Notes for Math 104: Introduction to Analysis
UC Berkeley Summer 2019

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Math 104: Introduction to Analysis **Summer 2019**

Lecture 1: Introduction to logic and sets

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Note: *LaTeX format adapted from template for lecture notes from CS 267, Applications of Parallel Computing, UC Berkeley EECS department.*

1.1 Course Overview

This class has three major sections:

1. The basics: logic, proofs, sets, numbers (Ross ch. 1)
2. Sequences and series (Ross ch. 2-4)
3. Derivatives and integrals (Ross ch. 5-6)

Along the way, we'll cover metric spaces (generalizations of $\mathbb{R}$), power series, and Riemann-Stieltjes integrals.

Some of the goals of this class include getting comfortable with proofs, with sequences and series, finding out when it's okay to swap limits, derivatives, and integrals, and overall developing a rigorous foundation for calculus.

To do well on this class, consistent exposure to proofs is key. Use the homework to test understanding and get practice; do problems and study examples, and ask questions frequently.

1.2 Logic

1.2.1 Statements

Definition 1.1. *A statement is a sentence which is either true or false.*

Example 1.1. p: "Every integer is even or odd" (true)

□

Example 1.2. q: "There are finitely many prime numbers" (false)

□

Example 1.3. $p(x)$: "$x > 5$" (depends on the variable: $p(3)$ is false, $p(6)$ is true) $\square$

1.2.2 Logical operators

Statements can be combined using logical operators. Let p and q be statements. p And q (alternatively written as $p \wedge q$) is true if p is true and q is true; p Or q ($p \vee q$) is true if p is true or if q is true; Not p ($\neg p$) is true if p is false; $p \implies q$ means p implies q, i.e. if p is true then q is true; $p \iff q$ means p is true if and only if q is true.

Remark 1.1. *1. $p \implies q$ is a statement, and is equivalent to $q \vee (\neg p)$.*

2. If p is false, $p \implies q$ is vacuously true.

3. $p \iff q$ is the same as $p \implies q$ and $q \implies p$.

1.2.3 Quantifiers

Logic is also described in terms of quantifiers: the symbols $\forall$, meaning "for all", and $\exists$, meaning "there exists", represent quantifiers. This set of logical operators, quantifiers, and the idea of a statement in total represent "first-order logic".

Example 1.4. $\forall x, p(x)$: "Every integer is greater than 5". This is a false statement. $\square$

Example 1.5. $\exists x, p(x)$: "There exists an integer greater than 5". This is a true statement. $\square$

1.2.4 Negation

Negation of statements described in terms of logical operators or quantifiers follow these rules:

$$\begin{aligned} \neg(p \wedge q) &\iff (\neg p) \vee (\neg q) \\ \neg(p \vee q) &\iff (\neg p) \wedge (\neg q) \end{aligned}$$

So negation turns 'or' into 'and', and vice versa.

$$\begin{aligned}
\neg(p \implies q) &\iff p \wedge (\neg q) \\
\neg(p \iff q) &\iff (p \wedge (\neg q)) \vee (q \wedge (\neg p)) \\
\neg(\forall x, p(x)) &\iff \exists x, (\neg p(x)) \\
\neg(\exists x, p(x)) &\iff \forall x, (\neg p(x))
\end{aligned}$$

Negation also turns 'for all' into 'there exists', and vice versa.

1.3 Sets

Definition 1.2. *A set is a collection of objects (numbers, functions, other sets, metric spaces, etc.) satisfying certain set axioms.*

Remark 1.2. *We'll use ZFC, i.e. Zermelo-Fraenkel set theory with the axiom of choice, but the exact statement of the ZF axioms will be skipped due to lack of relevance.*

(ref: Corteel, Math 113, Spring 2019) A set can be given explicitly, e.g. $S = \{2, 5, 7\}$, or it can be given by a characteristic property, e.g. T is the set of perfect squares, which is represented as $T = \{n^2 \mid n \text{ is an integer}\}$. Some common sets are

$\mathbb{N}$	set of all nonnegative integers	$\{0, 1, 2, \dots\}$
$\mathbb{Z}$	set of all integers	$\{\dots, -2, -1, 0, 1, 2, \dots\} = \{\pm n \mid n \in \mathbb{N}\}$
$\mathbb{Q}$	set of rational numbers	$\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$
$\mathbb{R}$	set of real numbers	difficult (part of the point of this class!)
$\mathbb{C}$	set of complex numbers	$\{a + bi \mid a, b \in \mathbb{R}\}$

Note that Ross does not include 0 in $\mathbb{N}$ but most other conventions do.

The superscript $+$ on any set, e.g. $\mathbb{Z}^+$, refers to the set of positive elements of the regular set (such as the positive integers), and the superscript $*$ means nonnegative. $\mathbb{Z}^+$ is equivalent to what Ross calls $\mathbb{N}$.

Some more general notation includes

$x \in S$	x is an element of S
$x \notin S$	x is not an element of S
$\varnothing = \{\}$	the empty set (the set with no elements)
$A \subseteq B$	A is a subset of B
$A \subset B$	A is a proper subset of B

If A is a subset of B, that means every element of A is in B: $\forall x \in A, x \in B$. If A is a proper subset of B, then $A \subseteq B$ and $A \neq B$.

Two subsets are equal if and only if they are subsets of each other: $S = T \iff S \subset T \wedge T \subset S \iff \forall s \in S, s \in T \wedge \forall t \in T, t \in S$.

Want to Show	**Approach**
$p \implies q$	Assume p holds then prove q is true
$p \iff q$	Prove p implies q then prove q implies p
$\forall x \in S, p(x)$	Let $x \in S$ be any element and use it to show $p(x)$
$\neg\forall x \in S, p(x)$	Give a counterexample (find $x \in S$ s.t. $\neg p(x)$)

Table 1.1: Basic Proof Approaches

Example 1.6. $\mathbb{Z} \subseteq \mathbb{R}$, $\mathbb{R} \not\subseteq \mathbb{Z}$. □

Example 1.7. $\forall S, \varnothing \subset S$ and $S \subset S$. □

Definition 1.3. *Let S and T be sets. The union of S and T is $S \bigcup T = \{x \mid x \in S \text{ or } x \in T\}$; the intersection of S and T is $S \bigcap T = \{x \mid x \in S \text{ and } x \in T\}$; and the difference of S and T is $S \setminus T = \{x \in S \mid x \notin T\}$.*

(to do when I'm better at tikz: a Venn diagram with S on the left, T on the right, the intersection is $S \bigcap T$, and the two sides are $S - T$ and $T - S$. The whole thing is $S \bigcup T$.)

Definition 1.4. *If $S \bigcap T = \varnothing$, we say S and T are disjoint.*

1.3.1 Proofs

The aim of a proof is to show that a statement is true or false. Generally, this is done by starting with some basic knowledge, and applying definitions, theorems, axioms and so on until it is proved.

Definition 1.5. *We say $x \in \mathbb{Z}$ is even if $\exists y$ s.t. $x = 2y$ and odd if $\exists y$ s.t. $x = 2y + 1$.*

Theorem 1.3. *(sort of a trivial one just to demonstrate a proof) $\forall x \in \mathbb{Z}, x$ even $\implies x^2$ even.*

Proof. Let $x \in \mathbb{Z}$ be even, i.e. $\exists y$ s.t. $x = 2y$. So

$$x^2 = (2y)^2 = 4y^2 = 2(2y^2)$$

Therefore x^2 is even by definition. □

1.3.2 Other Proof Techniques

1. Proof by cases

Example 1.8. $\forall x \in \mathbb{Z}, x(x+1)$ is even. □

Proof. Let $x \in \mathbb{Z}$. x is either even or odd.

Case. *x is even. So $\exists y$ s.t. $x = 2y$, so $x(x+1) = 2y(2y+1) = 2(y(2y+1))$, so $x(x+1)$ is even by definition.*

Case. *x is odd. So $\exists y$ s.t. $x = 2y+1$, so $x(x+1) = (2y+1)(2y+2) = 2((2y+1)(y+1))$, so $x(x+1)$ is even by definition.*

□

(To do: fix the 'case' numbering.)

Remark 1.4. *Sometimes we can proceed with an assumption 'without loss of generality' (WLOG) and avoid proving something by cases to get an assumption. We say 'we may assume...' (w.m.a.) for these cases)*

Example 1.9. Let $m, n \in \mathbb{Z}$. By swapping m and n if necessary, w.m.a. $m \leq n$. □

2. Proof by contradiction: assume $\neg p$ and get a contradiction.

Example 1.10. $\sqrt{2}$ is irrational, i.e. $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$. □

Proof. To get a contradiction, assume $\sqrt{2} \in \mathbb{Q}$. Therefore $\exists a, b \in \mathbb{Z}$ such that $\sqrt{2} = \frac{a}{b}$. By cancelling common factors, w.m.a. $\gcd(a,b) = 1$. Therefore

$$2 = \frac{a^2}{b^2} \implies 2b^2 = a^2 \implies a^2 \text{ is even} \implies a \text{ is even}$$

Therefore by definition, $\exists c$ s.t. $a = 2c$, so $2b^2 = a^2 = (2c)^2 = 4c^2$, so $b^2 = 2c^2$. Therefore b^2 is even, implying that b is even. So $\gcd(a,b) \geq 2$ which is a contradiction. Therefore $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$. □

3. Uniqueness proof

   To prove $\exists! x, p(x)$, i.e. there exists a unique x with some property, we proceed by first showing $\exists x, p(x)$, then letting x, y such that $p(x) = p(y)$ and showing that this implies $x = y$.

Example 1.11. $\forall x \in \mathbb{Z}$, if x is even, $\exists! y$ such that $x = 2y$. $\square$

Proof. y exists by the definition of evenness. If y, z are s.t. $x = 2y, x = 2z$, then $2y = x = 2z \implies y = z$. $\square$

1.3.3 Proof terminology

Please don't use these words

- "Clearly"
- "Obviously"
- "Trivial" (most of the time)

Instead, use "it follows immediately from...", "one can check directly from..." and cite a specific theorem or definition.

1.3.4 Numbers

Our first goal is to define $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ rigorously. To start with, we can obtain $\mathbb{N} = \{0, 1, 2, \dots\}$ by starting with 0 and adding 1 to it over and over. We have to understand what 0 and addition are for this.

Definition 1.6. $\mathbb{N}$ *is a set equipped with an element 0 and a function s called the successor function, such that the following axioms (the Peano axioms) hold:*

P1. $0 \in \mathbb{N}$

P2. $s : \mathbb{N} \to \mathbb{N}$ *is a function, i.e.* $\forall n \in \mathbb{N}, s(n) \in \mathbb{N}$ *(we would say $s(n) = n + 1$ but we don't know what addition is yet)*

P3. $\forall m, n \in \mathbb{N}, s(m) = s(n) \implies m = n$ *(s is "injective".)*

P4. $\nexists n \in \mathbb{N}$ *such that* $s(n) = 0$.

P5. $\forall S \subset \mathbb{N}$, *if* $0 \in S$ *and* $\forall n \in S, s(n) \in S$, *then* $S = \mathbb{N}$.

The last of the Peano axioms gives us the *principle of mathematical induction*: if $p_0, p_1, \dots$ are statements, and p_0 is true, and $p_n \implies p_{n+1}$, then p_n is true $\forall \mathbb{N}$. Mathematical induction does not have to start at zero; in general if p_m is true where m is the lowest number such that we know upfront that p_m is true, then p_n is true $\forall n \in \mathbb{N}, n \geq m$.

Example 1.12. $\forall a_1, \ldots, a_n \in \mathbb{R}$, $a_1 \ldots a_n = 0 \implies \exists i$ s.t. $a_i = 0$.

□

Proof. The proof is by induction on n.

Base case: If $n = 1$, we assume $a_1 = 0$ so the statement is trivial.

Inductive step: Suppose true for n. Let $a_1, \ldots, a_{n+1} \in \mathbb{R}$ such that $a_1 a_2 \ldots a_{n+1} = 0$.

$$(a_1 a_2 \ldots a_n)(a_{n+1}) = 0$$

We know that if the product of two numbers is zero, one of them is zero. Therefore either $a_1 a_2 \ldots a_n = 0$, or $a_{n+1} = 0$. If $a_{n+1} = 0$ the statement is true; if $a_1 a_2 \ldots a_n = 0$ then we know $\exists 1 \leq i \leq n$ s.t. $a_i = 0$ by the inductive hypothesis. Therefore the statement is true, and the proof is complete.

□

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Lecture 2: Defining $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$; Ordered Fields

Lecturer: Michael Christianson *25 June* *Aditya Sengupta*

2.1 The Well-Ordering Principle

Definition 2.1. *The Well-Ordering Principle states that any $S \subset \mathbb{N}, S \neq \varnothing$ contains a minimum element, i.e. $\exists s_0 \in S, r \geq s_0 \forall r \in S$.*

Note that this is not true of $\mathbb{R}$. For example, $(a, b) = \{r \in \mathbb{R} \mid a < r < b\}$ does not have a minimum element.

Theorem 2.1. *The Well-Ordering Principle is equivalent to the proposition P5 (mathematical induction.)*

The direction WOP to P5 is proved in Ross, or in Hutchings' notes. (come back and add this just for completeness). Here we will show that P5 implies the WOP.

Proof. We proceed by contraposition. Let $S \subset \mathbb{N}$ have no minimum element. By induction, we want to show that $S = \varnothing$; that is, $\forall n \in N, \nexists m < n, m \in S$ by induction on n.

Base case: If $n = 0, \nexists m < n$ by P4.

Inductive step: Suppose $\nexists m < n, m \in S$. If $\exists m < n+1, m \in S$, then $m = n$, so $n \in S$ is a minimal element. This is a contradiction because we assumed S not to have a minimal element. Therefore no such m exists, which implies that $S = \varnothing$. Therefore, based on mathematical induction (P5) we have shown that the well-ordering principle holds, i.e. that the only subset of $\mathbb{N}$ without a minimal element is the empty set. □

2.2 Defining Operations

The usual notation for $\mathbb{N}$ is $1 = s(0), 2 = s(s(0))$ and so on. To define addition and multiplication, we can proceed inductively (recursively) based on the rules $n+0 = n$ and $n+s(m) = s(m+n), \forall m, n \in \mathbb{N}$. This allows us to build up addition in general: $n+1 = n+s(0) = s(n+0) = s(n)$; $n+2 = n+s(1) = s(n+1) = s(s(n))$, and so on.

Similarly, we can build up multiplication based on the rules $n \cdot 0 = 0$ and $n \cdot s(m) = n \cdot m + n$. This gives us $n \cdot 1 = n \cdot s(0) = n \cdot 0 + n = 0 + n = n$; $n \cdot 2 = n \cdot s(1) = n \cdot 1 + n = n + n$, and so on.

We can also define subtraction in this way: $\forall n, m$ s.t. $n \geq m$, i.e. $n = s(s(\dots s(m)))$, then we define $n - m$ to be the unique element of $\mathbb{N}$ such that $(n-m)+m = n$. We can prove by induction on m that there exists a unique element $n - m$ such that this is true.

2.3 $\mathbb{Z}$

We can define $\mathbb{Z}$ based on $\mathbb{N}$. As a set,

$$\mathbb{Z} = \{0\} \bigcup \{n, -n \mid n \in \mathbb{N}, n > 0\} = \{0, 1, -1, 2, -2, \dots\}$$

We can define addition in $\mathbb{Z}$ for all $m, n \in \mathbb{N}$ by the following rules:

$$\begin{aligned} n + m &= \quad \text{same as in } \mathbb{N} \\ n + (-m) &= \begin{cases} n - m & n \geq m \\ -(m - n) & m > n \end{cases} \\ (-n) + (-m) &= \quad -(n + m) \end{aligned}$$

and multiplication by the following rules:

$$\begin{aligned} m \cdot n = & \quad \text{same as in } \mathbb{N} \\ m \cdot (-n) = & \quad -(m \cdot n) \\ (-m) \cdot n & \quad -(m \cdot n) \\ (-m) \cdot (-n) & \quad m \cdot n \end{aligned}$$

2.4 $\mathbb{Q}$

It is more difficult to define $\mathbb{Q}$ than $\mathbb{Z}$. A natural-seeming but incorrect definition would be to just make pairs of elements (we haven't defined division yet so this is strange, but that's not why it's incorrect): $\mathbb{Q} \neq \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$. This is not correct because elements of $\mathbb{Q}$ can be the same with different representations but this definition counts them as different elements, e.g. $\frac{1}{2} = \frac{2}{4}$. To resolve this we introduce an equivalence relation.

Define the equivalence relation $(a, b) \sim (c, d) \iff a \cdot d = b \cdot c$. Then this gives us the equivalence classes

$$[(a, b)] = \{(c, d) \mid (a, b) \sim (c, d), c, d \in \mathbb{Z}, d \neq 0\}$$

For example, the equivalence class corresponding to one half is $[(1, 2)] = \{(n, 2n) \mid n \in \mathbb{N}, n > 1\}$. Most other equivalence classes are defined similarly to this, e.g. (an, bn) where a/b is the ratio in its most reduced form. The exception is $[(0, 1)] = \{(0, n) \mid n \in \mathbb{N}, n \neq 0\}$.

So the correct definition of the rational numbers is as follows:

$$\mathbb{Q} = \{[(a, b)] \mid a, b \in \mathbb{Z}, b \neq 0\}$$

We can define addition and multiplication:

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)]$$

$$[(a,b)] \cdot [(c,d)] = [(ac, bd)]$$

A potential problem with this is the representative you choose for a fraction: for example, we would want $[(1,2)] + [(c,d)] = [(2,4)] + [(c,d)]$. This turns out to be the case, just because addition of fractions works.

2.5 Ordered Fields

Definition 2.2. *A field is a set F equipped with addition and multiplication such that $\forall a, b, c \in F$,*

- *A1. Associativity: $a + (b + c) = (a + b) + c$*
- *A2. Commutativity: $a + b = b + a$*
- *A3. Identity: $\exists! 0 \in F, 0 + a = a$*
- *A4. Inverses: $\exists! -a \in F$ s.t. $(-a) + a = 0$*

- *M1. Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$*
- *M2. Commutativity: $a \cdot b = b \cdot a$*
- *M3. Identity: $\exists! 1 \in F, 1 \cdot a = a$*
- *M4. Inverses: if $a \neq 0$, $\exists! a^{-1} \in F$ s.t. $a \cdot a^{-1} = 1$*

Additionally, the distributive law $a \cdot (b + c) = a \cdot b + a \cdot c$, and $1 \neq 0$: the multiplicative and additive identities are distinct. This is equivalent to saying F has more than one element.

Definition 2.3. *A <u>total order</u> on a set S is a relation $\leq$ such that the following hold:*

- *O1. Totality: $\forall a, b \in S$, either $a \leq b$ or $b \leq a$.*
- *O2. Anti-Symmetry: $\forall a, b \in S$, if $a \leq b$ and $b \leq a$ then $a = b$.*
- *O3. Transitivity: $\forall a, b, c \in S$, if $a \leq b$ and $b \leq c$ then $a \leq c$.*

Definition 2.4. *An <u>ordered field</u> is a field F equipped with a total order $\leq$ such that the following hold:*

- *O4 $\forall a, b, c \in F, a \leq b \implies a + c \leq b + c$*
- *O5 $\forall a, b, c \in F, a \leq b, 0 \leq c \implies ac \leq bc$.*

$\mathbb{Q}$ and $\mathbb{R}$ are ordered fields, which just means that arithmetic works the way you would expect.

Definition 2.5. *Let F be an ordered field and let $a \in F$. Define*

$$|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$$

Theorem 2.2. *Let $a, b \in F$. Then*

(a) $|a| \geq 0$

(b) $|a \cdot b| = |a| \cdot |b|$

(c) $|a + b| \leq |a| + |b|$

The last one is the triangle inequality, which is really useful in a lot of different places.

Proof. (of the Triangle Inequality: the others are proved in Ross.)

If $a \geq 0$, then $-|a| \leq 0 \leq a = |a|$, and if $a \leq 0$ then $-|a| = a \leq 0 \leq -a = |a|$. In either case $-|a| \leq a \leq |a|$. Likewise, $-|b| \leq b \leq |b|$. Therefore

$$-|a| + (-|b|) \leq a + (-|b|) \leq a + b \leq |a| + b \leq |a| + |b|$$

So

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

If $|a+b| = a+b$ then $|a+b| \leq |a| + |b|$ and we're done. If $|a+b| = -(a+b)$ then $|a| + |b| \geq -(a+b) = |a+b|$ and the inequality still holds. Therefore the proof is complete. $\square$

Intuitively, we can understand the Triangle Inequality as saying that the direct distance between two points is shorter than or of equal length to any other path described by some number of vectors. For example, say a and b are two vectors lined up tip-to-tail; the magnitude of their vector sum represents the direct path, and it is smaller than the sum of their magnitudes unless they are collinear, in which case they are equal.

2.6 Differences between $\mathbb{Q}$ and $\mathbb{R}$

Theorem 2.3. *$\forall x \in \mathbb{R}, x \in \mathbb{Q}$ iff x has a repeating or terminating decimal expansion.*

Example 2.1. π and e are irrational because their decimal expansions never repeat or terminate (though that is not how you would prove that they are irrational.) $\square$

Example 2.2. $\sum_{n=0}^{\infty} 10^{-2^n} = 10^{-1} + 10^{-2} + 10^{-4} + 10^{-8} + \cdots = 0.11010001000\ldots$ is irrational because its decimal expansion does not terminate or repeat. $\square$

Example 2.3. $\log_2 3 \notin \mathbb{Q}$.

