difference, and this force tries to pull the Moon out of the Earth's orbit. Let's quantify this. Let the Moon-Earth distance be x and the Sun-Earth distance be r, and let them all be collinear so that the Sun-Moon distance is $r - x$. We'll assume $x \ll r$, which is justified. For the moon to be able to stay in orbit, we need $F_{tidal} < F_{grav} = \frac{GM_\oplus}{x^2}$. The tidal force is just the differential component relative to the Earth-Sun gravitational force:

$$F_{tidal} = \frac{GM_\odot}{(r \pm x)^2} - \frac{GM_\odot}{r^2}. \tag{26.1}$$

Taylor expanding this for $x \ll r$, we get

$$F_{tidal} = \frac{GM_\odot}{r^2\left(1 \pm \frac{x}{r}\right)^2} - \frac{GM_\odot}{r^2} \approx \frac{GM_\odot}{r^2}\left(1 \mp 2\frac{x}{r} - 1\right) \tag{26.2}$$

so our criterion is (dropping the $\pm$ because we can just work with absolute values)

$$2\frac{GM_\odot}{r^3}x < \frac{GM_\odot}{x^2}. \tag{26.3}$$

The 2 should really be a 3, if you do some stuff with centrifugal forces. So we can get

$$x < r\left(\frac{M_\oplus}{3M_\odot}\right)^{1/3} = R_H. \tag{26.4}$$

R_H is known as the Hill radius, and it's also approximately equal to the Roche lobe radius. R_H quantifies the gravitational region of influence of (in this example) Earth.

If we're talking about disruption within a body, then the maass is $m = \frac{4}{3}\pi\rho_{int}x^3$, and if we say the central object has a mass $M = \frac{4}{3}\pi\rho_c R_c^3$, we get

$$r \lesssim R_C \left(\frac{\rho_C}{\rho_{int}} \right)^{1/3}. \tag{26.5}$$

This is the *Roche limit*, which tells us when tides become important for things like Saturn's rings. This also comes up for tidal disruption events for black holes at the centers of galaxies.

26.2 Jeans mass

The criterion for gravitational collapse of a self-gravitating gas cloud is

$$\frac{GM}{R^2} > \frac{1}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} \sim \frac{1}{\rho} \frac{P}{R} \tag{26.6}$$

to which we can apply the ideal gas law $P = \rho c_s^2$ to get

$$\frac{GM}{R} > c_s^2. \tag{26.7}$$

From this, we want to calculate the criterion for cloud collapse given ρ_{gas}, T, for example in the ISM. This gets us (after some symbol-pushing that I missed)

$$M > \left(\frac{5kT}{G\mu} \right)^{3/2} \left(\frac{4\pi}{3} \rho \right)^{-1/2} = M_J. \tag{26.8}$$