□

Proof. Suppose $\log_2 3 = \frac{m}{n} \implies 3 = 2^{m/n} \implies 3^n = 2^m$, but 3^n is odd and 2^m is even which is a contradiction. □

Math 104: Introduction to Analysis **Summer 2019**

Lecture 3: Irrational numbers; defining $\mathbb{R}$

Lecturer: Michael Christianson *26 June* *Aditya Sengupta*

3.1 Irrational and Transcendental Numbers

An irrational number is any element $x \in \mathbb{R}, x \notin \mathbb{Q}$. It is any number whose decimal expansion does not repeat or terminate. Examples include any $\log_m n, m, n \in \mathbb{N}, m, n > 1$ with no common factors; transcendental numbers, and some zeros of polynomials.

Definition 3.1. *$x \in \mathbb{R}$ is algebraic if there exists a polynomial $p(x) = \sum_{i=0}^{n} c_i x^i$ such that $p(x)$ is not constant (i.e. $c_i \neq 0$ for some $i > 0$), $c_i \in \mathbb{Z} \forall i$, and $p(a) = 0$. We say in this case that a is a root or zero of $p(x)$.*

If a is not algebraic, it is said to be transcendental. For example, e, π, e^{π} are all transcendental. In general, if a and b are algebraic, with $a \neq 0, 1$ and $b \notin \mathbb{Q}$, a^b is transcendental.

Remark 3.1. *$\forall \frac{m}{n} \in \mathbb{Q}$, $\frac{m}{n}$ is a root of $nx - m$, so $\frac{m}{n}$ is algebraic. Therefore any transcendental number is irrational.*

Every rational number is algebraic, but not every algebraic number is rational.

Theorem 3.2. *(Rational Root Theorem) Let $p(x) = \sum_{i=0}^{n} c_i x^i$ with $c_i \in \mathbb{Z} \forall i$. Let $r \in \mathbb{Q}$ s.t. $p(r) = 0$, i.e. r is a root of p. Then $r = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ s.t. $\gcd(m, n) = 1$. Then m divides c_0 and n divides c_n.*

Example 3.1. $\sqrt{2}$ is a root of $p(x) = x^2 - 2$, in which $c_n = c_2 = 1$ and $c_0 = -2$. We know by the rational root theorem that if $\sqrt{2}$ is rational, then $\sqrt{2} = \frac{m}{n}$ where m divides -2 and n divides 1. Therefore $m \in \{\pm 2, \pm 1\}$ and $n \in \{\pm 1\}$, so $\frac{m}{n} \in \{\pm 2, \pm 1\}$. But $p(\pm 2) = 2 \neq 0$ and $p(\pm 1) = -1 \neq 0$, so no rational roots of $p(x)$ exist. Therefore $\sqrt{2}$ is irrational.

$\square$

Corollary 3.3. *If $c_n = 1$, then any rational root of $p(x)$ has to be an integer.*

Example 3.2. The cube root of 12 is irrational. It is a zero of $p(x) = x^3 - 12$. If $\sqrt[3]{12} \in \mathbb{Q}$ then $\sqrt[3]{12} \in \mathbb{Z}$ dividing 12. However, we can check that $1^3 = 1, 2^3 = 8, 3^3 = 27$ exceeding 12. If $n \leq 0, n^3 \leq 0 \implies n^3 \neq 12$. If $n \geq 3, n^3 \geq 27 \implies n^3 \neq 12$. Therefore by exhaustion the cube root of 12 cannot be rational.

$\square$

Note that not all "weird square roots" are irrational (see Ross, ex. 2.4ish).

3.2 Defining $\mathbb{R}$ from $\mathbb{Q}$

There are a few different potential ways to define $\mathbb{R}$.

1. $\mathbb{R}$ is the set of all numbers that can be written in decimal notation.

$$\text{“}\mathbb{R} = \{k.d_1d_2d_3\cdots \mid k \in \mathbb{Z}, d_i \in \{0,1,2,3,4,5,6,7,8,9\}\}\text{”}$$

   A slight problem with this is duplication due to equal numbers with different decimal representations. For example, $0.\bar{9} = 1$. But you could imagine using the same kind of setup with equivalence classes as we did with $\mathbb{Q}$ to resolve that. The bigger problem here is that this definition does not say much about the properties of $\mathbb{R}$.

2. Using intuition from number lines and completeness, we can define $\mathbb{R}$ rigorously using Dedekind cuts. This tells us some of the properties of $\mathbb{R}$, but is tedious to prove.

3. $\mathbb{R}$ is the “smallest complete metric space” containing $\mathbb{Q}$. This is probably the best definition, but it needs Cauchy sequences and metric spaces.

Definition 3.2. *Let $S \subset \mathbb{R}$ be nonempty and let $s_0 \in S$.*

1. *s_0 is the maximum of S if $\forall r \in S, r \leq s_0$. This is denoted by $s_0 = \max s$. Not every set has a maximum.*
2. *s_0 is the minimum of S if $\forall r \in S, r \geq s_0$. This is denoted by $s_0 = \min s$. Not every set has a minimum.*
3. *If $M \in \mathbb{R}$ such that $\forall s \in S, s \leq M$, we say M is an upper bound for S. If $\exists$ such M, we say S is bounded above.*
4. *If $\exists m \in \mathbb{R}$ s.t. $\forall s \in S, s \geq m$, we say m is a lower bound for S. If $\exists$ such m, we say S is bounded below.*
5. *We say S is bounded if it is bounded above and bounded below.*
6. *If $\alpha \in \mathbb{R}$ is an upper bound for S and $\forall \gamma < \alpha, \gamma$ is not an upper bound, we say α is the supremum (or least upper bound) of S. This is denoted by $\alpha = \sup S$.*
7. *If $\beta \in \mathbb{R}$ is a lower bound for S and $\forall \gamma > \beta, \gamma$ is not a lower bound, we say β is the infimum (or greatest lower bound) of S. This is denoted by $\beta = \inf S$.*

Theorem 3.4. *If $S \subset \mathbb{R}$ is nonempty and finite, $\max S$ and $\min S$ exist. This can be proven by induction on the size of S.*

We often represent subsets of $\mathbb{R}$ in interval notation. Let a and b be real numbers; then

$$\begin{aligned}
(a,b) &= \{r \in \mathbb{R} \mid a < r < b\} && \text{open interval}\\
[a,b] &= \{r \in \mathbb{R} \mid a \leq r \leq b\} && \text{closed interval}\\
[a,b) &= \{r \in \mathbb{R} \mid a \leq r < b\} && \text{half-open interval}\\
(a,b] &= \{r \in \mathbb{R} \mid a < r \leq b\} && \text{half-open interval}
\end{aligned}$$

The open interval (a,b) has no maximum because we can get arbitrarily close to b without actually reaching b. Similarly it has no minimum. We can prove that $\sup((a,b)) = b$ and $\inf((a,b)) = a$.

In the closed interval $[a,b]$, the maximum is b and the minimum is a.

Theorem 3.5. *If $S \subset \mathbb{R}$ has a maximum, then $\max S = \sup S$. This is because $\max S$ is an upper bound by definition, and $\forall \alpha < \max S, \max S \in S \implies \alpha$ is not an upper bound.*

Theorem 3.6. *If $S \subset \mathbb{R}$ has a maximum, then $\min S = \inf S$. This is because $\min S$ is a lower bound by definition, and $\forall \beta > \min S, \min S \in S \implies \beta$ is not an upper bound.*

Example 3.3. $S = \{\frac{1}{n^2} \mid n \in \mathbb{N}, n \geq 3\}$. S is bounded above by $\frac{1}{9}$, and 0 is a lower bound. In fact, $\inf S = 0$ and $\min S$ does not exist.

□

The intuition we should have for extending $\mathbb{Q}$ to $\mathbb{R}$ is that $\mathbb{R}$ has no "gaps". This can be formalized in the Completeness Axiom:

Theorem 3.7. *$\forall S \subset \mathbb{R}$ nonempty and bounded above, then $\sup S$ exists and is in $\mathbb{R}$.*

Corollary 3.8. *$\forall S \subset \mathbb{R}$ nonempty and bounded below, $\inf S$ exists.*

Proof. Let $-S = \{-s \mid s \in S\}$. Then $-\sup(-S) = \inf S$. □

Theorem 3.9. *(Archimedean Property) Let $a, b > 0$ be real. Then $\exists n \in \mathbb{Z}_{>0}$ s.t. $na > b$.*

Proof. We proceed by contradiction. Suppose $na \leq b \forall n \in \mathbb{Z}_{>0}$. Then, define the set $S = \{na \mid n \in \mathbb{Z}_{>0}\}$. S is nonempty and b is an upper bound for S. Therefore by the completeness axiom, $\sup S = \alpha$ exists. Since $a > 0, \alpha - a < \alpha$. Since α is chosen to be the least upper bound, $\alpha - a$ is not an upper bound on S. Therefore $\exists n \in \mathbb{Z}_{>0}$ such that $na > \alpha - a$. Therefore $(n+1)a > \alpha$. But $(n+1)a \in S$, so α cannot be an upper bound for S. This is a contradiction; therefore the proof is complete. □

Theorem 3.10. *("Denseness" of $\mathbb{Q}$) Let $a, b \in \mathbb{R}, a < b$. Then $\exists r \in \mathbb{Q}$ s.t. $a < r < b$.*

Proof. By the Archimedean property, $b - a > 0 \implies \exists n \in \mathbb{Z}_{>0}$ s.t. $n(b-a) > 1 \implies nb > na + 1$. We want $m \in \mathbb{N}$ such that $nb > m > na \implies b > \frac{m}{n} > a$.

Intuitively, m should be the smallest integer greater than na. This can be shown explicitly: by the Archimedean property, $\exists k \in \mathbb{Z}_{\geq 0}$ such that $k \cdot 1 > |an|$. Let $S = \{z \in \mathbb{Z} \mid an \leq z \leq k\}$. Because $k \in S, S \neq \varnothing$. Also, S is finite, because $-k \leq an \leq k$. Moreover, $\min S$ exists because $S \subset \mathbb{Z}$ and it is finite. Let $m = \min S$. Then $m \in S \implies m > an$, but $m - 1 \leq a_n$, because otherwise $m - 1$ would be $\min S$. So

$$an < m \leq an + 1 < bn$$

Therefore m exists, meaning $\frac{m}{n}$ exists, so the proof is complete. □

Example 3.4. Consider again $S = \{\frac{1}{n^2} \mid n \in \mathbb{N}, n \geq 3\}$. We can show that $\inf S = 0$.

□

Proof. $\frac{1}{n^2} \geq 0 \forall n \implies 0$ is a lower bound. We want to show that $\forall \alpha > 0, \exists n \geq 3$ s.t. $0 < \frac{1}{n^2} < \alpha$. By denseness of $\mathbb{Q}, \exists \frac{m}{n} \in \mathbb{Q}, 0 < \frac{m}{n} < \alpha$. We may assume $m, n > 0$. So

$$\alpha > \frac{m}{n} \geq \frac{1}{n} \geq \frac{1}{n^2} >\geq \frac{1}{N^2} > 0$$

Take $N = \max\{n, 3\}$. Then $\frac{1}{N^2} \in S \implies \alpha$ is not a lower bound on S. $\square$

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Lecture 4: Completeness, infinity, and Dedekind cuts

Lecturer: Michael Christianson — *27 June* — *Aditya Sengupta*

4.1 More about inf and sup

If a minimum of a set exists, it is equal to its infimum; if a maximum of a set exists; it is equal to its supremum.

For example, consider $S = [a, b)$. We can see that $\min S = \inf S = a$ and $\sup S = b$ but no max exists.

Proof. a is a lower bound for S and b is an upper bound. $\forall \alpha > a$, we want $c \in S$ such that $\alpha > c \geq a$. This is satisfied by $c = a$. Therefore $\inf S = a$.

$\forall \alpha < b$, we want $c \in S, \alpha < c$. Let $s = \max\{\alpha, a\}$ and let $c = \frac{s+b}{2}$. Then

$$\alpha < \frac{\alpha + b}{2} \leq \frac{s+b}{2} < b$$
$$a < \frac{s+b}{2} < b \implies \frac{s+b}{2} \in S$$

Therefore we have found c such that $\alpha < c < b$ for any $\alpha \in S$, which tells us that no maximum exists.

Alternatively, we can proceed using the denseness of $\mathbb{Q}$: $\forall \alpha < b, \exists r \in \mathbb{Q}$ such that $\max\{\alpha, a\} < r < b$. Then $r \in [a, b)$, and $r > \alpha$, so α is not an upper bound. $\square$

4.2 Notes on $\pm\infty$

1. $\forall a \in \mathbb{R}$, we define $-\infty < a < +\infty$.
2. Math with $\pm\infty$ has the following rules: $\forall a \in \mathbb{R}$,

$$\begin{aligned}
a + \infty &= +\infty \\
a - \infty &= -\infty \\
\infty + \infty &= \infty \\
-\infty - \infty &= -\infty \\
a \cdot \infty = \operatorname{sgn} a \cdot \infty &= \begin{cases} +\infty & a > 0 \\ -\infty & a < 0 \end{cases} \\
a \cdot -\infty = -\operatorname{sgn} a \cdot \infty &= \begin{cases} -\infty & a > 0 \\ +\infty & a < 0 \end{cases} \\
\pm\infty \cdot \pm\infty &= +\infty \\
\pm\infty \cdot \mp\infty &- \infty
\end{aligned}$$

Note that $0 \cdot \infty$, $\infty - \infty$, and $\frac{\pm\infty}{\pm\infty}$ are not defined. $\pm\infty \notin \mathbb{R}$; this is clear from the fact that $\pm\infty$ do not follow the common properties of $\mathbb{R}$, such as $a \cdot 0 = 0 \forall a \in \mathbb{R}$.

3. $\pm\infty$ can be used in interval notation, with the following conventions:

$$
\begin{aligned}
(-\infty, a) &= \{x < a\} \\
(-\infty, a] &= \{x \leq a\} \\
(a, +\infty) &= \{x > a\} \\
[a, +\infty) &= \{x \geq a\} \\
(-\infty, \infty) &= \mathbb{R}
\end{aligned}
$$

4. Suprema and infima:

Definition 4.1. *Let $S \subset \mathbb{R}$ be nonempty. If S is not bounded above, define $\sup S = +\infty$. If S is not bounded below, define $\inf S = -\infty$.*

Lemma 4.1. *Let $S \subset \mathbb{R}$ be nonempty. Then $\inf S \leq \sup S$.*

Proof. $S \neq \varnothing \implies \exists a \in S$. Then by definition

$$\inf S \leq a \leq \sup S$$

This holds even if $\inf S = -\infty$ or $\sup S = +\infty$. □

Remark 4.2. *Often proofs like this require cases for $\sup S = \infty$ or $\inf S = -\infty$.*

Definition 4.2. *We define $\sup \varnothing$. $\forall a \in \mathbb{R}, \forall x \in \varnothing, a \geq x$ vacuously. $\forall x \in \varnothing, -\infty \geq x$. Similarly, $\inf \varnothing = +\infty$.*

Remark 4.3. *This is weird, and we'll almost never use it.*

4.3 Dedekind cuts

Lemma 4.4. *Let $a \in \mathbb{R}$. Define $S_a = \{r \in \mathbb{Q} \mid r < a\} = (-\infty, a) \bigcap \mathbb{Q}$, the rational numbers less than a. Then, $\sup S_a = a$.*

Proof. This is true because of the density of $\mathbb{Q}$.

$$
\begin{aligned}
\forall r \in S_a, a > r \\
\forall \alpha < a, \exists r \in \mathbb{Q}, \alpha < r < a \\
\therefore r \in \S_a \text{ by definition}
\end{aligned}
$$

□

Dedekind cuts are a way of defining $\mathbb{R}$ in terms of these sorts of subsets whose suprema are the corresponding element of $\mathbb{R}$. If we can define S_a without using a, then S_a somehow tells us what a is (by the fact that $\sup S_a = a$).

Definition 4.3. *A Dedekind cut is a subset* $S \subset \mathbb{Q}$ *such that*

(i) $S \neq \mathbb{Q}$ *and* $S \neq \varnothing$

(ii) $\forall s \in S, r \in \mathbb{Q}, r < s \implies r \in S$.

(iii) $\max S$ *does not exist.*

Proposition 4.5. $\forall a \in \mathbb{R}, S_a$ *is a Dedekind cut.*

Proposition 4.6. $\forall S \subset \mathbb{Q}$ *such that* S *is a Dedekind cut, let* $a = \sup S \in \mathbb{R}$*. Then* $S = S_a$*.*

Proof. Let $S \subset \mathbb{Q}$ be a Dedekind cut. Write $a = \sup S \in \mathbb{R}$ (because S is bounded above, $a \neq +\infty$.) $\forall r \in S, r \in \mathbb{Q}, r \leq a$. But if $r = a$ then $a \in S \implies \max S = a$. So $r \neq a \implies r < a \implies r \in S_a$. Therefore $S \subset S_a$. Conversely, let $r \in S_a$. $r \in \mathbb{Q}, r < a$. Since $a = \sup S$, r is not an upper bound. This means $\exists s \in S, s > r$. By the second property of Dedekind cuts, $r \in S$. So $S_a \subset S \implies S = S_a$. $\square$

Corollary 4.7. *The function* $\{S \subset \mathbb{Q} \mid S \text{ a Dedekind cut}\} \to \mathbb{R}, S \to \sup S$ *has an inverse given by* $a \to S_a$*. That is, there exists a bijection between the set of Dedekind cuts and the set of real numbers.*

Definition 4.4. *We define* $\mathbb{R} = \{S \subset \mathbb{Q} \mid S \text{ a Dedekind cut}\}$ *to be a set.*

4.4 Structure of $\mathbb{R}$

4.4.1 $\mathbb{Q}$ is a subset of $\mathbb{R}$

$\mathbb{Q} \subset \mathbb{R} : \forall r \in \mathbb{Q}, S_r = \{s \in \mathbb{Q} \mid s < r\}$ is a Dedekind cut and $S_r \in \mathbb{R}$. Identify r with S_r, then $\{S_r \mid r \in \mathbb{Q}\} \subset \mathbb{R}$.

4.4.2 Ordering

There is a notion of ordering in $\mathbb{R}$. Let S and T be Dedekind cuts. We say $S \leq T$ if $S \subseteq T$. If $S = S_a, T = S_b$, then $S \subseteq T \iff a \leq b$.

4.4.3 Addition

$$\forall S, T \text{Dedekind cuts}, S + T = \{s + t \mid s \in S, t \in T\}.$$
$$\sup(S + T) = \sup S + \sup T$$

i.e. if $S_a = S, S_b = T$, then $S + T + S_{a+b}$.

4.4.4 Additive inverses

Additive inverses: Let $S_a = (-\infty, a) \bigcap \mathbb{Q}$. Then $S_{-a} = (\infty, -a) \bigcap \mathbb{Q} = \{-b \mid b \in (a, +\infty)\} = \{-b \mid b \notin (\infty, a]\}$.

This definition *almost* works; it gives us $-S = \{-r \mid r \notin S, r \in \mathbb{Q}\}$. The problem with this is if $S = S_r, r \in \mathbb{Q}$, then $-r \in -S$. This is a problem, because $-S$ now has a maximum element which violates the definition of a Dedekind cut. Therefore, the correct definition is

$$-S = \{-r \mid r \in \mathbb{Q}, \exists r' \in Q \text{s.t.} r' < r, r' \notin S\}$$

One can check that S is a Dedekind cut and that $S + (-S) = S_0$, and $S_0 + S = S$.

4.4.5 Multiplication

A reasonable-looking definition for the product of two Dedekind cuts is $S \cdot T = \{s \cdot t \mid s \in S, t \in T\}$, but this is not a Dedekind cut. Instead, for $S, T > S_0$, $S \cdot T = \{s \cdot t \mid s \in S, t \in T, s \geq 0 \vee t \geq 0\}$. We can check that $S_a \cdot S_b = S_{a \cdot b}$, and that $S \cdot T$ is a Dedekind cut.

We can deal with the other cases by taking the negative (additive inverse) of a Dedekind cut. For example, let $S < S_0, T > S_0$. $-S > S_0$, so $-((-S) \cdot T) = S \cdot T$.

4.4.6 Multiplicative inverses

$$S^{-1} = \{r \in \mathbb{Q} \mid \exists r' > r, \exists s \in S, r' \cdot s < 1\}$$

We can check that S^{-1} is a Dedekind cut, and that $S \cdot S^{-1} = S_1$.

Note that now that we have definitions of addition and multiplication with well-defined inverses, we can state that

Theorem 4.8. *$\mathbb{R}$ is an ordered field.*

4.4.7 Completeness

Theorem 4.9. *Let $A \subset \mathbb{R}$ be nonempty and bounded above. Then $\sup A \in \mathbb{R}$.*

Proof. Let $S = \bigcup_{T \in A} T \subset \mathbb{Q}$. We want to prove that S is a Dedekind cut and that $S = \sup A$. These can be shown from Dedekind cut properties (see notes/Ross and finish up the proof. Or do it.) □

4.5 Sequences

Definition 4.5. *A sequence is a function whose domain is $\{n \in \mathbb{Z} \mid n \geq m\}$ for some m.*

Remark 4.10. *Usually, $m = 0$ or $m = 1$.*

We denote a sequence by $S : \{n \in \mathbb{Z} \mid n \geq m\} \to \mathbb{R}$, the individual elements by $s(n) = s_n$, the whole sequence or subsets thereof by $s = (s_n)_{n=m}^{+\infty} = (s_m, s_{m+1}, s_{m+2}, \dots)$.

Example 4.1.

$$s_n = \frac{1}{n^2}, n \in \mathbb{Z}_{>0} (s_n)_{n \in \mathbb{Z}_{>0}} = (1, \frac{1}{4}, \frac{1}{9}, \dots)$$

□

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Lecture 5: Sequences and Series

Lecturer: Michael Christianson *1 July* *Aditya Sengupta*

5.1 Sequences

Definition 5.1. *A sequence is a function whose domain is $\{n \in \mathbb{Z} \mid n \geq m\}$ for some $m \in \mathbb{Z}$. (Usually $m = 0$ or $m = 1$, i.e. the domain is $\mathbb{N}$ or $\mathbb{Z}_{>0}$.*

Remark 5.1. *Sequences that start at $m > 0$ can be shifted by the bijection from the integers to the integers $\mathbb{Z} \to \mathbb{Z}, n \to n - m$*

Example 5.1. Let $s_n = \frac{1}{n^2} \forall n \in \mathbb{Z}_{>0}$.

$$(s_n)_{n \in \mathbb{Z}_{>0}} = (s_n) = (s_1, s_2, s_3, \dots) = \left(1, \frac{1}{4}, \frac{1}{9}, \dots\right)$$

□

Example 5.2. Let $a_n = (-1)^n \forall n \in \mathbb{N}$.

$$(a_n)_{n \in \mathbb{N}} = (a_n) = (1, -1, 1, -1, \dots)$$

□

Example 5.3. Let $c_n = \sqrt[n]{n}, n \in \mathbb{Z}_{>0}$.

$$(c_n) = \left(1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \dots\right)$$

$c_{100} \approx 1.0471$ and $c_{1000} \approx 1.0069$, so it seems like the sequence is going to 1.

□

Example 5.4. Let $b_n = \left(1 + \frac{1}{n}\right)^n, n \in \mathbb{Z}_{>0}$.

$$(b_n)_{n \in \mathbb{Z}_{>0}} = \left(2, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \ldots\right)$$

$b_{100} \approx 2.7048, b_{1000} \approx 2.7169$. This sequence ends up going to e.

$\square$

In the above two examples, we had a notion of the sequence getting closer and closer to something. We can formalize this by defining *convergence*.

Definition 5.2. *Let (s_n) be a sequence of real numbers and let $s \in \mathbb{R}$. We say (s_n) converges to s if*

$$\forall \epsilon > 0, \epsilon \in \mathbb{R}, \exists N \in \mathbb{R} \,\text{s.t.}\, \forall n \in N, |s_n - s| < \epsilon.$$

In this case, we say s is the limit of (s_n), and we write

$$\lim_{n \to +\infty} s_n = s$$

or $s_n \to s$ as $n \to \infty$.

If no such s exists, we say (s_n) diverges.

Remark 5.2. *We can consider $|s_n - s|$ to be a "distance from s_n to s". Based on this, we can interpret the definition as saying "s_n gets with ϵ of s once n is big enough."*

For example, the limit of the sequence $s_n = \frac{1}{n^2}$ is 0, as we would expect, because we showed that $\inf\{s_n \mid n \in \mathbb{Z}_{>0}\} = 0$ and the sequence is decreasing. We will be able to formally prove this soon. However, $a_n = (-1)^n$ diverges.

Lemma 5.3. *Let (s_n) be a sequence. If (s_n) converges, its limit is unique.*

Proof. Let $s, t \in \mathbb{R}$ be such that $s_n \to s$ and $s_n \to t$. We prove $s = t$.

$$\begin{aligned} \forall \epsilon > 0, \exists N_1 \in \mathbb{R} \,\text{s.t.} \\ \forall n > N_1, |s_n - s| < \frac{\epsilon}{2} \\ \forall n > N_2, |s_n - t| < \frac{\epsilon}{2} \end{aligned}$$

So, $\forall n > \max\{N_1, N_2\}$, we have

$$\begin{aligned} |s-t| &= |(s-s_n)+(s_n-t)| && \text{adding and subtracting } s_n \\ &\le |s_n-s|+|s_n-t| && \text{Triangle Inequality} \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon \end{aligned}$$

Therefore

$$\forall \epsilon > 0, |s-t| < \epsilon.$$

By definition, $\forall a \in \mathbb{R}, |a| \ge 0$, but we know $|s-t| < \epsilon$ for any $\epsilon > 0$. Therefore

$$\begin{aligned} |s-t| &= 0 \\ s &= t \end{aligned}$$

□

Remark 5.4. *1. $n \in \mathbb{Z}$ in this definition.*

2. Can assume $\mathbb{N} \in \mathbb{Z}_{>0}$

3. Notice that N might depend on ϵ.

Example 5.5. Prove $\lim_{n\to+\infty} \frac{1}{n^2} = 0$.

□

Start with $\epsilon > 0$ and $|s_n - 0| < \epsilon$. We want to solve for n in terms of ϵ:

$$\begin{aligned} |s_n-0| &= \left|\frac{1}{n^2}\right| = \frac{1}{n^2} \\ \frac{1}{n^2} < \epsilon &\iff 1 < \epsilon \cdot n^2 \\ &\iff \frac{1}{\epsilon} < n^2 \\ &\iff \frac{1}{\sqrt{\epsilon}} < n \end{aligned}$$

Proof. Let $\epsilon > 0$. Define $N = \frac{1}{\sqrt{\epsilon}}$. Then, $\forall n > N = \frac{1}{\sqrt{\epsilon}}$, we want to show $|s_n - s| < \epsilon$.

$$\begin{aligned} n &> \frac{1}{\sqrt{\epsilon}} \\ n^2 &> \frac{1}{\epsilon} \\ n^2\epsilon &> 1 \\ \epsilon &> \frac{1}{n^2} \\ \epsilon > \frac{1}{n^2} &= \left|\frac{1}{n^2} - 0\right| \\ \epsilon &> |s_n - 0| \end{aligned}$$

So by definition of the limit, $\frac{1}{n^2} \to 0$. Therefore

$$\lim_{n\to+\infty} \frac{1}{n^2} = 0$$

□

Example 5.6. Prove

$$\lim_{n\to+\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$$

□

Start with $\epsilon > 0$, and

$$\begin{aligned} \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| &< \epsilon \\ \left|\frac{(3n+1)7 - 3(7n-4)}{7(7n-4)}\right| &< \epsilon \\ \left|\frac{19}{7(7n-4)}\right| &< \epsilon \end{aligned}$$

Note that we can assume $n \geq 1$ here, and in general it is okay to consider $n > N$ for any $N \in \mathbb{N}$, because we are working in the limit where n is sufficiently large that this is true.

So when $n > 1$, we have

$$\begin{aligned} \frac{19}{7(7n-4} < \epsilon &\iff \frac{19}{7\epsilon} < 7n - 4 \\ &\iff \frac{19}{49\epsilon} + \frac{4}{7} < n \end{aligned}$$

Proof. Let $\epsilon > 0$. Define N $= \max\{1, \frac{19}{49\epsilon} + \frac{4}{7}\}$. Then, $\forall n > N$,

$$\begin{aligned} n &> \frac{19}{49\epsilon} + \frac{4}{7} \\ \iff \epsilon &> \frac{19}{7(7n-4)} \end{aligned}$$

Because $n > 1$, we know that $\frac{19}{7(7n-4)} > 0$, so

$$\epsilon > \frac{19}{7(7n-4)} = \left|\frac{19}{7(7n-4)}\right| = \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right|$$

Therefore by the definition of a limit,

$$\frac{3n+1}{7n-4} \to \frac{3}{7}$$

□

Example 5.7. Prove

$$\lim_{n\to+\infty} \frac{4n^3+3n}{n^3-6} = 4$$

□

We start with $\epsilon > 0$ and

$$\begin{aligned} \left|\frac{4n^3+3n}{n^3-6} - 4\right| &< \epsilon \\ \left|\frac{3n+24}{n^3-6}\right| &< \epsilon \\ \frac{3n+24}{n^3-6} &< \epsilon \qquad \text{when } n \geq 2 \end{aligned}$$

Here, it seems like we're stuck because it's difficult to invert this expression. To resolve this, we replace this expression with something similar that is at least as big.

$$\begin{aligned} 3n + 24 &\leq 3n + 24n = 27n \\ n^3 - 6 &\geq n^3 - \frac{1}{2}n^3 = \frac{n^3}{2} \\ \text{want } \frac{n^3}{2} \geq 2 \iff n^3 &\geq 12 \iff n \geq 3 \end{aligned}$$

Therefore, for $n \geq 3$, we have

$$\frac{3n+24}{n^3-6} \leq \frac{54}{n^2}$$

This is easily inverted for an inequality on n,

$$n > \sqrt{\frac{54}{\epsilon}}$$

Then, the usual proof can be set up.

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Lecture 6: Limit theorems

Lecturer: Michael Christianson *2 July* *Aditya Sengupta*

6.1 Proving a limit

In general, we want to prove a statement of the form

$$\lim_{n\to+\infty} s_n = s \in \mathbb{R}$$

To do this, we first "work backwards" to find N in terms of ϵ. Then, we start with $\epsilon > 0$ and $|s_n - s| < \epsilon$. Next, we want to simplify $|s_n - s|$ to solve for n in terms of ϵ in $|s_n - s| < \epsilon$. Tactics here:

(a) Algebraic manipulation (get common denominators, factors, etc).

(b) Can assume $n > a$ for any $a \in \mathbb{R}$.

(c) Can replace $|s_n - s|$ with any expression for n that is $\geq |s_n - s|$.

After this, we "solve" for n in terms of ϵ. We should get something of the form

$$n > f(\epsilon)$$

Finally, we can start the proof.

Proof. Let $\epsilon > 0$. Define $N = \max\{a, f(\epsilon)\}$. Then, $\forall n > N$,

$$n > f(\epsilon) \to \text{ reverse everything } \to |s_n - s| < \epsilon$$

Therefore, by definition of the limit, $\lim\limits_{n\to+\infty} s_n = s$. □

6.2 Proving a sequence diverges

1. By contradiction. Assume $\exists s \in \mathbb{R}\,\text{s.t.}\, s_n \to s$. Pick a specific $\epsilon > 0$. By assumption, $\exists N\,\text{s.t.}\,\forall n > N, |s_n - s| < \epsilon$. Use this inequality to get a contradiction. (This may need multiple values of ϵ.)
2. By a counterexample. Let $s \in \mathbb{R}$; we prove $s \neq \lim\limits_{n\to+\infty} s_n$. Pick a specific $\epsilon > 0$, show $\forall N, \exists n > N\,\text{s.t.}\,|s_n - s| \geq \epsilon$.

Example 6.1. Prove $a_n = (-1)^n$ diverges. □

We first do a proof by contradiction.

Proof. Suppose $a = \lim_{n\to\infty} a_n$ exists. By definition, with $\epsilon = 1$,

$$\exists N \text{ s.t. } |(-1)^n - a| = |a_n - a| < 1$$
$$\begin{cases} |1-a| & n \text{ even} \\ |1+a| & n \text{ odd} \end{cases} < 1$$

Therefore if such a limit exists, it satisfies $|1-a| < 1$ and $|1+a| < 1$. Combining these two and using the Triangle Inequality, we get

$$2 = |1+1| = |1-a+a+1| \leq |1-a| + |1+a| < 1+1 < 2$$

Therefore, $2 < 2$. But $2 = 2$.

Therefore a does not exist. □

Next, we do a proof by counterexample.

Proof. Let $a \in \mathbb{R}$. We show $a \neq \lim_{n\to+\infty} a_n$. There are two cases:

Case. $a \neq 1$. *Let* $0 < \epsilon < |1-a|$ *be any real number. Then,* $\forall N \in \mathbb{R}$, *pick* $n > N$ *even. Then*

$$|a_n - a| = |1-a| > \epsilon$$

Therefore $a \neq \lim_{n\to+\infty} a_n$ *by definition.*

Case. $a = 1$. *Let* $0 < \epsilon < |-1-a| = |1+a| = 2$, *i.e. pick any positive* $\epsilon < 2$. *Then,* $\forall N \in \mathbb{R}$, *pick* $n > N$ *odd. Then*

$$|a_n - a| = |-1-a| > \epsilon$$

Therefore $|a_n - a| > \epsilon$, *i.e.* a *is not the limit.*

□

6.3 Bounded sequences

Definition 6.1. *A sequence $(s_n)_{n=m}^{\infty}$ is bounded if the set $\{s_n \mid n \geq m\}$ is a bounded set.*

Remark 6.1. *A set $S \subset \mathbb{R}$ is bounded iff $\exists M \in \mathbb{R}\,\text{s.t.}\, |s| \leq M \forall s \in S$.*

Theorem 6.2. *Any convergent sequence is bounded.*

Proof. Let $(s_n)_{n=m}^{+\infty}$ be a convergent sequence, and let $s = \lim\limits_{n\to+\infty} s_n$. Fix $\epsilon > 0$. By definition of the limit, $\exists N \in \mathbb{Z}_{>0}\,\text{s.t.}\, \forall n > N$,

$$\begin{aligned} |s_n| - |s| \leq ||s_n| - |s|| \leq |s_n - s| < \epsilon \\ \implies |s_n| < |s| + \epsilon \end{aligned}$$

This proves that $\forall n > N$, the sequence is bounded. Only finitely many elements of the sequence remain. Therefore, let

$$M = \max\{|s_m|, |s_{m+1}|, \ldots, |s_N|, |s| + \epsilon\}$$

Then, $|s_n| \leq M \forall n$, so (s_n) is bounded. □

This is a useful result, because an unbounded sequence cannot converge.

Theorem 6.3. *Let (s_n) and (t_n) be convergent sequences.*

(a) $\lim\limits_{n\to+\infty}(s_n + t_n) = \lim\limits_{n\to+\infty} s_n + \lim\limits_{n\to+\infty} t_n$

(b) $\lim\limits_{n\to+\infty} s_n t_n = \lim\limits_{n\to+\infty} s_n \lim\limits_{n\to+\infty} t_n$

(c) If $s_n \neq 0 \forall n$ and $\lim\limits_{n\to+\infty} s_n \neq 0$, then

$$\lim_{n\to+\infty} \frac{t_n}{s_n} = \frac{\lim\limits_{n\to+\infty} t_n}{\lim\limits_{n\to+\infty} s_n}$$

Proof. (a) Let $\epsilon > 0$. We want to find some $N\,\text{s.t.}\, \forall n > N$,

$$\begin{aligned} |s_n + t_n - s - t| < \epsilon \\ |(s_n - s) + (t_n - t)| < \epsilon \end{aligned}$$

Because s_n and t_n converge, we know that

$$\begin{aligned}\exists N_1, N_2 \in \mathbb{R}\,\text{s. t.}\\ \forall n > N_1, |s_n - s| < \frac{\epsilon}{2}\\ \forall n > N_2, |t_n - t| < \frac{\epsilon}{2}\end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Then $\forall n > N$ we know $n > N_1$ and $n > N_2$. Therefore

$$\begin{aligned}|s_n + t_n - s - t| = |(s_n - s) + (t_n - t)| &\leq |s_n - s| + |t_n - t|\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

Therefore, $|s_n + t_n - s - t| < \epsilon$ and so $\lim\limits_{n\to+\infty}(s_n + t_n) = s + t$.

(b) Intuitively, we know that

$$\begin{aligned}|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st|\\ &\leq |s_n(t_n - t)| + |t(s_n - s)|\\ &= |s_n||t_n - t| + |t||s_n - s|\end{aligned}$$

We expect $|s_n|$ to be finite (because any convergent sequence is bounded) and $|t_n - t|$ to be arbitrarily small (because $t_n \to t$) and the same with the other two terms, i.e. the sum should go to zero.

Formally, let $\epsilon > 0$. (s_n) converges $\implies \exists M \in \mathbb{R}, M > 0\,\text{s. t.}\,|s_n| \leq M \forall n$. Then

$$\begin{aligned}\exists N_1, N_2 \in \mathbb{R}\,\text{s. t.}\\ \forall n > N_1, |t_n - t| < \frac{\epsilon}{2M}\\ \forall n > N_2, |s_n - s| < \frac{\epsilon}{2(|t| + 1)}\end{aligned}$$

where we take $|t| + 1$ so that we do not end up dividing by zero. Then, let $N = \max\{N_1, N_2\}$.

$$\begin{aligned}\forall n > N, |s_n t_n - st| &\leq |s_n||t_n - t| + |t||s_n - s|\\ &< M\frac{\epsilon}{2M} + (|t| + 1)\frac{\epsilon}{2(|t| + 1)}\\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

(c) This is difficult to explicitly prove. First, we want to show that $s \neq 0 \implies m = \inf\{|s_n| \mid \forall n\} > 0$. (See this in example 6, page 43 of Ross.)

Next, let $\epsilon > 0. \exists N \in \mathbb{R}\,\text{s. t.}\, n > N\,\text{s. t.}\,|s_n - s| < \epsilon| < \epsilon \cdot m \cdot |s|$. So $\forall n > N$,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s \cdot s_n}\right| = \frac{|s - s_n|}{|s| \cdot |s_n|}$$

We know from the statement on m above that $|s_n| \geq m$ and therefore

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{|s - s_n|}{|s| \cdot m} \leq \frac{\epsilon \cdot m \cdot |s|}{|s| \cdot m} = \epsilon$$

Therefore $\lim_{n \to +\infty} \frac{1}{s_n} = \frac{1}{s}$. Therefore we can consider the sequence $\frac{1}{s_n}$ and use the above part to show the limit,

$$\lim_{n \to +\infty} \frac{t_n}{s_n} = \lim_{n \to +\infty} t_n \cdot \frac{1}{s_n} = t \cdot \frac{1}{s} = \frac{t}{s}$$

$\square$

Theorem 6.4. *(a)* $\forall a \in \mathbb{R}, a_n = a$ *converges to* a.

(b) $\forall p > 0, \lim_{n \to +\infty} \frac{1}{n^p} = 0$.

(c) $\forall a \in \mathbb{R}\,\text{s.t.}\, |a| < 1, \lim_{n \to +\infty} a^n = 0$.

(d) $\lim_{n \to +\infty} n^{1/n} = 1$.

(e) $\forall a > 0, \lim_{n \to +\infty} a^{1/n} = 1$.

Example 6.2. Show $\lim_{n \to +\infty} \frac{n^3+6n^2+7}{4n^2+3n-4} = \frac{1}{4}$.

$\square$

Proof.

$$\frac{n^3 + 6n^2 + 7}{4n^2 + 3n - 4} = \frac{1 + \frac{6}{n} + \frac{7}{n^2}}{4 + \frac{3}{n} - \frac{4}{n^2}}$$

Because $\forall p > 0, \lim_{n \to +\infty} \frac{1}{n^p} = 0$, we see that the limit of the numerator is $1 + 0 + 0 = 1$ and the limit of the denominator is $4 + 0 + 0 = 4$. Therefore, the limit of the quotient is the quotient of the limits, so the limit is $\frac{1}{4}$. $\square$

6.4 Diverging to $\pm\infty$

Definition 6.2. *Let* (s_n) *be a sequence. We say* (s_n) *diverges to* $+\infty$*, i.e.* $\lim_{n \to +\infty} s_n = +\infty$ *if*

$$\forall M > 0, \exists N \in \mathbb{R}\,\text{s.t.}$$
$$\forall n > N, s_n > N, s_n > M$$

and we say (s_n) diverges to $-\infty$ if

$$\begin{aligned} &\forall M < 0, \exists N \in \mathbb{R}\,\mathrm{s.\,t.} \\ &\quad \forall n > N, s_n < M \end{aligned}$$

Remark 6.5. *We say $\lim_{n\to+\infty} s_n$ exists if s_n converges or if it diverges to $\pm\infty$.*

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Lecture 7: Divergence, monotone and Cauchy sequences

Lecturer: Michael Christianson *3 July* *Aditya Sengupta*

7.1 Divergence

Definition 7.1. *We say* (s_n) *diverges to* $+\infty$ *if* $\forall M > 0, \exists N \in \mathbb{R}\,\text{s.t.}\,\forall n > N, s_n > M$.

Example 7.1. $\lim_{n \to +\infty} n^2 = +\infty$

□

Example 7.2. $\lim_{n \to +\infty} (-n) = -\infty$

□

Example 7.3. $\lim_{n \to +\infty} (-1)^n n$ DNE.

□

Example 7.4. $\lim_{n \to +\infty} \sqrt{n} + 7 = +\infty$

□

To prove this, we start with $M > 0$ and $s_n > M$.

$$\begin{aligned} \sqrt{n} + 7 &> M \\ \sqrt{n} &> M - 7 \\ n &> (M-7)^2 \text{ if } M > 7 \end{aligned}$$

For $M < 7$, $\sqrt{n} + 7 > M \forall n$, so this doesn't affect the proof.

Proof. Let $M > 0$. If $M < 7$ then $\sqrt{n} + 7 > M \forall n$, so we can choose $N = 0$. If $M \geq 7$, pick $N = (M-7)^2$. Then

$$\begin{aligned} \forall n > N, n > (M-7)^2 \\ \sqrt{n} > \sqrt{(M-7)^2} = |M-7| = M-7 \\ \sqrt{n} + 7 > M \end{aligned}$$

Therefore, by definition, $\sqrt{n} + 7 \to +\infty$. □

Example 7.5. Prove $\lim\limits_{n \to +\infty} \frac{n^2+3}{n+1} = +\infty$. □

Start with $M > 0$:

$$\frac{n^2+3}{n+1} > M$$

We replace the expression with something simpler that is less than or equal to the sequence, but still greater than M. We make the numerator smaller by setting it to n^2, and the denominator bigger by setting it to $2n$ (as long as $n \geq 1$). So

$$\begin{aligned} \frac{n^2+3}{n+1} \geq \frac{n^2}{2n} \\ \frac{n}{2} > M \implies n > 2M \end{aligned}$$

Proof. Let $M > 0$. Define $N = \max\{1, 2M\}$.

$$\forall n > N, n > 2M \implies \frac{n^2}{2n} > M$$

Also, $n > 1$, so

$$n^2 + 3 > n^2, n + 1 \leq 2n \therefore \frac{n^2+3}{n+1} \geq \frac{n^2}{2n} > M$$

Therefore, by definition, $\lim\limits_{n \to +\infty} \frac{n^2+3}{n+1} = +\infty$. □

Theorem 7.1. *Let (s_n) and (t_n) be sequences such that $s_n \to +\infty$.*

(a) If $\lim_{n\to+\infty} t_n > -\infty$ *then* $\lim_{n\to+\infty}(s_n + t_n) = +\infty$.

(b) If $\lim_{n\to+\infty} t_n > 0$, *then* $\lim_{n\to+\infty}(s_n \cdot t_n) = +\infty$.

(c) If $t_n > 0 \forall n$, *then* $t_n \to +\infty \iff \frac{1}{t_n} \to 0$.

7.2 Monotone and Cauchy sequences

Definition 7.2. *Let* (s_n) *be a sequence of real numbers.*

1. (s_n) *is increasing (or non-decreasing) if* $s_n \leq s_{n+1} \forall n$.
2. (s_n) *is decreasing (or non-increasing) if* $s_n \geq s_{n+1} \forall n$.
3. (s_n) *is monotonic (or monotone) if it is increasing or decreasing.*

Example 7.6. $a_n = 1 - \frac{1}{n}$ is increasing. □

Example 7.7. $b_n = \frac{1}{n^2}$ is decreasing. □

Example 7.8. $c_n = (-1)^n n$ is not monotonic. □

Theorem 7.2. *Let* $(s_n)_{n\in\mathbb{N}}$ *be monotonic. Define* $S = \{s_n \mid n \in \mathbb{N}\}$*; then*

(a) if (s_n) *is increasing and bounded, then* (s_n) *converges to* $\sup S$.

(b) If (s_n) *is decreasing and bounded, then* (s_n) *converges to* $\inf S$.

(c) If (s_n) *is increasing and unbounded, then* $\lim_{n\to+\infty} s_n = +\infty$.

(d) If (s_n) *is decreasing and unbounded, then* $\lim_{n\to+\infty} s_n = -\infty$

Remark 7.3. (s_n) *increasing* $\implies s_0 \leq s_1 \leq s_2 \leq \dots$, *which tells us that* s_0 *is a lower bound for* S. *If* (s_n) *is bounded, then* $\sup S \in \mathbb{R}$ *is "right next to"* S. (s_n) *will not converge to* s *if the sequence ends up oscillating below* $\sup S$.

Proof. (a) Suppose (s_n) is increasing and bounded. Let $s = \sup S$. Let $\epsilon > 0$. $s - \epsilon$ is not an upper bound for S, i.e.

$$\exists N \,\mathrm{s.t.}\, s_N > s - \epsilon$$

Since (s_n) is increasing, $\forall n > N$,

$$\begin{aligned} s \geq s_n \geq s_N > s - \epsilon \\ \epsilon > 0 \geq s_n - s > -\epsilon \iff |s_n - s| < \epsilon \end{aligned}$$

Therefore $s_n \to s$.

(b) Almost the same with inf instead of sup.

(c) If (s_n) is unbounded and increasing, $\forall M > 0, M$ is not an upper bound for S. Therefore $\exists N \,\mathrm{s.t.}\, s_N > M$. (s_n) is increasing, so $s_n \geq s_N > M$. So $\forall n > N$, $\lim_{n \to +\infty} s_n = +\infty$.

$\square$

Example 7.9. Define (s_n) by $s_1 = 5, s_n = \frac{s_{n-1}^2 + 5}{2s_{n-1}}$. We claim that (s_n) is bounded below by 0.

Proof. We prove this by induction.

<u>Base case:</u> $s_1 = 5 > 0$. <u>Inductive step:</u> Suppose $s_n > 0$ for some $n \geq 1$. Then

$$s_{n+1} = \frac{s_n^2 + 5}{2s_n} > 0$$

because $s_n > 0$. $\square$

We claim that s_n is decreasing. (ref. Ross section 10ish)

$\square$

Definition 7.3. *A sequence is Cauchy if* $\forall \epsilon > 0, \exists N \in \mathbb{R} \,\mathrm{s.t.}\, m, n > N \implies |s_m - s_n| < \epsilon$.

Lemma 7.4. *Any convergent sequence is Cauchy.*

Proof. Let (s_n) be a convergent sequence, and let

$$s = \lim_{n \to +\infty} (s_n) \forall \epsilon > 0, \exists N \,\mathrm{s.t.}\, |s_n - s| < \frac{\epsilon}{2} \forall n$$

Then, $\forall m, n > N$,

$$|s_m - s_n| = |(s_m - s) + (s - s_n)| \qquad \leq |s_m - s| + |s_n - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\square$

It turns out that any Cauchy sequence is convergent as well, but this is difficult to prove.

Lemma 7.5. *Cauchy sequences are bounded.*

Proof. Let (s_n) be Cauchy. Set $\epsilon = 1$. Then

$$\exists N \in \mathbb{Z}_{>0} \,\text{s. t.}\, \forall m, n > N, |s_n - s_m| < 1$$

In particular,

$$\begin{aligned} |s_n| - |s_{N+1}| \le |s_n - s_{N+1}| < 1 \forall n > N \\ |s_n| < 1 + |s_{N+1}| \end{aligned}$$

Now, let $M = \max\{|s_1|, \ldots, |s_N|, 1 + |s_{N+1}\}$. Then $|s_n| \le M \forall n$, so (s_n) is bounded. $\square$

We're still not ready to prove that a Cauchy sequence necessarily converges; to do that, we need to build the machinery of $\limsup$ and $\liminf$. The idea of this is to take the limits of monotonic sequences made up from (s_n).

Let $u_n = \sup\{s_n \mid n > N\}$ and $l_n = \inf\{s_n \mid n > N\}$.

Definition 7.4.

$$\limsup(s_n) = \lim_{n \to +\infty} u_n = \lim_{n \to +\infty} \sup\{s_n \mid n > N\}$$
$$\liminf(s_n) = \lim_{n \to +\infty} l_n = \lim_{n \to +\infty} \inf\{s_n \mid n > N\}$$

Note that $\limsup(s_n) \le \sup\{s_n \mid n \in \mathbb{N}\}$ but they are not "usually" equal.

Intuitively, we can think of $\limsup$ as the number that infinitely many of the s_ns get close to.

Remark 7.6. $\liminf s_n \le \limsup s_n$ *because* $l_n \le u_n \forall n$.

Theorem 7.7. *Let* (s_n) *be bounded.*

(a) *If* $\lim_{n \to +\infty} s_n$ *exists, then* $\lim_{n \to +\infty} s_n = \limsup s_n = \liminf s_n$.

(b) *Conversely, if* $\limsup s_n = \liminf s_n$ *then* $\lim_{n \to +\infty} s_n$ *exists.*

Proof. We prove part (b); (a) is proved in Ross. For all n, if we define $l_n = \inf\{s_{n+1}, s_{n+2}, \ldots\}$ and $u_n = \sup\{s_{n+1}, s_{n+2}, \ldots\}$, then

$$l_n \le s_{n+1} \le u_n$$

Taking limits,

$$\lim l_n = \liminf s_n = \limsup s_n = \lim u_n$$

So by the squeeze lemma, $\lim_{n \to +\infty} s_n = \lim_{n \to +\infty} s_{n+1} = \liminf s_n = \limsup s_n$. $\square$

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Lecture 8: Subsequences

Lecturer: Michael Christianson *8 July* *Aditya Sengupta*

Theorem 8.1. *A sequence is convergent if and only if it is Cauchy.*

Theorem 8.2. *Let* (s_n) *be a sequence. Then* $\lim_{n\to+\infty} s_n$ *exists iff* $\limsup s_n = \liminf s_n$ *and in this case* $\lim_{n\to+\infty} s_n = \limsup s_n = \liminf s_n$

Definition 8.1. *Let* $(s_n)_{n\in\mathbb{N}}$*; a subsequence is a sequence of the form* $(s_{n_k})_{k\in\mathbb{N}}$*, where* $0 \leq n_0 < n_1 < n_2 < \dots$ *are natural numbers.*

Example 8.1. Let $(s_n) = (-1)^n n^2$.

$$s_n = (0, -1, 4, -9, 16, -25, \dots)$$

The subsequence $(s_{2n})_{n\in\mathbb{N}}$ is

$$(s_{2n}) = (0, 4, 16, 36, 64, \dots)$$

and the subsequence $(s_{2n+1})_{n\in\mathbb{N}}$ is

$$(s_{2n+1}) = (-1, -9, -25, -49, \dots)$$

$\lim_{n\to+\infty} s_n$ does not exist, but $\lim_{n\to+\infty} s_{2n} = +\infty$ and $\lim_{n\to+\infty} s_{2n+1} = -\infty$.

□

Example 8.2. Let $t_n = \frac{1}{n}, n \in \mathbb{Z}_{>0}$; we see that $(t_n) \to 0$. We can take subsequences such as $(t_{n+5})_{n\in\mathbb{Z}_{>0}} = (t_6, t_7, t_8, \dots)$, or $(t_{3n})_{n\in\mathbb{Z}_{>0}} = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \dots\right)$

□

Proposition 8.3. *Let* (s_n) *be a sequence such that* $\lim_{n\to+\infty} s_n$ *exists (i.e. is in* $\mathbb{R}$ *or is* $\pm\infty$*). For all subsequences* (s_{n_k}) *of* (s_n)*,* $\lim_{k\to\infty} s_{n_k} = \lim_{n\to+\infty} s_n$*.*

Proof. First, we claim $n_k \geq k \forall k$. We show this by induction:

<u>Base case:</u> $n_0 \geq 0$ by definition. <u>Inductive step:</u> if $n_k \geq k$, $n_{k+1} > n_k \geq k \implies n_{k+1} \geq k+1$.

Case. $\lim_{n\to+\infty} s_n = s \in \mathbb{R}$.

Then $\forall\epsilon > 0, \exists N$ such that $|s_n - s| < \epsilon \forall n > N$. Then, $\forall k > n, n_k \geq k > N$, so $|s_{n_k} - s| < \epsilon$. Therefore $\lim_{k\to+\infty} s_{n_k} = s$.

$\square$

Theorem 8.4. *Let (s_n) be a sequence.*

(a) For any $t \in \mathbb{R}$, there exists a subsequence of (s_n) that converges to t iff $\forall\epsilon$, the set

$$S_\epsilon = \{n \in \mathbb{N} \mid |s_n - t| < \epsilon\}$$

is infinite. (In this case, the subsequence can be taken to be monotonic.)

(b) If (s_n) is not bounded above, there exists a subsequence diverging to $+\infty$.

(c) If (s_n) is not bounded below, there exists a subsequence diverging to $-\infty$.

Proof. (a) The rough idea of the proof is that there is some sequence (s_n) that we place on a number line. We want to show that infinitely many of these s_ns are arbitrarily close to t. If $\exists(s_{n_k})$ converging to t, then $\forall\epsilon > 0, \exists N \,\text{s.t.}\, \forall n > N, |s_n - s| < \epsilon \implies n \in S_\epsilon \forall n > N$. This proves the theorem in the forward direction. In the other direction, let $k \in \mathbb{N}$. The set $S_{1/k}$ is infinite, therefore $\exists n_k \,\text{s.t.}\, s_{n_k} \in S_{1/k}$. Then $|s_{n_k} - t| < \frac{1}{k}$.

(b) Suppose (s_n) is not bounded above. $\forall k \in \mathbb{N}, k$ is not an upper bound for $\{s_n \mid n \in \mathbb{N}\} \implies \exists n_k \,\text{s.t.}\, s_{n_k} \geq k$. Then $(s_{n_k}) \to +\infty$. $\forall M > 0$, let $N = M$. Then $\forall k > N, s_{n_k} \geq K > N = M$, so $\lim_{k\to+\infty} s_{n_k} = +\infty$.

$\square$

Theorem 8.5. *(Bolzano-Weierstrass theorem): every bounded sequence of real numbers has a convergent subsequence.*

The idea is that we want to prove that there exists a monotonic subsequence. Then, a bounded monotonic sequence converges.

Definition 8.2. *Let (s_n) be a sequence. A subsequential limit is some $a \in \mathbb{R}\bigcup\{\pm\infty\}$ such that a is the limit of a subsequence of (s_n).*

Example 8.3. If $\lim_{n\to+\infty} s_n$ exists, every subsequence has the same limit, so the set of subsequential limits is $\{\lim_{n\to+\infty} s_n\}$.

$\square$

Example 8.4. $s_n = (-1)^n n^2$ has subsequential limits $\{+\infty, -\infty\}$ from the subsequences (s_{2n}) and (s_{2n+1}).

$\square$

We claim that in the second example, $\pm\infty$ are the only subsequential limits. We can show this as follows: for all subsequences (s_{n_k}), the subsequence is unbounded. So, some subsequence of (s_{n_k}) diverges to either $\pm\infty$. So if $\lim_{n\to+\infty}(s_{n_k})$ exists, then $\lim_{n\to+\infty} s_{n_k} = \pm\infty$ because every subsequence has the same limit.

Lemma 8.6. *Let (s_n) be a sequence.*

(a) There exists a subsequence whose limit is $\limsup s_n$.

(b) There exists a subsequence whose limit is $\liminf s_n$.

Theorem 8.7. *Let (s_n) be a sequence of real numbers; then define*

$$S = \{\textit{subsequential limits of } (s_n)\} = \{a \mid \exists (s_{n_k})\ \mathrm{s.\,t.} \lim_{k\to+\infty} s_{n_k} = a\}$$

(a) $\limsup s_n, \liminf s_n \in S$. *In particular,* $S \neq \varnothing$.

(b) $\sup S = \limsup S, \inf S = \liminf S$.

(c) $\lim_{n\to+\infty} s_n$ *exists* $\iff$ S *contains a single element.*

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Lecture 9: Subsequences, lim sup and lim inf

Lecturer: Michael Christianson *9 July* *Aditya Sengupta*

9.1 More about subsequences

For the midterm, it is important to understand a proof of the following:

Proposition 9.1. *If $\lim_{n\to+\infty} s_n$ exists, then every subsequence of (s_n) has a limit $\lim_{n\to+\infty} s_n$.*

To prove this, we showed that for any (s_{n_k}), $n_k \geq k \forall k$. Then we applied the definition of a limit.

Proposition 9.2. *If (s_n) is not bounded above, there exists a subsequence (s_{n_k}) diverging to $+\infty$.*

Proof. $\forall k$, pick n_k such that s_{n_k} does what we want, i.e. cause s_{n_k} to be big when k is big. The simplest choice here is $s_{n_k} \geq k$. Then k is not an upper bound on $\{s_n \mid n \in \mathbb{N}\}$, that is, there exists n_k such that $s_{n_k} > k$. Now apply the definition of a limit to show that $\lim_{k\to+\infty} s_{n_k} = +\infty$. □

Important statements that may be used in other proofs, but which we won't have to prove:

Theorem 9.3. *(Bolzano-Weierstrass) Every bounded sequence has a convergent subsequence.*

Theorem 9.4. *Let (s_n) be a sequence, and let S be the set of all subsequential limits.*

(a) $\limsup s_n, \liminf s_n \in S$.

(b) $\sup S = \limsup s_n, \inf S = \liminf s_n$

(c) $\lim_{n\to+\infty} s_n$ *exists iff S contains one element.*

Proof. (c) If $\lim_{n\to+\infty} s_n$ exists, then every subsequence has limit $\lim_{n\to+\infty} s_n$, therefore $S = \{\lim_{n\to+\infty} s_n\}$. If S contains one element, $\limsup s_n = \liminf s_n$, therefore $\lim_{n\to+\infty} s_n$ exists

□

Example 9.1. Let $s_n = 1 + (-1)^n$, $t_n = n\cos\left(\frac{n\pi}{3}\right)$.

(a) What are the subsequences of (s_n) and (t_n)?

$(s_{2k})_{k\in\mathbb{N}} = (2, 2, 2, \dots)$, which converges to 2, and $(s_{2k+1})_{k\in\mathbb{N}} = (0, 0, 0, \dots)$, so $S = \{0, 2\}$ is the set of all subsequential limits.

For all subsequences (s_{n_k}) of (s_n), either infinitely many of n_k are even or infinitely many of n_k are odd. So, there exists a subsequence of (s_{n_k}) converging to either 0 or 2. If $\lim_{n\to+\infty} s_{n_k}$ exists, every subsequence has the same limit, so

$\lim_{n\to+\infty} s_{n_k} \in \{0, 2\}$.

For the subsequences of t_n, we do not see any constant terms; any subsequence grows or decays to $\pm\infty$. Since t_n is not bounded above or below, we see that $\pm\infty$ are subsequential limits. So $T = \{, \pm\infty\}$.

(b) What are lim sup and lim inf of both?

$\limsup s_n = \sup S = 2$ and $\liminf s_n = \inf S = 0$; $\limsup t_n = \sup T = +\infty$ and $\liminf t_n = \inf T = -\infty$.

(c) Prove (s_n) and (t_n) have no limit.

The sets S and T have more than one element, but this is not the case if $\lim_{n\to+\infty} s_n$ or $\lim_{n\to+\infty} t_n$ exist.

(d) Which of (s_n) and (t_n) are bounded?

We expect that (s_n) is bounded and (t_n) is not. If a sequence is not bounded above, we expect that there exists a subsequence such that the limit of that subsequence is $+\infty$, i.e. $+\infty$ is in the set of subsequential limits. The same holds for being bounded below and the limit being $-\infty$. Therefore, because $\pm\infty \in T$, (t_n) is not bounded, and because $\pm\infty \notin S$, (s_n) is not bounded.

$\square$

9.2 Using sequence properties

We can show that the rational numbers are countable in the form of a sequence argument,

Proposition 9.5. *There exists a sequence* $(r_n)_{n\in\mathbb{N}}$ *such that* $\forall q \in \mathbb{Q}, \exists! n \,\text{s.t.}\, r_n = q$

Proof. Cantor diagonalization. $\square$

We can extend this to the following,

Proposition 9.6. $\forall a \in \mathbb{R}, \epsilon > 0, \exists$ *infinitely many* $q \in \mathbb{Q} \,\text{s.t.}\, q \in (a - \epsilon, a + \epsilon)$.

Therefore the set

$$S_\epsilon = \{q \in \mathbb{Q} \mid |q - a| < \epsilon\} \equiv \{n \in \mathbb{N} \mid |r_n - a| < \epsilon\}$$

is infinite (there is a bijection between the two sets). Therefore, there exists a subsequence of (r_n) converging to a. For all k, there exists an $r_{n_k} \in \mathbb{Q}$ such that $a - \frac{1}{n} < r_{n_k} < a$ by denseness of $\mathbb{Q}$. So every real number is a subsequential limit of (r_n). Also, (r_n) is not bounded above or below, so $\pm\infty$ are also subsequential limits.

9.3 $\limsup$ and $\liminf$

Recall that for a sequence (s_n), if it is bounded above, its $\limsup$ is $\limsup s_n = \lim_{N\to\infty}(\sup\{s_n \mid n > N\})$ and $\liminf$ is defined similarly if it is bounded below. If (s_n) is not bounded above, $\limsup s_n = +\infty$ and if it is not bounded below, $\liminf s_n = -\infty$.

Let $l_N = \inf\{s_n \mid n > N\}$ and $u_N = \sup\{s_n \mid n > N\}$. We see that $l_N \leq u_N \forall N$, and therefore $\lim_{N\to+\infty} l_N \leq \lim_{N\to+\infty} u_N$, so $\liminf s_n \leq \limsup s_n$.

For all N, $\{s_n \mid n > N+1\} \subseteq \{s_n \mid n > N\}$, therefore $u_{N+1} \leq u_N$ and similarly $l_{N+1} \geq l_N$. So u_N is decreasing, i.e. $\lim_{N\to+\infty}$ exists, and l_N is increasing, so $\lim_{N\to+\infty} l_N$ exists.

$$\limsup s_n = \lim_{N\to+\infty} u_N \leq u_1 = \sup\{s_n \mid n > 1\} < +\infty$$

Therefore if (s_n) is bounded above, $\limsup s_n \in \mathbb{R}$ or is $-\infty$. Similarly, if (s_n) is bounded below, $\liminf s_n \geq l_1 = \inf\{s_n \mid n > 1\} > -\infty$.

Theorem 9.7. *For all decreasing sequences (s_n), $\lim_{n\to+\infty} s_n = \inf\{s_n \mid n \in \mathbb{N}\}$. For all increasing sequences (s_n), $\lim_{n\to+\infty} s_n = \sup\{s_n \mid n \in \mathbb{N}\}$.*

An alternative way of dealing with $\limsup$ is

$$\begin{aligned}\limsup s_n &= \lim_{N\to+\infty} u_N = \inf\{u_N\} = \inf\{\sup\{s_n \mid n > N\} \mid N \in \mathbb{N}\}\\ \liminf s_n &= \lim_{N\to+\infty} l_N = \sup\{\inf\{s_n \mid n > N\} \mid N \in \mathbb{N}\}\end{aligned}$$

Theorem 9.8. *Let (s_n) be a sequence. Then, $\lim_{n\to+\infty} s_n$ exists iff $\limsup s_n = \liminf s_n$ and in that case*

$$\limsup s_n = \liminf s_n = \lim_{n\to+\infty} s_n$$

Intuitively, it is important to understand that $\limsup s_n$ and $\liminf s_n$ are like $\lim_{n\to+\infty} s_n$ but are always defined.

Theorem 9.9. *Let (s_n) be a sequence and let S be the set of its subsequential limits. Then*

(a) $\limsup s_n, \liminf s_n \in S$

(b) $\sup S = \limsup s_n, \inf S = \liminf s_n$

(c) If for all subsequences s_{n_k} of (s_n) such that its limit exists, we have $\lim_{k\to+\infty} s_{n_k} \leq \alpha \in \mathbb{R}$, then $\limsup s_n \leq \alpha$.

This theorem has been written in lecture like three times, so I may go back and clean this up.

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Lecture 10: Ratios and Roots, Review

Lecturer: Michael Christianson *10 July* *Aditya Sengupta*

10.1 $\limsup$ and $\liminf$ combinations

Proposition 10.1. *Let (s_n) be a convergent sequence and write $s = \lim_{n\to+\infty} s_n$.*

(a) For all sequences (t_n), $\limsup(s_n + t_n) = s + \limsup t_n$.

(b) If $s > 0$, then for all sequences (t_n), $\limsup(s_n t_n) = s \cdot \limsup t_n$.

(c) If $s < 0$, then for all sequences (t_n), $\limsup s_n t_n = s \cdot \liminf t_n$

Proof. (of (a)): We claim $\limsup(s_n + t_n) \geq s + \limsup t_n$. First, there exists a subsequence (t_{n_k}) such that $\lim_{k\to+\infty} t_{n_k} = \limsup t_n$. Then $(s_{n_k}) \to s$, so $s > -\infty$ implies

$$\lim_{n\to+\infty}(s_{n_k} + t_{n_k}) = \lim_{n\to+\infty} s_{n_k} s_{n_k} + \lim_{n\to+\infty} t_{n_k} = s + \limsup t_n$$

Therefore $\limsup(s_n + t_n) \geq \lim_{n\to+\infty}(s_{n_k} + t_{n_k}) = s + \limsup t_n$.

Next, we show that $\limsup(s_n + t_n) \leq s + \limsup t_n$. It suffices to show that there exists a subsequence $(s_{n_k} + t_{n_k})$ of $(s_n + t_n)$ whose limit exists, such that

$$\lim_{k\to+\infty}(s_{n_k} + t_{n_k}) \leq s + \limsup t_n$$

Let $(s_{n_k} + t_{n_k})$ be a subsequence whose limit exists. Then

$$\begin{aligned}\lim_{k\to+\infty} t_{n_k} &= \lim_{k\to+\infty}(s_{n_k} + t_{n_k} - s_{n_k}) \\ &\lim_{k\to+\infty}(s_{n_k} + t_{n_k}) - \lim_{k\to+\infty} s_{n_k}\end{aligned}$$

So $\lim t_{n_k}$ exists, and $\limsup t_n \geq \lim t_{n_k}$. Add $s = \lim s_{n_k}$ to both sides, and we get

$$s + \limsup t_n \geq s + \lim_{k\to+\infty} t_{n_k} = \lim s_{n_k} + \lim t_{n_k} = \lim_{k\to+\infty}(s_{n_k} + t_{n_k})$$

$\square$

Proof. (of c given b): consider the sequence $(-s_n)$; $\limsup s_n = -\liminf(-s_n)$.

Now, $(-s_n) \to -s > 0$ because $s < 0$. So, applying part (b) to $(-s_n)$ and $(-t_n)$, we get

$$\limsup s_n t_n = \limsup((-s_n)(-t_n)) = -s \limsup(-t_n) = s \liminf t_n$$

$\square$

10.2 Ratios and Roots

Theorem 10.2. *Let (s_n) be a sequence such that $s_n \neq 0$ $\forall n$. Then*

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}|\right|$$

Proof. We show the first part of this; let $\alpha = \liminf |s_n|^{1/n}$ and $L = \liminf \left|\frac{s_{n+1}}{s_n}\right|$. If $L = -\infty$, $L \leq \alpha$. So we may assume $L > -\infty$.

If $\forall L' < L, L' < \alpha$, then $L \leq \alpha$, because α is then an upper bound on $(-\infty, L)$ and the supremum of $(-\infty, L)$ is L. Therefore α must be greater than or equal to the supremum, i.e. $L \leq \alpha$.

Let $L' < L$; we show $L' < \alpha$. There exists a subsequence of $|s_n|^{1/n}$ convergent to α. But $|s_n|^{1/n} \geq 0$ for all n, so $\alpha \geq 0$. So if $L' < 0$, $\alpha > L'$ and we are done. Therefore, consider $L' \geq 0$.

By definition,

$$L' < L = \lim_{N\to+\infty} \inf\{\left|\frac{s_{n+1}}{s_n}\right| \mid n > N\}$$

So $\exists N \in \mathbb{Z}_{>0}$ s. t. $L' < \inf\{\left|\frac{s_{n+1}}{s_n}\right| \mid n > N\}$. So, $\forall n > N$, we have

$$L' < \left|\frac{s_{n+1}}{s_n}\right|$$

Now $\forall n > N$,

$$\begin{gathered}|s_n| = \left|\frac{s_n}{s_{n-1}}\right| \cdot \left|\frac{s_{n-1}}{s_{n-2}}\right| \cdot \ldots \left|\frac{s_{N+1}}{s_N}\right| \cdot |s_N| \\ \implies |s_n| > (L')^{n-N}\end{gathered}$$

This last step uses the fact that $L' > 0$.

Now, let $a = (L')^{-N}|s_N|$. Then $a \in \mathbb{R}$, and

$$|s_n| > (L')^n a \implies L' \cdot a^{1/n} < |s_n|^{1/n}$$

As $n \to +\infty, a^{1/n} \to 1$. Therefore $\liminf |s_n|^{1/n} > L' \lim_{n\to\infty} a^{1/n}$. $\square$

10.3 Review

A list of important things:

- Proofs by induction
- Irrational numbers, rational root theorem
- Definitions of supremum and infimum
- Completeness axiom
- Archimedean property
- Denseness of $\mathbb{Q}$
- How to work with plus or minus infinity: don't do $\infty - \infty$, $\frac{\infty}{\infty}$, $\infty \cdot 0$.
- Combining limits: sums, products, etc
- Definition of finite limits (epsilon definition)
- Definition of infinite limits (distance greater than any finite M)
- Limit theorems for convergent sequences
- Limit theorems for infinite limits
- Examples of limits (theorem 9.7)
- Proving limits using limit theorems
- Types of sequences, subsequences, $\limsup$ and $\liminf$
- Monotonic and Cauchy sequences
- Subsequences
- Know the theorems! (ex 10.10)
- Definitions of lim sup and lim inf
- Theorems in sections 10, 11

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Lecture 12: Series

Lecturer: Michael Christianson *16 July* *Aditya Sengupta*

12.1 Comparison, Root, Ratio Tests

Proposition 12.1. *Let $\sum a_n$ and $\sum b_n$ be series.*

(a) If $|b_n| \leq a_n\ \forall n$ and $\sum a_n$ converges, then $\sum b_n$ converges absolutely.

(b) If $b_n \geq a_n\ foralln$ and $\sum a_n = +\infty$, then $\sum b_n = +\infty$.

Theorem 12.2. *(Root Test) Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$.*

(a) If $\alpha < 1$, then $\sum a_n$ is absolutely convergent.

(b) If $\alpha > 1$, then $\sum a_n$ diverges.

(c) If $\alpha = 1$, then the test is inconclusive.

Corollary 12.3. *(Ratio Test) Let $\sum a_n$ be a series.*

(a) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ converges absolutely.

(b) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ diverges.

(c) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$, then the test is inconclusive.

Proof. From section 12, we know that

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{1/n} \leq \limsup |a_n|^{1/n} \leq \left|\frac{a_{n+1}}{a_n}\right|$$

Therefore (a) and (b) follow from this inequality and the Root Test.

For part (c), let $a_n = \frac{1}{n^p}$ for some $p > 0$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^p}{(n+1)^p} = \left(\frac{n}{n+1}\right)^p$$
$$\lim \left(\frac{n}{n+1}\right)^p = 1^p = 1$$
$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \limsup \left|\frac{a_{n+1}}{a_n}\right| = \liminf \left|\frac{a_{n+1}}{a_n}\right|$$

But $\sum a_n$ diverges when $p = 1$ and converges when $p = 2$. $\square$

Remark 12.4. *The Root Test works when the Ratio Test does, and the Root Test sometimes works when the Ratio Test doesn't. But if the limit of the ratio of successive terms is 1, then both give no information.*

A few useful limits to know for the Root Test are $\lim_{n\to+\infty} n^{1/n} = 1$, $\lim_{n\to+\infty} a^{1/n} = 1$ for all $a > 0$, and $\lim_{n\to+\infty} (n!)^{1/n} = +\infty$ (which can be proved by the Ratio Test).

Example 12.1. Consider the series $\sum \frac{2^n}{n!}$. $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{n+1}$, which in the limit is 0. Therefore by the Ratio Test, this series converges.

□

Example 12.2. Consider the series $\sum_{n=0}^{\infty} n^{-2+(-1)^n} = 0 + 1 + \frac{1}{2} + \frac{1}{3^3} + \frac{1}{4} + \frac{1}{5^3} + \dots$. The Ratio Test gives us

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{-2+(-1)^{n+1}}}{n^{-2+(-1)^n}} = \begin{cases} \frac{n}{(n+1)^3} & n \text{even} \\ \frac{n^3}{n+1} & n \text{odd} \end{cases}$$

So the series of ratios of terms has a subsequence with limit 0 and a subsequence with limit $+\infty$. Therefore the lim inf is less than 1 and the lim sup is greater than 1, meaning the ratio test cannot give any information.

We consider the Root Test:

$$\begin{aligned} |a_n|^{1/n} &= (n^{-2} \cdot n^{(-1)^n})^{1/n} \\ &= (n^2)^{1/n} \cdot (n^{(-1)^n})^{1/n} \\ &= \left(\frac{1}{n^{1/n}}\right)^2 \cdot \left(n^{1/n}\right)^(- 1)^n \end{aligned}$$

which in the limit is 1, so the root test gives no information.

□

Example 12.3. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$. This looks roughly like $\sum \frac{1}{n^2}$, so we use the comparison test:

$$\left|\frac{(-1)^n}{n^2+1}\right| = \frac{1}{n^2+1} \le \frac{1}{n^2} \ \forall n$$

Therefore, $\sum \frac{1}{n^2}$ converges $\implies \sum \frac{(-1)^n}{n^2+1}$ converges by the comparison test. $\square$

12.2 Integral Test

Consider $\sum \frac{1}{n}$; we can visually see by drawing the coarse Riemann sum that the sum is greater than the corresponding integral $\int_1^{n+1} \frac{dx}{x}$. This integral evaluates to $\ln(n+1)$. Therefore, if we define $s_n = \sum_{k=1}^n \frac{1}{k}$, then $s_n \geq \ln(n+1)\ \forall n$, and $\lim\limits_{n\to+\infty} \ln(n+1) = +\infty$. (for all $M > 0$, let $N = e^M$. Then $\forall n > N, \ln(n+1) > \ln n > \ln e^M = M$.) So $\lim\limits_{n\to+\infty} s_n = +\infty$, i.e. the sum diverges to $+\infty$.

Theorem 12.5. *Let $p > 0$. The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.*

Proof. The case $p = 1$ is shown above. If $p < 1$, then $n^p \leq n]foralln$, so $\frac{1}{n^p} \geq \frac{1}{n}\ \forall n$. Therefore, $\sum \frac{1}{n^p}$ diverges to $+\infty$ by the comparison test. For $p > 1$, let $s_n = \sum_{k=1}^n \frac{1}{k^p}$. This turns out to be less than the integral $\int_1 \frac{dx}{x^p}$ when offset by 1:

$$\begin{aligned} s_n = 1 + \sum_{k=2}^{n} \frac{1}{n^p} &\leq 1 + \int_1^n \frac{dx}{x^p} \\ &= 1 + \frac{1}{1-p} x^{1-p} \,|_1^n < 1 + \frac{1}{p-1} = \frac{p}{p-1} \end{aligned}$$

So $s_n < \frac{p}{p-1}$ for all n, therefore (s_n) is increasing and is bounded above by $\frac{p}{p-1}$ and below by s_1. Therefore s_n converges. $\square$

The integral test should be used when other tests don't work, $a_n \geq 0\ \forall n$, and there exists a nice function $f : [1, \infty) \to \mathbb{R}$ such that $a_n = f(n)\ \forall n$, and f is decreasing, i.e. $f(x_1) \geq f(x_2) \forall x_1 \leq x_2$. Further, it is required that it is possible to compute or estimate $\int_1^n f(x)dx$ for all n.

Theorem 12.6. *Let $f : [1, \infty) \to \mathbb{R}$ be a function such that f is decreasing, and $\int_1^n f(x)dx$ is finite for all $n \in \mathbb{Z}_{>0}$. Let $a_n = f(n) \forall n \in \mathbb{Z}_{>0}$.*

(a) If $\lim\limits_{n\to+\infty} \int_1^n f(x)dx$ converges, then $\sum a_n$ converges.

(b) If $\lim\limits_{n\to+\infty} \int_1^n f(x)dx$ diverges to $+\infty$, then $\sum a_n = +\infty$.

Definition 12.1. *An alternating series is one of the form $\sum(-1)^n a_n$ where $a_n \leq 0 \forall n$ or $a_n \geq 0 \forall n$.*

Theorem 12.7. *Alternating Series Test: let $\sum(-1)^n a_n$ be an alternating series such that $a_n \geq 0\ \forall n$, (a_n) is a decreasing sequence, and $\lim\limits_{n\to+\infty} a_n = 0$. Then $\sum(-1)^n a_n$ converges, and for all n, $|s - s_n| < a_n$, where $s = \sum(-1)^n a_n$ and $s_n = \sum_{k=0}^n (-1)^k a_k$.*

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Lecture 13: Alternating Series, Continuity

Lecturer: Michael Christianson *17 July* *Aditya Sengupta*

Proof. (sketch of a proof of the Alternating Series Test) Because (a_n) is a decreasing sequence, we know that (s_{2k}) is decreasing. For all k, $a_{2k+3} \leq a_{2k+2}$, so

$$s_{2k+3} = s_{2k+1} + a_{2k+2} - a_{2k+3} \geq s_{2k+1}$$

Therefore (s_{2k+1}) is increasing.

Next, we show that for all $m, n \in \mathbb{N}$, $s_{2m+1} \leq s_{2n}$. When $m = n$, we get $s_{2n+1} = s_{2n} - a_{2n+1} \leq s_{2n}$; when $m \leq n$, $s_{2m+1} \leq s_{2n+1} \leq s_{2n}$; when $m \geq n$, $s_{2n} \geq s_{2m} \geq s_{2m+1}$.

Now, for all $k \in \mathbb{N}$, $s_1 \leq s_3 \leq s_5 \leq \cdots \leq s_{2k-1} \leq s_{2k+1} \leq s_{2k} \leq s_{2k-2} \leq s_{2k-4} \leq \cdots \leq s_4 \leq s_2$.

(s_{2k}) is decreasing to $s_1 \leq s_{2k} \leq s_2 \ \forall k$, so (s_{2k}) is bounded and therefore it converges to some t_1. Likewise, (s_{2k+1}) converges to some t_2. Then

$$\begin{aligned} t_2 - t_1 &= \lim_{k \to +\infty} s_{2k+1} - \lim_{k \to +\infty} s_{2k} = \lim_{k \to +\infty} (s_{2k+1} - s_{2k}) \\ &= \lim_{k \to +\infty} (-1)^{2k+1} a_{2k+1} = 0 \end{aligned}$$

Therefore $t_2 = t_1$, and we can show that $\lim_{n \to +\infty} s_n = t_2 = t_1$. □

Corollary 13.1. *Let $\sum_{n=0}^{\infty} (-1)^n a_n$ be an alternating series. Now, for all $k \in \mathbb{N}$, if the sequence $(|a_n|)$ is decreasing and $\lim_{n \to +\infty} a_n = 0$, then the series converges.*

Example 13.1. Consider $a_n = (1 + 2 \cdot (-1)^n) \cdot 2^{-n}$; by computing a few terms we see that this series is alternating. $\lim_{n \to +\infty} a_n = 0$, but $|a_{2k}| = 3 \cdot 2^{-2k} > 2 \cdot 2^{-2k} = 2^{-(2k-1)} = |a_{2k-1}$, so $(|a_n|)$ is not decreasing. But $a_n \leq 3 \cdot 2^{-n} \ \forall n$, which converges because it is a geometric series, i.e. a_n converges absolutely by the comparison test. □

13.1 Continuity

Definition 13.1. *Let S and T be sets. A function (or map) from S to T is a choice of $f(s) \in T \ \forall s \in S$. We denote this by $f : S \to T$.*

Definition 13.2. *Let S and T be sets and let $f : S \to T$. S is the domain of $f(\operatorname{dom} F)$ and T is the codomain of f.*

For us, all functions are of the form $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}$. If $\operatorname{dom}(f)$ is not specified, it is the largest subset of $\mathbb{R}$ where f is well defined.

Example 13.2. $f(x) = \frac{1}{x}, \operatorname{dom}(f) = \mathbb{R} \setminus \{0\}$

□

Example 13.3. $g(x) = \sqrt{4 - x^2}, \operatorname{dom}(g) = [-2, 2]$

□

Definition 13.3. *Let $f : S \to \mathbb{R}$ be a function and let $a \in S$. We say f is continuous at a if there exist sequences (a_n) such that $a_n \in S\ \forall n$ and $\lim\limits_{n \to +\infty} a_n = a$. We have*

$$\lim_{n \to +\infty} f(a_n) = f(a) = f(\lim_{n \to +\infty} a_n)$$

We say f is discontinuous at a if f is not continuous at a.

We say f is continuous if f is continuous at x for all $x \in S$.

Theorem 13.2. *Let $f : S \to \mathbb{R}$ and let $a \in S$. Then, f is continuous at a if and only if*

$$\begin{aligned} \forall \epsilon > 0, \exists \delta > 0 \,\text{s.t.} \\ \forall x \in S \,\text{s.t.}\, |a - x| < \delta, |f(a) - f(x)| < \epsilon \end{aligned}$$

Proof. Suppose the $\delta - \epsilon$ property holds; we want to prove that f is continuous at a.

Let (a_n) be a sequence in S converging to a. Then, $\forall \epsilon > 0, \exists \delta \,\text{s.t.}\, \forall x \in S$ with $|a - x| < \delta$, $|f(a) - f(x)| < \epsilon$.

Since $\lim\limits_{n \to +\infty} a_n = a$, there exists some N such that $|a - a_n| < \delta$. Therefore $|f(a) - f(a_n)| < \epsilon \forall n > N$. Therefore $\lim\limits_{n \to +\infty} f(a_n) = f(a)$, so f is continuous at a.

In the reverse direction, we proceed by contraposition, i.e. if the $\delta - \epsilon$ property does not hold, then f is not continuous at a. If the $\delta - \epsilon$ property doesn't hold, then

$$\begin{aligned} \exists \epsilon > 0 \,\text{s.t.}\ \forall \delta > 0, \exists x \in S \,\text{s.t.} \\ |a - x| < \delta \text{ but } |f(a) - f(x)| \geq \epsilon \end{aligned}$$

For all $n \in \mathbb{Z}_{>0}$, pick $\delta = \frac{1}{n}$. Then there exists $a_n \in S$ such that $|a - a_n| < \delta = \frac{1}{n}$, but $|f(a) - f(a_n)| \geq \epsilon$. Then (a_n) is a sequence converging to a, but $|f(a) - f(a_n)| < \epsilon$ is never true for any n. So $\lim\limits_{n \to +\infty} f(a_n) \neq f(a)$, so f is discontinuous at a. □

Example 13.4. Let $f(x) = 2x^2 + 1$; we want to prove f is continuous.

Let $a \in \mathbb{R}$. For any sequence (a_n) converging to a, we have

$$\lim_{n\to+\infty} f(a_n) = \lim_{n\to+\infty} 2a_n^2 + 1 = 2\left(\lim_{n\to+\infty} a_n\right)^2 + 1 = 2a^2 + 1 = f(a)$$

Therefore f is continuous at a.

We can prove the same thing using $\delta - \epsilon$: work backwards from

$$|f(a) - f(x)| < \epsilon$$

We want to solve for x or $|x|$ or $|a - x|$ in terms of things that do not involve x:

$$|f(a) - f(x)| = |2a^2 + 1 - 2x^2 - 1| = |2(a + x)(a - x)| = 2|a - x| \cdot |a + x|$$

We replace $|a + x|$ with something larger. Assume $|a - x| < 1$. Then

$$1 > |a - x| \geq ||a| - |x|| = ||x| - |a|| \geq |x| - |a|$$

Therefore

$$\begin{aligned} |x| &< 1 + |a| \\ |a + x| \leq |a| + |x| &< 2|a| + 1 \end{aligned}$$

So

$$\begin{aligned} |f(a) - f(x)| < 2|a - x|(2|a| + 1) &< \epsilon \\ |a - x| &< \frac{\epsilon}{2(2|a| + 1)} \end{aligned}$$

Now we can begin the proof.

Proof. Let $\epsilon > 0$. Define $\delta = \min\{1, \frac{\epsilon}{2(2|a|+1)}\}$. Then $\forall x \in \mathbb{R}$ such that $|a - x| < \delta$, $2|a - x|(2|a| + 1) < \epsilon$. Also $|a - x| < 1$, so $|x + a| < 2|a| + 1$. Therefore

$$\epsilon > 2|a-x|(2|a|+1) > 2|a-x||x+a| = |f(a)-f(x)|$$

So f is continuous at a by the $\delta - \epsilon$ definition. $\square$

$\square$

Example 13.5. Consider the function

$$f(x) = \begin{cases} \frac{1}{x}\sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that f is discontinuous at 0.

We want to find some (a_n) such that $\lim\limits_{n\to+\infty} a_n = 0$, but $\lim\limits_{n\to+\infty} f(a_n) \neq f(0) = 0$. We want to find an (a_n) such that $\sin\left(1/a_n^2\right) = 1$. This gives us

$$a_n = \frac{1}{\sqrt{\frac{\pi}{2} + 2\pi n}}$$

Then, $\lim\limits_{n\to+\infty} a_n = 0$, but $f(a_n) = \frac{1}{a_n}\sin\left(\frac{1}{a_n^2}\right) = \frac{1}{a_n}$. Therefore $\lim\limits_{n\to+\infty} f(a_n) = \lim\limits_{n\to+\infty} \frac{1}{a_n} = +\infty \neq f(0)$.

Therefore f is discontinuous at 0.

$\square$

Most familiar functions are continuous everywhere: $e^x, \sin x, \cos x, \log_b x \forall b > 0, x^p \forall p \in \mathbb{R}$. Note that when we say a function is continuous everywhere, we mean throughout its domain.

Example 13.6. Consider $f(x) = |x|$. To show that this is continuous, we want to show that for all (a_n) converging to a, $||a_n| - |a|| \leq |a_n - a| < \epsilon$ (by the triangle inequality). This is true because $f(x) = x$ is continuous.

$\square$

13.2 Combining functions

Given $f : S \to \mathbb{R}, g : T \to \mathbb{R}$,

1. $(f+g)(x) = f(x) + g(x)$, with domain $S \cap T$
2. $(fg)(x) = f(x)g(x)$, with domain $S \cap T$
3. $\forall k \in \mathbb{R}, (kf)(x) = kf(x)$.
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, with domain $\{x \in S \cap T, g(x) \neq 0\}$
5. $\max(f,g)(x) = \max\{f(x), g(x)\}$; $\min(f,g)(x) = \min\{f(x), g(x)\}$, with domain $S \cap T$.
6. $(g \circ f)(x) = g(f(x))$, with domain $\{x \in S \mid f(x) \in T\}$.

Theorem 13.3. *Let $f : S \to \mathbb{R}, g : T \to \mathbb{R}, a \in \mathbb{R}$.*

(a) f and g continuous at a implies $f+g, fg$ are continuous at a.

(b) $\forall k \in \mathbb{R}, f$ continuous at a implies kf is continuous at a.

(c) f and g continuous at a and $g(a) \neq 0$ implies $\frac{f}{g}$ is continuous at a.

(d) f and g are continuous at a, $\max(f,g)$ and $\min(f,g)$ are continuous at a.

(e) f continuous at a and g continuous at $f(a)$ implies $g \circ f$ is continuous at a.

Math 104: Introduction to Analysis **Summer 2019**

Lecture 14: Bijective functions and continuity

Lecturer: Michael Christianson *18 July* *Aditya Sengupta*

14.1 Bounded functions

Definition 14.1. *Let $f : S \to \mathbb{R}$ be a function. We say f is bounded if its image $f(S) = \{f(s) \mid s \in S\}$ is a bounded set, i.e. $\exists M \in \mathbb{R}$ such that $|f(s)| \leq M \ \forall s \in S$.*

Theorem 14.1. *(Extreme Value Theorem) Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded, and f assumes its maximum and minimum values on $[a, b]$, i.e. $\exists l, u \in [a, b]$ such that $f(l) \leq f(x) \leq f(u)$ for all $x \in [a, b]$.*

Example 14.1. Consider $f(x) = \frac{1}{x}$ on $(0, 1)$. $\square$

Example 14.2. Consider $g(x) = x^2$ on $(-1, 1)$. For all $0 < a < 1$, $g(\sqrt{a}) = a$, but $g(x) < 1 \ \forall x$. $\square$

Proof. We claim f is bounded and prove it by contradiction. Suppose f is not bounded. Then $\forall n \in \mathbb{Z}_{>0}$, $\exists x_n \in [a, b]$ s. t. $|f(x_n)| > n$. Then (x_n) is bounded, so the Bolzano-Weierstrass theorem implies that there exists a subsequence (x_{n_k}) of (x_n) that converges to some $y \in \mathbb{R}$. Therefore $x_{n_k} \in [a, b] \forall k \implies y \in [a, b]$.

f is continuous at y, so $f(y) = \lim_{k \to +\infty} f(x_{n_k})$, so $(f(x_{n_k}))$ is bounded. Therefore $(|f(x_{n_k})|)$ is bounded. But

$$\lim_{k \to +\infty} |f(x_{n_k})| = \lim_{n \to +\infty} |f(x_n)| = +\infty$$

This is a contradiction, therefore f is bounded.

Let

$$M = \sup\{f(s) \mid s \in [a, b]\} \in \mathbb{R}$$

We find some $u \in [a, b]$ such that $f(u) = M$. For all $n \in \mathbb{Z}_{>0}$, $M - \frac{1}{n}$ is not an upper bound on $f([a, b])$. Therefore there exists $y_n \in [a, b]$ s. t. $M - \frac{1}{n} < f(y_n) < M$, which can also be written as

$$
\begin{aligned}
-\frac{1}{n} < f(y_n) - M \leq 0 < \frac{1}{n} \\
|f(y_n) - M| < \frac{1}{n}
\end{aligned}
$$

So (y_n) is a sequence in $[a, b]$ such that $\lim_{n\to+\infty} f(y_n) = M$. $y_n \in [a, b] \forall n$ implies that (y_n) is bounded, so there exists a convergent subsequence (y_{n_k}). Write $u = \lim y_{n_k}$; then $u \in [a, b]$ be3cause $y_{n_k} \in [a, b] \forall k$. f is continuous at u, therefore by continuity,

$$
f(u) = \lim f(y_{n_k}) = \lim_{n\to+\infty} f(y_n) = M
$$

Repeat this argument with $-f$ in place of f for the minimum. We get that $\exists l \in [a, b]$ such that $-f$ assumes its maximum at l.

$$
\begin{aligned}
-f(l) \geq -f(x) \forall x \in [a, b] \\
f(l) \leq f(x) \forall x \in [a, b]
\end{aligned}
$$

So f assumes its minimum value at l. □

Definition 14.2. *A subset $I \subseteq \mathbb{R}$ is an interval if $\exists a < b$ in $\mathbb{R} \cup \{\pm\infty\}$ such that I has one of the following forms: $(a, b), [a, b], [a, b), (a, b]$.*

Theorem 14.2. *(Intermediate Value Theorem) Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a continuous function. For all $a < b$ in I and any y lying between $f(a)$ and $f(b)$, i.e. $f(a) < f(y) < f(b)$ or $f(b) < f(y) < f(a)$, there exists $x \in I$ such that $a < x < b$ and $f(x) = y$.*

Proof. We prove the statement for $f(b) < y < f(a)$. Define

$$
S = \{x \in [a, b] \mid f(x) > y\}
$$

$a \in S$ implies $S \neq \varnothing$, i.e. $\sup S = x_0$ is a real number. Since $S \subseteq [a, b], x_0 \in [a, b] \subseteq I$. We show $f(x_0) = y$. First, we show that $f(x_0) \geq y$.

For all $n \in \mathbb{Z}_{>0}$, $x_0 - \frac{1}{n}$ is not an upper bound on S. Therefore, there exists $s_n \in S$ such that $x_0 - \frac{1}{n} < s_n \leq x_0$, i.e. $|s_n - x_0| < \frac{1}{n}$, so $\lim_{n\to+\infty} s_n = x_0$.

Then, f is continuous at x_0, so $f(x_0) = \lim_{n\to+\infty} f(s_n) \geq y$. $s_n \in S \forall n$, so $f(s_n) > y$.

Next, we show that $f(x_0) \leq y$. For all $n \in \mathbb{Z}_{>0}, t_n = \min\{b, x_0 + \frac{1}{n}\}$. Then, $x_0 < b$ and $x_0 < x_0 + \frac{1}{n}$, so

$$
x_0 - \frac{1}{n} < x_0 < t_n \leq x_0 + \frac{1}{n}
$$

So $|t_n - x_0| < \frac{1}{n}$, i.e. $\lim_{n\to+\infty} t_n = x_0$.

For all n, $t_n > x_0 \implies t_n \notin S$, so $f(t_n) \leq y$. So continuity of f at x_0 gives us

$$f(x_0) = \lim_{n\to+\infty} f(t_n) \leq y.$$

Therefore $f(x_0) = y$. $\square$

Definition 14.3. *Let $f : S \to T$ be a function.*

1. *f is injective if $\forall s, s' \in S\,\text{s.t.}\, f(s) = f(s')$, we have $s = s'$.*
2. *The image of f is $f(S) = \{f(s) \mid s \in S\}$.*
3. *f is surjective or onto if $f(S) = T$, ki.e. $\forall t \in T, \exists s \in S\,\text{s.t.}\, f(s) = t$.*
4. *f is bijective if it is injective and surjective.*

Corollary 14.3. *Lert $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a continuous function. Then $f(I)$ is either an interval or a single point.*

Proof. If $\sup f(I) = \inf f(I)$, then $f(I) = \{\sup f(I)\}$. For all $y \in f(I), \inf f(I) \leq y \leq \sup f(I)$. If instead $\inf f(I) < \sup f(I)$, then I is an interval with end points $\inf f(I)$ and $\sup f(I)$. For all y such that $\inf f(I) < y < \sup f(I)$, there exists $y_0, y_1 \in f(I)$ such that $y_0 < y < y_1$. By the intermediate value theorem, there exists $x \in I$ such that $f(x) = y \implies y \in f(I)$. So

$$(\inf f(I), \sup f(I)) \subseteq f(I) \subseteq [\inf f(I), \sup f(I)]$$

$\square$

Example 14.3. Let $f : [0,1] \to [0,1]$ be a continuous function. We show f has a fixed point, i.e. $\exists a \in [0,1]\,\text{s.t.}\, f(a) = a$.

Let $g(x) = f(x) - x$ on $[0,1]$. Then

$$\begin{aligned} g(0) &= f(0) - 0 = f(0) \geq 0 \\ g(1) &= f(1) - 1 \leq 1 - 1 = 0 \end{aligned}$$

If $g(0) \neq 0$ and $g(1) \neq 0$, then $g(0) > 0 > g(1)$, so $\exists x \in (0,1)\,\text{s.t.}\, g(x) = 0$ by the intermediate value theorem, i.e. $f(x) = x$.

$\square$

Definition 14.4. *Let $f : S \to \mathbb{R}$.*

1. *f is strictly increasing if $\forall x_1 < x_2 \in S, f(x_1) < f(x_2)$.*

2. *f is strictly decreasing if $\forall x_1 < x_2 \in S, f(x_1) > f(x_2)$.*

Theorem 14.4. *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a continuous function. The following are equivalent:*

1. *f is continuous and injective.*
2. *$f(I)$ is an interval, and f is either strictly increasing or strictly decreasing.*

Proof. In the forward direction, f being continuous and injective implies that $f(I)$ is an interval or a point. But f being injective means it must be an interval.

We claim that $\forall a < b < c$ in I, $f(b)$ lies between $f(a)$ and $f(c)$ (strict inequalities because the function is injective.) Suppose this is false; then either $f(b) > \max\{f(a), f(c)\}$, or $f(b) < \min\{f(a), f(c)\}$.

If $f(b) > \max\{f(a), f(c)\}$, pick $y \in \mathbb{R}$ such that $f(b) > y > \max\{f(a), f(c)\}$. By IVT on $[a, b]$ and $[b, c]$, $\exists x_1 \in (a, b)$ and $x_2 \in (b, c)$ such that $f(x_1) = f(x_2) = y$. But this implies $x_1 = x_2$ because f is injective, which is a contradiction.. Therefore $f(b) < \max\{f(a), f(c)\}$.

If $f(b) < \min\{f(a), f(c)\}$, pick $y \in \mathbb{R}$ such that $\min\{f(a), f(c)\} < y < f(b)$. By IVT on $[a, b]$ and $[b, c]$, $\exists x_1 \in (a, b)$ and $x_2 \in (b, c)$ such that $f(x_1) = f(x_2) = y$. But this implies $x_1 = x_2$ because f is injective, which is a contradiction.. Therefore $f(b) > \min\{f(a), f(c)\}$.

In the backward direction, f strictly increasing or strictly decreasing implies that f is injective, because $x_1 < x_2 \implies f(x_1) < f(x_2)$ or $x_1 < x_2 \implies f(x_1) > f(x_2)$ both imply that $x_1 < x_2 \implies f(x_1) \neq f(x_2)$. Therefore it suffices to prove continuity. We prove the case where f is strictly decreasing, and assume the proof for the case f is strictly increasing follows similarly. (see Ross section 18).

Let $a \in I$ and let $\epsilon > 0$. Roughly, choose ϵ sufficiently small that it is in the image of the function around a then finish the proof. $\square$

4

Math 104: Introduction to Analysis **Summer 2019**

Lecture 15: Uniform continuity, extensions

Lecturer: Michael Christianson *22 July* *Aditya Sengupta*

Definition 15.1. *Let $f : S \to T$ be an injective function. The inverse of f is the function*

$$f^{-1} : f(S) \to S, f^{-1}(t) = s$$

for the unique s such that $f(s) = t$.

Equivalently, f^{-1} is the unique function such that

$$f^{-1} \circ f(s) = s \ \forall s \in S$$
$$f \circ f^{-1}(t) = t \ \forall t \in f(S)$$

Lemma 15.1. *Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be an injective function. If f is continuous, then $f^{-1} : f(I) \to \mathbb{R}$ is continuous and injective. Moreover, f and f^{-1} are either both strictly increasing or strictly decreasing.*

Proof. f is either strictly increasing or strictly decreasing. If f is strictly increasing, $\forall x < y$ in $f(S)$. Let $a, b \in I$ such that $f(a) = x, f(b) = y$. Then,

$$a \geq b \implies f(a) \geq f(b) \text{but} \qquad f(a) = x < f(b) = y$$

Therefore $a < b$, so $a = f^{-1}(x) < b = f^{-1}(y)$, which implies that f^{-1} is strictly increasing. ☐

The image of $f^{-1} = f^{-1}(f(I)) = I$ is an interval, and f^{-1} is strictly increasing or strictly decreasing, therefore f^{-1} is continuous.

Example 15.1. Let $m \in \mathbb{Z}_{>0}, f(x) = x^m$ on $[0, \infty)$. f is continuous and strictly increasing, which can be shown by induction; if $m = 1, f(x) = x$ which is strictly increasing. If x^m is strictly increasing, then

$$x < y \implies x^m < y^m$$
$$x^{m+1} < xy^m < yy^m = y^{m+1}$$

So $f^{-1} : [0, \infty) \to [0, \infty), f^{-1}(y) = \sqrt[m]{y}$ is injective and continuous. Therefore, for all $y \in [0, \infty), y$ has a unique mth root. This depends on the image being the domain, i.e. $f([0, \infty)) = [0, \infty)$

$$\begin{aligned} \forall x, f(x) \geq 0 &\implies I \subseteq [0, \infty) \\ \forall y \in [0, \infty), &\exists b > 0 \,\text{s. t.} \\ y \leq b^m (\text{take }) &b = \max\{1, y\} \end{aligned}$$

Then, $0 = f(0) \leq y \leq b^m = f(b)$. So by the intermediate value theorem, $y \in I$; specifically, $\exists 0 \leq x \leq b \,\text{s. t.}\, f(x) = y$.

□

Definition 15.2. *$f : S \to \mathbb{R}$ is uniformly continuous on a subset $T \subseteq S$ if $\forall \epsilon > 0, \exists \delta > 0 \,\text{s. t.}\, \forall x, y \in T, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.*

f is uniformly continuous if f is uniformly continuous on S.

Example 15.2. Consider $f(x) = \frac{1}{x^2}$ on $(0, \infty)$. We claim that $f(x)$ is continuous on its domain.

Let $a = (0, \infty)$, and we start to work backwards:

$$\begin{aligned} |f(x) - f(a)| &< \epsilon \\ |f(x) - f(a)| &= \left| \frac{(a + x)(a - x)}{x^2 a^2} \right| \\ &= \frac{|a - x||a + x|}{|x|^2 a^2} \end{aligned}$$

To proceed, we try to assume $|x - a| < 1$, which gives us

$$|a| - |x| \leq ||a| - |x|| \leq |x - a| < 1$$

However, this gives us $|x| > |a| - 1$, which is bad for the case $a = 1$. Therefore, we take $|x - a| < \frac{a}{2}$, which gives us

$$|x - a| < \frac{a}{2} \implies |x| < |a| + \frac{a}{2}$$

and

$$|x + a| \leq |x| + |a| < \frac{3a}{2} + a = \frac{5a}{2}$$

Therefore

$$
\begin{aligned}
|f(x)-f(a)| &< \frac{|a-x|\left(\frac{5a}{2}\right)}{\left(\frac{a}{2}\right)^2 a^2} \\
&= |a-x| \cdot \frac{4 \cdot 5}{2a^3} = |a-x| \cdot \frac{10}{a^3} < \epsilon
\end{aligned}
$$

Let $\delta = \min\{\frac{a}{2}, \frac{a^3\epsilon}{10}\}$. Then, $\forall x$ s. t. $|x-a| < \delta$, $|f(x)-f(a)| < \epsilon$, so f is continuous at a.

Now, we can discuss uniform continuity. We claim that $\forall b > 0$, f is uniformly convergent on $[b, \infty)$. For all $x, y \in [b, \infty)$,

$$
f(x) - f(y) = \frac{(x-y)(x+y)}{x^2y^2} = (x-y) \cdot \left(\frac{x+y}{x^2y^2}\right)
$$

We want to bound $\frac{x+y}{x^2+y^2}$ in terms of $[b, \infty)$.

$$
\begin{aligned}
\frac{x+y}{x^2y^2} &= \frac{1}{xy^2} + \frac{1}{x^2y} \\
&\leq \frac{1}{b^3} + \frac{1}{b^3} = \frac{2}{b^3}
\end{aligned}
$$

So

$$
|f(x)-f(y)| \leq |x-y| \cdot \frac{2}{b^3}
$$

Let $\delta = \frac{\epsilon b^3}{2}$. Then $|x-y| < \delta = \frac{\epsilon b^3}{2}$ implies

$$
\epsilon > |x-y| \cdot \frac{2}{b^3} \geq |f(x)-f(y)|
$$

Therefore f is uniformly continuous on $[b, \infty)$.

Finally, we claim that $f(x)$ is *not* uniformly continuous on $(0, \infty)$. Intuitively, we can think of this in terms of the derivative, which blows up to $-\infty$ as $x \to 0$. Because the rate of change of $f(x)$ is unbounded, the function cannot be uniformly continuous.

$\square$

Theorem 15.2. *Let $f : [a, b] \to \mathbb{R}$ be a function. f is continuous if and only if f is uniformly continuous.*

Proof. If f is uniformly continuous then f is continuous by definition. Suppose f is continuous but not uniformly continuous. Then there exists some ϵ such that for every δ, there exist $x, y \in [a, b]$ s. t. $|x-y| < \delta$

but $|f(x) - f(y)| \geq \epsilon$.

For all n, take $\delta = \frac{1}{n}$. Then, there exist $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < \frac{1}{n}$, but $|f(x_n) - f(y_n)| \geq \epsilon$. (x_n) is bounded, so by the Bolzano-Weierstrass theorem, there exists (x_{n_k}) convergent. Let $z = \lim\limits_{k \to +\infty} x_{n_k}$. Then (y_{n_k}) converges to z as well. $x_{n_k}, y_{n_k} \in [a, b] \forall k, z \in [a, b]$. So f is continuous at z, which implies

$$\begin{aligned} \lim f(x_{n_k}) &= f(z) = \lim f(y_{n_k}) \\ \therefore \lim f(x_{n_k}) &- f(y_{n_k}) = 0 \end{aligned}$$

But $|f(x_{n_k}) - f(y_{n_k}) \geq \epsilon$ for all k, which is a contradiction. Therefore f must be uniformly continuous. $\square$

Definition 15.3. *Let $f : S \to \mathbb{R}$. We say $\widetilde{f} : T \to \mathbb{R}$ is an extension of f to T if $S \subseteq T$ and $\widetilde{f}(x) = f(x)\ \forall x \in S$.*

Example 15.3. Let $f(x) = x \sin \frac{1}{x}$ on $\left(0, \frac{1}{\pi}\right]$. An extension of f to include 0 would be

$$\widetilde{f}(x) = \begin{cases} x \sin \frac{1}{x} & 0 < x \leq \frac{1}{\pi} \\ 0 & x = 0 \end{cases}$$

$\square$

Example 15.4. Let $g(x) = \sin \frac{1}{x}$ on $\left(0, \frac{1}{\pi}\right]$. No extension of g to $\left[0, \frac{1}{\pi}\right]$ is continuous.

$\square$

Theorem 15.3. *Let $f : (a, b) \to \mathbb{R}$ be a function. f is uniformly continuous iff there exists an extension $\widetilde{f} : [a, b] \to \mathbb{R}$ of f such that $\widetilde{f}$ is continuous. Moreover, in this case, the extension $\widetilde{f}$ is the unique continuous extension.*

Proof. (Only in the forward direction) If $\widetilde{f}$ exists, then $\widetilde{f}$ is uniformly continuous, so $\widetilde{f}$ is uniformly continuous on (a, b), so f is uniformly continuous. $\square$

Lemma 15.4. *Let $f : S \to \mathbb{R}$ be uniformly continuous. For all (s_n) in S, if (s_n) is Cauchy then $(f(s_n))$ is also Cauchy.*

Proof. For all $\epsilon > 0$, we want N such that $\forall m, n > N$, $|f(s_n) - f(s_m)| < \epsilon$. From the definition of continuity, we know that exists $\delta > 0$ such that $\forall x, y \in S$, $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Because (s_n) is Cauchy, there exists an N such that $|s_n - s_m < \delta \forall m, n > N$, and so $|f(s_n) - f(s_m)| < \epsilon \forall m, n > N$. $\square$

Example 15.5. Consider $f(x) = \frac{1}{x^2}$ on $(0, \infty)$. Let $s_n = \frac{1}{n}$; (s_n) is Cauchy. But $f(s_n) = n^2$, so $\lim_{n \to +\infty} f(s_n) = +\infty$. Therefore $(f(s_n))$ is not Cauchy and f is not uniformly continuous.

□

Now, we can do the backward-direction half of the proof of the unique existence of a continuous function extension.

Proof. Assume $\widetilde{f}$ is uniformly continuous. Let (s_n) be a sequence in (a, b) such that $\lim_{n \to +\infty} s_n = a$. Then, (s_n) is Cauchy, so $(f(s_n))$ is Cauchy, i.e. $(f(s_n))$ converges. So, define $\widetilde{f}(a) = \lim_{n \to +\infty} f(a_n)$. Likewise, let (s'_n) be a sequence in (a, b) converging to b. Then define $\widetilde{f}(b) = \lim_{n \to +\infty} f(s'_n)$ and let $\widetilde{f}(s) = f(s)\ \forall a < s < b$. We show $\widetilde{f}$ is continuous at a and b; let (t_n) be a sequence in $[a, b]$ converging to a. Define a new sequence,

$$(u_n) = (s_0, t_0, s_1, t_1, s_2, t_2, \dots)$$

Then $\lim_{n \to +\infty} u_n = a$ because (u_n) is a combination of two sequences that converge to a. So by continuity of f, $(f(u_n))$ is Cauchy by the lemma above, so it converges, so its limit is the same as the limit of any subsequence. Since (s_n) and (t_n) are both convergent to a, $\lim_{n \to +\infty} f(u_n) = \lim_{n \to +\infty} f(s_n) = \lim_{n \to +\infty} f(t_n) = f(a) = \lim_{n \to +\infty} \widetilde{f}(t_n)$. Therefore $\widetilde{f}$ is continuous at a. □

Math 104: Introduction to Analysis **Summer 2019**

Lecture 16: Limits of functions, derivatives

Lecturer: Michael Christianson *23 July* *Aditya Sengupta*

16.1 Uniform continuity from the derivative

Theorem 16.1. *Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be continuous. Let I^o be I without any endpoints. If f is differentiable on I^o and f' is bounded on I^o, then f is uniformly continuous on I.*

Example 16.1. Consider $f(x) = \frac{1}{x^2}$ on $[b, \infty)$ for $b > 0$.

$$|f'(x)| = \left|-\frac{1}{x^3}\right| = \frac{1}{x^3} \leq \frac{1}{b^3} \forall x \in [b, \infty)$$

So $f'(x)$ is bounded on (b, ∞), and f is continuous on $[b, \infty)$. Therefore f is uniformly continuous on $[b, \infty)$.

□

16.2 Limits of functions

16.2.1 Definitions

Definition 16.1. *Let $f : S \to \mathbb{R}$, and let $a, L \in \mathbb{R} \cup \{\pm\infty\}$. Further, let $T \subseteq S$. We write*

$$\lim_{x \to a^T} f(x) = L$$

if $\forall (t_n) \in T$ converging to a, $\lim_{n \to +\infty} f(t_n) = L$.

Remark 16.2. *$f : S \to \mathbb{R}$ is continuous at $a \in S$ iff $\lim_{x \to a^S} f(x) = f(a)$.*

Definition 16.2. *Let $f : S \to \mathbb{R}$ and let $L \in \mathbb{R} \cup \{\pm\infty\}$.*

1. *We write $\lim_{x \to a} f(x) = L$ if $\lim_{x \to a^I} f(x) = L$, where $I = J \setminus \{a\}$ and J is an open interval containing a. This is the two-sided limit.*
2. *$\lim_{x \to a^+} f(x) = L$ if $\lim_{x \to a^I} f(x) = L$, where $I = (a, b)$ for some $b > a$. This is the right-hand limit.*

3. $\lim_{x \to a^-} f(x) = L$ *if* $\lim_{x \to a^I} f(x) = L$, *where* $I = (c, a)$ *for some* $c < a$. *This is the right-hand limit.*
4. *We say* $\lim_{x \to +\infty} f(x) = L$ *if* $\lim_{x \to +\infty^I} f(x) = L$ *for* $I = (b, \infty)$. *(and the same on the other side with* $-\infty$.*)*

Example 16.2. To show that $\lim_{x\to\infty} \frac{1}{(x-2)^3} = 0$, we pick $b = 2$. Then, we want to show that for all (s_n) in $(2, \infty)$ such that $\lim s_n = \infty$, $\lim_{n \to +\infty} f(s_n) = 0$.

□

16.2.2 Limit theorems

These work like they do with sequences. For example,

$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a^S} f(x) + \lim_{x \to a^S} g(x)$$

The above holds as long as it does not create an undefined expression like $\infty - \infty$.

Remark 16.3. $f : S \to \mathbb{R}$ *is continuous at* $a \in S$ *if and only if* $\lim_{x \to a^S} f(x) = f(a)$.

Theorem 16.4. *Let* $f : S \to \mathbb{R}, a \in \mathbb{R}$ *such that there exists a sequence in* S *converging to* a. *Let* $L \in \mathbb{R}$. *Then*

$$\begin{aligned} \lim_{x \to a^S} f(x) = L \iff & \forall \epsilon > 0, \exists \delta > 0 \,\text{s.t.} \\ & \forall x \in S, |x - a| < \delta \implies |f(x) - f(a)| < \epsilon. \end{aligned}$$

Corollary 16.5.
1. $\lim_{x \to a} f(x) = L$ *if and only if* $\forall \epsilon > 0, \exists \delta > 0 \,\text{s.t.}\, \forall x \neq a, a - \delta < x < a + \delta \implies |f(x) - L| < \epsilon$.
2. $\lim_{x \to a^+} f(x) = L$ *if and only if* $\forall \epsilon > 0, \exists \delta \,\text{s.t.}\, a < x < a + \delta \implies |f(x) - L| < \epsilon$ *(and the same for* $x \to a^-$*).*
3. $\lim_{x \to \infty} f(x) = L$ *iff* $\forall \epsilon > 0, \exists M$ *such that* $\forall x > M, |f(x) - L| < \epsilon$.

Lemma 16.6. $\lim_{x \to a} f(x)$ *exists iff* $\lim_{x \to a^-} = \lim_{x \to a^+} f(x)$, *and in this case,* $\lim_{x \to a}$ *is equal to both.*

16.3 Differentiation

Definition 16.3. *Let* $f : S \to \mathbb{R}, a \in S$. *The derivative of* f *at* a *is*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Definition 16.4. *The derivative of f is the function f' given by $a \to f'(a)$.* $\operatorname{dom} f' \subseteq \operatorname{dom} f$.

Definition 16.5. *f is differentiable at a if $f'(a)$ exists and is finite.*

Remark 16.7. *f is differentiable at a if and only if $a \in \operatorname{dom}(f')$.*

Proposition 16.8. *Let $f : S \to \mathbb{R}$ be a function and let $a \in S$. f is differentiable at a implies f is continuous at a.*

Proof. We want $\lim\limits_{x\to a} f(x) = f(a)$.

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}(x-a)\left(\frac{f(x)-f(a)}{x-a}\right) + f(a) \\ &\lim_{x\to a}(x-a)\cdot\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right) + \lim_{x\to a} f(a)\end{aligned}$$

The above statement is valid because the limits all exist and are finite. We get

$$\lim_{x\to a} f(x) = 0 \cdot f'(a) + f(a) = f(a)$$

Therefore f is continuous at a. □

Example 16.3. Let $g(x) = x^2$.

$$\begin{aligned}g'(2) = \lim_{x\to 2}\frac{g(x)-g(2)}{x-2} &= \lim_{x\to 2}\frac{x^2-4}{x-2} \\ &= \lim_{x\to 2} x+2 = 4\end{aligned}$$

□

Proposition 16.9. *Let f, g be functions, $a \in \mathbb{R}$. Suppose f and g are differentiable at a.*

(a) $\forall c \in \mathbb{R}$, *$c \cdot f$ is differentiable at a, and $(cf)'(a) = c \cdot f'(a)$.*

(b) $f + g$ is differentiable at a, and $(f+g)'(a) = f'(a) + g'(a)$.

(c) fg is differentiable at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$ (Product Rule).

(d) $\frac{f}{g}$ is differentiable at a, and if $g(a) \neq 0$,

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$$

(Quotient Rule)

Proof. The first two subparts follow from limit theorems and definitions.

For the product rule, $\forall x \neq a$, the following holds:

$$\begin{aligned}
\frac{f(x)g(x) - f(a)g(a)}{x-a} &= \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x-a} \\
&= f(x) \cdot \left(\frac{g(x) - g(a)}{x-a} \right) + g(a) \left(\frac{f(x) - f(a)}{x-a} \right)
\end{aligned}$$

In the limit $x \to a$, this becomes

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a)$$

For the quotient rule,

$$\begin{aligned}
\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(a) &= \frac{f(x)}{g(x)} - \frac{f(a)}{g(a)} \\
&= \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)} \\
&= \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{g(x)g(a)} \\
&= g(a)\frac{f(x) - f(a)}{g(a)g(x)} + f(a) \cdot \frac{g(a) - g(x)}{g(x)g(a)} \\
&= \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}
\end{aligned}$$

□

Example 16.4. Let $f(x) = c$ for some $c \in \mathbb{R}$. Then for all $a \in \mathbb{R}$,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} = \lim_{x \to a} \frac{0}{x-a} = 0$$

Therefore the derivative of a constant is 0 everywhere.

□

Example 16.5. Let $g(x) = x^n, n \in \mathbb{Z}_{>0}$. We claim that $g'(x) = nx^{n-1}$.

□

Proof. We prove this by induction on n. When $n = 1$,

$$g'(a) = \lim_{x \to a} \frac{x - a}{x - a} = \lim_{x \to a} 1 = 1$$

Next, suppose $x^n = nx^{n-1}$ for some n. Then

$$(x^{n+1})' = (x \cdot x^n)' = 1 \cdot x^n + nx^{n-1} \cdot x = (n + 1)x^{(n-1)+1}$$

□

16.4 Chain Rule

Theorem 16.10. *Let f be differentiable at a and g be differentiable at $f(a)$. Then*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Proof.

$$\frac{g(f(x)) - g(f(a))}{x - a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a}$$

The first term simplifies to $g'(f(a))$ and the second simplifies to $f'(a)$. □

16.5 Critical Points

Theorem 16.11. *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, and let $a \in I$. If f assumes a min or max at a, i.e. $f(x) \leq f(a) \forall x \in I$ or $f(a) \leq f(x) \forall x \in I$, and if f is differentiable at a, then $f'(a) = 0$.*

Proof. Suppose $f'(a) > 0$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} > 0$$

There exists δ such that $x < a - \delta < a + \delta < y$ and

$$\frac{f(z) - f(a)}{z - a} > 0 \; \forall z \in (a - \delta, a + \delta)$$

For any $a < z < a + \delta$,

$$f(z) - f(a) > 0 \implies f(z) > f(a)$$

Therefore $f(a)$ is not a maximum. Similarly, $f(a)$ is not a minimum (with the hypothesis $f'(a) < 0$) if we consider $a - \delta < z < a$. Therefore this is a contradiction and $f'(a) = 0$. $\square$

Math 104: Introduction to Analysis **Summer 2019**

Lecture 17: Differentiation, MVT, L'Hopital's Rule

Lecturer: Michael Christianson *24 July* *Aditya Sengupta*

Lemma 17.1. *Let $g : S \to \mathbb{R}$. If $\lim_{x \to a} g(x) > b$, then $\exists \delta > 0 \,\text{s.t.}\, \forall x \in S, |x - a| < \delta \implies g(x) > b$.*

Proof. By the $\delta - \epsilon$ definition of $\lim_{x \to a} f(x) = L$, we know that (taking $\epsilon = L - b > 0$),

$$\begin{aligned} \exists \delta > 0 \,\text{s.t.} \forall |x - a| < \delta, & \\ |g(x) - L| &< \epsilon = L - b \\ b - L < g(x) - L &< L - b \\ b &< g(x) \end{aligned}$$

$\square$

We can use the expression $\lim_{x \to a} f(x)$ in a few different ways: using the epsilon-delta definition, using the statement that $\lim_{x \to a} f(x) = f(a) \iff f$ is continuous at a (assuming f is defined on some interval containing a), or using the sequence definition.

Theorem 17.2. *Let $f : I \to \mathbb{R}$ be a function and $I = (a, b)$ be an open interval. If f assumes a max or min at x, then either $f'(x_0)$ does not exist or $f'(x_0) = 0$.*

Proof. Assuming $f'(x_0)$ exists, we show that $f'(x_0) = 0$. By replacing f by $-f$, we may assume $f(x_0)$ is a maximum.

Suppose $f'(x_0) > 0$. Then

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

There exists δ such that $(x_0 - \delta, x_0 + \delta) \subseteq I$ and

$$|x - x_0| < \delta \implies \frac{f(x) - f(x_0)}{x - x_0} > 0$$

Then, for any x such that $x_0 < x < x_0 + \delta$,

$$\begin{aligned} x - x_0 > 0 &\implies f(x) - f(x_0) > 0 \\ &\implies f(x) > f(x_0) \end{aligned}$$

This is a contradiction, so $f'(x_0)$ is not greater than 0.

If $f'(x_0) < 0$, $\exists\delta$ such that

$$|x - x_0| < \delta \implies \frac{f(x) - f(x_0)}{x - x_0} < 0$$

Pick $x_0 - \delta < x < x_0$; then

$$\begin{aligned} x - x_0 < 0 &\implies f(x) - f(x_0) > 0 \\ &\implies f(x) > f(x_0) \end{aligned}$$

This is a contradiction, so $f'(x_0)$ is not less than 0.

Therefore, if $f'(x_0)$ is finite, it must be 0. $\square$

Theorem 17.3. *(Rolle's Theorem): Let $f : [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b). If $f(a) = f(b)$, then $\exists c \in [a,b]$ s.t. $f'(c) = 0$.*

Proof.

$$\begin{gathered} \exists x_0, y_0 \in [a,b] \text{ s.t.} \\ f(x_0) \le f(x) \le f(y_0)\ \forall x \in [a,b] \end{gathered}$$

Case. *If $x_0, y_0 \in \{a,b\}$, then $f(a) = f(b)$, so $f(x) = f(x_0) = f(y_0) \forall x$, i.e. f is constant. This implies that $f'(c) = 0$ for all $c \in [a,b]$.*

Case. *If x_0 or y_0 are not in $\{a,b\}$, then either $x_0 \in (a,b)$ or $y_0 \in (a,b)$, i.e. either x_0 or y_0 is a minimum or maximum of the function on $[a,b]$. Therefore $f'(x_0) = 0$ or $f'(y_0) = 0$.*

$\square$

Rolle's theorem is not inherently that useful, but it can be used to prove the Mean Value Theorem:

Theorem 17.4. *(Mean Value Theorem): let $f : [a,b] \to \mathbb{R}$ be a continuous function that is differentiable on (a,b). Then $\exists c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Let $L(x)$ be the function whose graph is the line passing through $(a, f(a))$ and $(b, f(b))$, which can be explicitly given by

$$L(x) = \left(\frac{f(b) - f(a)}{b - a}\right) \cdot (x - a) + f(a)$$

Define $g(x) = f(x) - L(x)$. g is continuous on $[a,b]$ and differentiable on (a,b), and

$$g(a) = f(a) - L(a) = 0$$
$$g(b) = f(b) - L(b) = 0$$

Therefore Rolle's theorem says there exists c such that

$$\begin{aligned} g'(c) &= 0 \\ f'(c) - L'(c) &= 0 \\ f'(c) - \frac{f(b) - f(a)}{b - a} &= 0 \\ f'(c) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

□

Corollary 17.5. *Let $f : (a, b) \to \mathbb{R}$ be differentiable. If $f'(x) = 0\ \forall x \in (a, b)$, then f is constant.*

Proof. Suppose f is not constant. Then,

$$\exists x_1 < x_2 \in (a, b) \,\text{s. t.}\, f(x_1) \neq f(x_2)$$

Then, by the MVT on $[x_1, x_2], \exists c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

This is a contradiction. Therefore f must be constant. □

Corollary 17.6. *If $f, g : (a, b) \to \mathbb{R}$ are such that $f'(x) = g'(x) \forall x \in (a, b)$, then*

$$\exists c \in \mathbb{R} \,\text{s. t.}\, f(x) = g(x) + c \forall x \in (a, b).$$

Proof.

$$(f - g)' = f' - g' = 0$$

Therefore $f - g$ is constant. □

Corollary 17.7. *Let $f : (a, b) \to \mathbb{R}$ be differentiable.*

(a) If $f'(x) > 0\ \forall x \in (a, b)$, then f is strictly increasing.

(b) If $f'(x) \geq 0\ \forall x \in (a, b)$, then f is increasing.

(c) *If $f'(x) \leq 0\ \forall x \in (a,b)$, then f is decreasing.*

(d) *If $f'(x) < 0\ \forall x \in (a,b)$, then f is strictly decreasing.*

Proof. Let $x_1 < x_2$ in (a,b). By MVT, there exists $c \in (x_1, x_2)$ such that

$$0 \leq f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

For (a), $f'(c) > 0$ and $x_2 - x_1 > 0$ implies $f(x_2) - f(x_1) > 0$, which tells us that f is strictly increasing. The other parts follow similarly. $\square$

Theorem 17.8. *(IVT for derivatives): let $f : (a,b) \to \mathbb{R}$ be differentiable. Let $x_1 < x_2 \in (a,b)$. For all y such that $f(x_1) < y < f(x_2)$ or $f(x_2) < y < f(x_1)$,*

$$\exists x \in (x_1, x_2)\,\text{s. t.}\, f(x) = y.$$

Remark 17.9. *If f' is continuous, this is immediate from the intermediate value theorem. But there are continuous functions whose derivatives have discontinuities, e.g. $f(x) = x \sin \frac{1}{x}$ is discontinuous at 0.*

Proof. By replacing f by $-f$ and y by $-y$, we may assume $f'(x_1) < y < f(x_2)$. Define

$$\begin{aligned} g(x) &= f(x) - y \cdot x \\ g'(x) &= f'(x) - y \end{aligned}$$

Then, $g(x)$ assumes a minimum on $[x_1, x_2]$ at some $x_0 \in [x_1, x_2]$. (This is sometimes called the extreme value theorem.) We want to show that $g'(x_0) = 0 \implies f'(x_0) = y$. For this, we show that $x_0 \in (x_1, x_2)$.

Note that $g'(x_1) = f'(x_1) - y < 0$, and $g'(x_2) = f'(x_2) - y > 0$. Therefore

$$\exists \delta_1\,\text{s. t.}\, |x - x_1| < \delta_1 \implies \frac{g(x) - g(x_1)}{x - x_1} < 0$$

So, for any $x_1 < x < x_1 + \delta_1$, $g(x) - g(x_1) < 0$, so $g(x_1)$ is not a minimum on $[x_1, x_2]$. Therefore $g(x_1) \neq g(x_0)$ and $x_1 \neq x_0$. Likewise, $g(x_2) > 0$ gives us

$$\exists \delta_2\,\text{s. t.}\, |x - x_2| < \delta_2 \implies \frac{g(x) - g(x_2)}{x - x_2} > 0$$

and $(x_2 - \delta_2, x_2 - \delta_2) \subseteq (x_1, b)$.

So

$$\begin{aligned} \forall x\,\text{s. t.}\, x_1 < x_2 - \delta_2 < x < x_2, x \in [x_1, x_2] \wedge x - x_2 < 0 \\ \therefore g(x) - g(x_2) < 0 \end{aligned}$$

So $g(x_2)$ is not a minimum for g on $[x_1, x_2]$, therefore $x_0 \neq x_2$.

Therefore g has a minimum in (x_1, x_2). $\square$

We can consider derivatives of function inverses. Let $f : I \to \mathbb{R}$ be differentiable and injective. Then, we can find $(f^{-1})'$. Assuming it exists, we can find it using the chain rule:

$$\begin{aligned} 1 = (f^{-1} \circ f)'(x) &= (f^{-1})' f(x) \cdot f'(x) \\ \therefore (f^{-1})'(f(x)) &= \frac{1}{f'(x)} \end{aligned}$$

Proposition 17.10. *Let $f : I \to \mathbb{R}$ be a continuous injective function with I an open interval. If f is differentiable at a and $f'(a) \neq 0$, then f^{-1} is differentiable at $f(a)$ and*

$$(f^{-1})'(f(a)) = \left.\frac{df^{-1}}{dx}\right|_{x=f(a)} = \frac{1}{f'(a)}$$

Example 17.1. Let $n \in \mathbb{Z}_{>0}$ and let $g(y) = \sqrt[n]{y}$ if n is odd, and $[0, \infty)$ if n is even. g is the inverse of $f(x) = x^n$. So for all $a \neq 0$ in $\operatorname{dom}(g)$, let $a = x_0^n = f(x_0)$.

$$\begin{aligned} g'(a) = \frac{1}{f'(x_0)} = \frac{1}{n x_0^{n-1}} &= \frac{1}{n} \cdot \frac{1}{(a^{1/n})^{n-1}} \\ &= \frac{1}{n} \cdot \frac{1}{a^{1-1/n}} = \frac{1}{n} a^{1/n-1} \end{aligned}$$

$\square$

Theorem 17.11. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). There exists $c \in (a, b)$ s.t.*

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Remark 17.12. *This reduces to the MVT if $g(x) = x$.*

Proof. Use Rolle's theorem on

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

$\square$

Theorem 17.13. *(L'Hopital's Rule) Let s signify $a, a^+, a^-, -\infty$ or $+\infty$ ($a \in \mathbb{R}$). Let f and g be differentiable functions such that $g'(x) \neq 0$ "near" s and such that*

$$\lim_{x \to s} \frac{f'(x)}{g'(x)} = L$$

If either

$$\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$$

or

$$\lim_{x \to s} |g(x)| = +\infty$$

then

$$\lim_{x \to s} \frac{f(x)}{g(x)} = L.$$

Proof. Consider $s = a^+$.

We claim that if $-\infty \leq L < \infty$, then for all $L_1 > L, \exists \alpha_1 > a$ such that $\forall a < x < \alpha_1$, we have $\frac{f(x)}{g(x)} < L_1$.

We also claim that if $-\infty < L \leq +\infty$, then $\forall L_2 < L, \exists \alpha_2 > a$ such that $\forall a < x < \alpha_2$, $\frac{f(x)}{g(x)} > L_2$.

If L is finite, both claims imply that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

If $L = -\infty$, the first claim implies that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

If $L = +\infty$, the second claim implies the same thing. Therefore, it is sufficient to prove the two claims. We prove the first one, and assume the other one follows by symmetry.

To prove the first claim, we find (a, b) such that

$$\begin{aligned} g'(x) &< 0 \forall x \in (a, b) \\ g'(x) &\neq 0 \forall x \in (a, b) \\ (a, b) \subseteq \mathrm{dom}(f) &\cap \mathrm{dom}(g) \end{aligned}$$

To do this, we pick k such that $L < k < L_1$. Then there exists $\delta = \alpha_1 > 0$ such that

$$a < x < \alpha_1 = \delta \implies \left| \frac{f'(x)}{g'(x)} - L \right| < k - L = \epsilon$$

This is true because the limit from above of the ratio of the derivatives is L. This gives us

$$\frac{f'(x)}{g'(x)} < k$$

For all $a < x < y < \alpha_1$, the generalized MVT tells us $\exists z \in (x, y)$ such that

$$k > \frac{f'(z)}{g'(z)} = \frac{f(y) - f(x)}{g(y) - g(x)}$$

and so

$$\lim_{x \to a^+} = \frac{f(y)}{g(y)} \leq k < L$$

So

$$\frac{f(y)}{g(y)} < L$$

as desired. $\square$

Math 104: Introduction to Analysis **Summer 2019**

Lecture 18: Integration

Lecturer: Michael Christianson *25 July* *Aditya Sengupta*

18.1 Using L'Hopital's Rule

L'Hopital's Rule can be used to resolve indeterminate forms of the type $\frac{0}{0}, \pm\frac{\infty}{\infty}, 0 \cdot \infty, 0^0, \infty^0, 1^\infty$.

Example 18.1. Consider the limit $\lim_{x \to 0^+} x \ln x$; we can rewrite this as

$$\begin{aligned}
\lim_{x \to 0^+} x \ln x &= \lim_{x \to 0^+} \frac{\ln x}{1/x} \\
&\overset{H}{=} \lim_{x \to 0^+} \frac{1/x}{-1/x^2} \\
&= \lim_{x \to 0^+} -x = 0
\end{aligned}$$

□

Example 18.2. Consider the limit $\lim_{x \to 0^+} x^x$; we solve this as follows:

$$\begin{aligned}
\lim_{x \to 0^+} x^x &= \lim_{x \to 0^+} e^{\ln x^x} \\
&= \lim_{x \to 0^+} e^{x \ln x} \\
&= e^0 = 1
\end{aligned}$$

The last step used the limit composition theorem (20.5 in Ross), which states that if $\lim_{x \to a} f(x)$ exists and if g is continuous at $f(a)$, then

$$\lim_{x \to a} g(f(x)) = g(\lim_{x \to a} f(x))$$

□

18.2 Integration

Definition 18.1. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. For all $S \subseteq [a, b]$, write*

$$M(f, S) = \sup\{f(x) \mid x \in S\}$$
$$m(f, S) = \inf\{f(x) \mid x \in S\}$$

Definition 18.2. *A partition of $[a, b]$ is a finite set $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$.*

Definition 18.3. *The upper and lower Darboux sums of a function over a partition of its domain are*

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$
$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

Definition 18.4.

$$U(f) = \inf\{U(f, P) \mid P \textit{ is a partition of } [a, b]\}$$
$$L(f) = \sup\{L(f, P) \mid P \textit{ is a partition of } [a, b]\}$$

Definition 18.5. *We say f is integrable if $U(f) = L(f)$. In this case, we write*

$$\int_a^b f = \int_a^b f(x)dx = U(f) = L(f)$$

where $\int_a^b f$ is the Darboux integral of f from a to b.

The conceptual difference between a Darboux integral and a Riemann sum is that a Riemann sum picks a specific point in each interval to sample the function (e.g. one of the endpoints or the middle) whereas the Darboux integral picks the maximum or minimum function value over the interval (as denoted by the infimum or supremum of each $U(f, P)$ or $L(f, P)$).

Example 18.3. Let $f(x) = c$. We expect $\int_a^b c = (b - a)c$.

$$M(f, S) = m(f, S) = c \ \forall S \subseteq [a, b]$$
$$\implies U(f, P) = \sum_{k=1}^{n} c(t_k - t_{k-1}) \qquad = c(t_1 - t_0 + t_2 - t_1 + \cdots + t_n - t_{n-1})$$
$$= c(t_n - t_0) = c(b - a)$$

Similarly, $L(f,P) = c(b-a)$. Therefore,

$$U(f) = \sup\{c(b-a)\} = c(b-a) = \inf\{c(b-a)\} = L(f)$$

This tells us that $f(x) = c$ is integrable.

□

Example 18.4.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

For any partition $P = \{a = t_0 < t_1 < \cdots < t_n < b\}$,

$$U(f,P) = \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1}) = b - a$$
$$L(f,P) = \sum_{k=0}^{n} 0 \cdot (t_k - t_{k-1}) = 0$$

So $U(f) = b - a$ and $L(f) = 0$, which implies f is not integrable.

□

18.3 Properties of Darboux integrals

Lemma 18.1. *Let $f : [a,b] \to \mathbb{R}$ be bounded, and let P and Q be partitions of $[a,b]$. If $P \subseteq Q$, then*

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$

Proof. We show that $U(f,Q) \leq U(f,P)$; the other subparts are proved in Ross.

We may assume that $Q \setminus P = \{u\}$. The idea is that this can be a base case for an induction proof, but it is sufficient to just show the case where P and Q differ by one point, because both sets are necessarily finite; for some finite set of k distinct points q_i in Q not in P, we can define $P_j = P \cup \{q_i \mid 1 \leq i \leq j\}$, and show that $U(f,Q) \leq U(f,P_k) \leq U(f,P_{k-1}) \leq \cdots \leq U(f,P_1) \leq U(f,P)$.

Let P and Q be defined as follows,

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$
$$Q = \{a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b\}$$

Then, we know that

$$\begin{aligned} [t_{k-1}, u] &\subseteq [t_{k-1}, t_k] \\ M(f, [t_{k-1}, u]) &\leq M(f, [t_{k-1}, t_k]) \end{aligned}$$

Therefore

$$\begin{aligned} U(f,P) - U(f,Q) &= M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) - M(f,[t_{k-1},u]) \cdot (u_{t_{k-1}}) - M(f,[u,t_k]) \cdot (t_k - u) \\ &\geq M(f,[t_{k-1},t_k]) \cdot [(t_k - t_{k-1}) - (u - t_{k-1} + t_k - u)] = 0 \end{aligned}$$

Therefore $U(f,P) \geq U(f,Q)$. □

Lemma 18.2. *Let $f : [a,b] \to \mathbb{R}$ be bounded, and let P, Q be partitions of $[a,b]$. Then $L(f,P) \leq U(f,Q)$.*

Proof. $P \cup Q$ is a partition with $P, Q \subseteq P \cup Q$. Therefore,

$$L(f,P) \leq L(f,P \cup Q) \leq U(f,P \cup Q) \leq U(f,Q).$$

□

Proposition 18.3. $L(f) \leq U(f)$

Proof.

$$\begin{aligned} \forall P, Q, L(f,P) &\leq U(f,Q) \\ L(f,P) &\leq \inf\{U(f,Q)\} = U(f)\ \forall P \\ \therefore U(f) &\geq \sup\{L(f,P)\} = L(f) \end{aligned}$$

□

18.4 The Cauchy criterion for integrability

Theorem 18.4. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. f is integrable if and only if $\forall \epsilon > 0, \exists$ a partition P such that*

$$U(f,P) - L(f,P) < \epsilon$$

Proof. Suppose f is integrable. $\forall \epsilon > 0$, there exist P_1, P_2 such that

$$\begin{aligned} L(f,P_1) &> L(f) - \frac{\epsilon}{2} \\ U(f,P_2) &< U(f) + \frac{\epsilon}{2} \end{aligned}$$

due to the definitions of inf and sup.

Let $P = P_1 \cup P_2$. Then $U(f,P) \le U(f,P_2)$ and $L(f,P) \ge L(f,P_1)$, so

$$\begin{aligned} U(f,P) - L(f,P) &\le U(f,P_2) - L(f,P_1) \\ &< U(f) + \frac{\epsilon}{2} - \left(L(f) - \frac{\epsilon}{2}\right) = U(f) - L(f) + \epsilon = \epsilon \end{aligned}$$

as required.

In the other direction, let $\epsilon > 0$, and pick P such that $U(f,P) - L(f,P) < \epsilon$.

$$\begin{aligned} U(f) \le U(f,P) &= U(f,P) - L(f,P) + L(f,P) \\ &< \epsilon + L(f) \end{aligned}$$

Therefore, $U(f)$ is a lower bound on $(L(f), \infty)$ so $*(f) \le \inf((L(f),\infty)) = L(f)$. But we know $L(f) \le U(f)$, so this implies $L(f) = U(f)$ and so f is integrable. $\square$

18.5 Riemann sums

Definition 18.6. *Let $f : [a,b] \to \mathbb{R}$ bounded and let P be a partition of $[a,b]$. A Riemann sum for f is a sum*

$$\sum_{k=1}^{n} f(x_k) \cdot (t_k - t_{k-1})$$

where $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ and $x_k \in [t_{k-1}, t_k]$.

Definition 18.7. *We define* $\operatorname{mesh}(P) = \max\{t_k - t_{k-1} \mid 1 \le k \le n\}$.

Definition 18.8. *f is Riemann integrable if there exists $r \in \mathbb{R}$ such that*

$$\begin{aligned} \forall \epsilon > 0, \exists \delta > 0 \,\text{s.t.} \\ \forall \textit{ partitions of } [a,b] P \,\text{s.t.}\, \operatorname{mesh} P < \delta \\ \textit{and any Riemann sum } S \textit{ associated to } P, |S - r| < \epsilon \end{aligned}$$

Here, r is the Riemann integral of f.

Theorem 18.5. *f is integrable if*

$$\forall \epsilon > 0, \exists \delta > 0 \,\text{s.t.}\, \operatorname{mesh}(P) < \delta \implies U(f,P) - L(f,P) < \epsilon.$$

Theorem 18.6. *f is integrable if and only if f is Riemann integrable, and in this case the integrals are equal.*

Lemma 18.7. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Let (s_n) be a sequence of Riemann sums such that*

$$\lim_{n\to+\infty} \mathrm{mesh}(P_n) = 0$$

where P_n is the partition associated to s_n. Then

$$\lim_{n\to+\infty} s_n = \int_a^b f.$$

Theorem 18.8. *Let $f : [a, b] \to \mathbb{R}$ be bounded. If f is monotonic, then f is integrable.*

Proof. We prove the case where f is decreasing. For all $x \in [a, b]$,

$$f(b) \leq f(x) \leq f(a)$$

So if $f(b) = f(a)$, then f is constant, therefore it is integrable. Therefore, we can assume $f(b) < f(a)$.

Let $\epsilon > 0$. We want P such that $U(f, P) - L(f, P) < epsilon$. Pick $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ such that $\mathrm{mesh}(P) = \frac{\epsilon}{f(a)-f(b)}$. Then

$$\begin{aligned}\forall k, &M(f, [t_{k-1}, t_k]) = f(t_{k-1})\\ &m(f, [t_{k-1}, t_k]) = f(t_k)\end{aligned}$$

So

$$\begin{aligned}U(f, P) - L(f, P) &= \sum_{k=1}^{n} (M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])) \cdot (t_k - t_{k-1})\\ &< \sum_{k=1}^{n} (f(t_{k-1}) - f(t_k)) \cdot \frac{\epsilon}{f(a) - f(b)}\\ &= \frac{\epsilon}{f(a) - f(b)} (f(t_0) - f(t_n)) = \epsilon\end{aligned}$$

Therefore f is integrable.

□

Math 104: Introduction to Analysis **Summer 2019**

Lecture 19: Proving A Lot Of Things Are Integrable

Lecturer: Michael Christianson *29 July* *Aditya Sengupta*

19.1 Continuous Functions are Integrable

Recall the Cauchy criterion for integrability,

Theorem 19.1. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then, f is integrable if $\forall \epsilon > 0, \exists$ a partition P of $[a, b]$ such that*

$$U(f, P) - L(f, P) < \epsilon.$$

Also, recall the following consequence of that criterion, that monotonic functions are integrable. We can use this to show that any continuous function is integrable.

Theorem 19.2. *Let $f : [a, b] \to \mathbb{R}$. If f is continuous, then f is integrable.*

Proof. Let $\epsilon > 0$. f is uniformly continuous on $[a, b]$. Therefore there exists $\delta > 0$ such that

$$\forall x, y \in [a, b], |x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}$$

Pick a partition such that $\mathrm{mesh}(P) < \delta$. Then, for all k, there exists $x_0, y_0 \in [t_{k-1}, t_k]$ such that

$$f(x_0) \leq f(x) \leq f(y_0) \ \forall x \in [t_{k-1}, t_k]$$

Then,

$$M(f, [t_{k-1}, t_k]) = f(y_0)$$
$$m(f, [t_{k-1}, t_k]) = f(x_0)$$

and

$$|x_0 - y_0| \leq t_k - t_{k-1} \leq \mathrm{mesh}(P) < \delta$$

Therefore, by choice of δ,

$$
\begin{aligned}
|f(y_0) - f(x_0)| &< \frac{\epsilon}{b-a} \\
M(f, [t_{k-1} - t_k]) - m(f, [t_{k-1}, t_k]) &< \frac{\epsilon}{b-a}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
U(f, P) - L(f, P) &= \sum_{k=1}^{n} (M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])) \cdot (t_k - t_{k-1}) \\
&< \sum_{k=1}^{n} \frac{\epsilon}{b-a}(t_k - t_{k-1}) \\
&= \frac{\epsilon}{b-a} \sum_{k=1}^{n} (t_k - t_{k-1}) \\
&= \frac{\epsilon}{b-a} \cdot (b-a) = \epsilon
\end{aligned}
$$

Therefore f is integrable. □

19.2 Smashing Integrable Things Together Creates Integrable Things

Proposition 19.3. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable.*

(a) $\forall c \in \mathbb{R}$, cf is integrable, and

$$\int_a^b cf = c \int_a^b f$$

(b) $f + g$ is integrable, and

$$\int_a^b f + g = \int_a^b f + \int_a^b g$$

(c) fg, $\min(f, g)$, and $\max(f, g)$ are integrable.

Proof. (a) If $c = 0$ then $c \cdot f = 0$, therefore cf is a constant function whose integral is $c(b-a) = 0(b-a) = 0$, which is the same as $0 \cdot \int_a^b f = 0$

Case. *Consider $c > 0$. For all $S \subseteq [a, b]$, write $cS = \{cs \mid s \in S\}$. Then*

$$
\begin{aligned}
M(cf, S) &= c \cdot M(f, S) \\
m(cf, S) &= c \cdot m(f, S)
\end{aligned}
$$

So, for all partitions P,

$$\begin{aligned} U(cf, P) &= c \cdot U(f, P) \\ L(cf, P) &= c \cdot L(f, P) \end{aligned}$$

Therefore

$$\begin{aligned} L(cf) &= \sup\{c \cdot L(f, P) \mid P \text{ a partition } \} \\ &= c \cdot \sup\{L(f, P) \mid P \text{ a partition } \} \\ &= c \cdot L(f) \end{aligned}$$

and similarly, $U(cf) = cU(f)$. Therefore, cf integrates to $U(cf) = L(cf) = cU(f) = cL(f) = c\int_a^b f$ as required.

Case. *Consider $c = -1$. Use $-\sup(-S) = \inf S$; we get $U(-f) = -L(f)$ and $L(-f) = -U(f)$. Therefore,*

$$\int_a^b f = U(f) = L(f) \implies \int_a^b (-f) = U(-f) = L(-f) = -L(f) = -U(f) = -\int_a^b f$$

Case. *Consider $c < 0$. Use cases 1 and 2:*

$$\begin{aligned} \int_a^b cf = \int_a^b (-1) \cdot |c| f &= -\int_a^b |c| f \\ &= -|c| \int_a^b f \\ &c \int_a^b f \end{aligned}$$

(b) Let $\epsilon > 0$. We want P such that

$$U(f+g, P) - L(f+g, P) < \epsilon$$

Because f and g are integrable, there exist partitions P_1, P_2 of $[a, b]$ such that

$$\begin{aligned} U(f, P_1) - L(f, P_1) &< \frac{\epsilon}{2} \\ U(g, P_2) - L(g, P_2) &< \frac{\epsilon}{2} \end{aligned}$$

Let $P = P_1 \cup P_2$. $P_1 \subseteq P$ implies

$$\begin{aligned} U(f,P) - L(f,P) \le U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2} \\ U(g,P) - L(g,P) \le U(g,P_1) - L(g,P_1) < \frac{\epsilon}{2} \end{aligned}$$

For all $S \subseteq [a,b]$,

$$\begin{aligned} M(f+g,S) \le M(f,S) + M(g,S) \\ m(f+g,S) \ge m(f,S) + m(g,S) \end{aligned}$$

Therefore

$$\begin{aligned} U(f+g,P) \le U(f,P) + U(g,P) \\ L(f+g,P) \ge L(f,P) + L(g,P) \end{aligned}$$

So

$$\begin{aligned} U(f+g,P) - L(f+g,P) &\le (U(f,P) + U(g,P)) - (L(f,P) - L(g,P)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore $f+g$ is integrable.

Next, we want to show that $f+g$ integrates to the sum of the integrals of f and g.

$$\begin{aligned} \int_a^b f+g = U(f+g) &\le U(f+g,P) \le U(f,P) + U(g,P) \\ &< L(f,P) + L(g,P) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &\le L(f) + L(g) + \epsilon \\ &= \int_a^b f + \int_a^b g + \epsilon \end{aligned}$$

Also,

$$\begin{aligned} \int_a^b f+g = L(f+g) &\ge L(f+g,P) \ge L(f,P) + L(g,P) \\ &> U(f,P) - \frac{\epsilon}{2} + U(g,P) - \frac{\epsilon}{2} \\ &\ge U(f) + U(g) - \epsilon \\ &= \int_a^b f + \int_a^b g - \epsilon \end{aligned}$$

Therefore, $\int_a^b f+g$ is arbitrarily close to $\int_a^b f + \int_a^b g$ in both directions. This implies

$$\int_a^b f+g = \int_a^b f + \int_a^b g$$

(c) To be done on HW 6.

□

19.3 Splitting Integrable Things Up Creates Integrable Things

Lemma 19.4. *Let* $f : [a,b] \to \mathbb{R}$.

(a)

$$\begin{aligned} &\forall [c,d] \subseteq [a,b], \\ f \text{ integrable on } [a,b] &\implies f \text{ integrable on } [c,d] \end{aligned}$$

(b)

$$\begin{aligned} &\forall a < c < b, \\ f \text{ integrable on } [a,c] \wedge f \text{ integrable on } [c,b] &\implies f \text{ integrable on } [a,b] \\ &\int_a^b f = \int_a^c f + \int_c^b f \end{aligned}$$

Proof. (a) See Ross 32.8

(b) f is bounded on $[a,c]$ and on $[c,b]$, therefore f is bounded on $[a,b]$.
Let $\epsilon > 0$. Pick P_1 a partition of $[a,c]$, P_2 a partition of $[c,b]$.

$$\begin{aligned} U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2} \\ U(f,P_2) - L(f,P_2) < \frac{\epsilon}{2} \end{aligned}$$

Let $P = P_1 \cup P_2$. Then

$$\begin{aligned} U(f,P) = U(f,P_1) + U(f,P_2) \\ L(f,P) = L(f,P_1) + L(f,P_)2 \\ U(f,P) - L(f,P) = &\quad U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) \\ < &\quad \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ = \epsilon & \end{aligned}$$

Therefore, f is integrable on $[a,b]$. Further,

$$\begin{aligned}\int_a^b f = U(f) \le U(f,P) &= U(f,P_1) + U(f,P_2) \\ &< L(f,P_1) + \frac{\epsilon}{2} + L(f,P_2) + \frac{\epsilon}{2}\end{aligned}$$

and similarly in the other direction, $\int_a^b f$ is arbitrarily close to and therefore equal to $\int_a^c f + \int_c^b f$.

□

Proposition 19.5. *Let $f, g : [a,b] \to \mathbb{R}$ be integrable.*

(a) If $f(x) \le g(x)$ $\forall x \in [a,b]$, then

$$\int_a^b f \le \int_a^b g$$

(b) If g is continuous, $g(x) \ge 0$ $\forall x$, and $\int_a^b g = 0$, then $g(x) = 0 \forall x \in [a,b]$.

(c) $|f|$ is integrable, and it integrates to something greater than or equal to the absolute value of the integral of f.

Proof. (a) Let $h = g - f$. Then, $h(x) \ge 0 \forall x$. This implies

$$L(h,P) \ge 0 \ \forall P$$

So

$$0 \le L(h,P) \le L(h) = \int_a^b h = \int_a^b g - \int_a^b f$$

So

$$\int_a^b f \le \int_a^b g$$

(b) We proceed by contraposition. Suppose $g(x_0) \neq 0$ for some $x_0 \in [a,b]$. Because g is continuous,

$$\lim_{x \to x_0} g(x) = g(x_0) > 0$$

Therefore there exists a δ such that $|x - x_0| < \delta$ implies $g(x) > 0$. Let $c = x_0 - \delta$ and $d = x_0 + \delta$. We may assume $[c,d] \subseteq [a,b]$. g is integrable on $[c,d]$, and

$$\int_a^b g = \int_a^c g + \int_c^d g + \int_d^b g \ge \int_c^d g$$

We know this because $g \geq 0$ on $[a, c]$ and $[d, b]$.

Then, $g(x) \geq g(x_0) \forall x \in [c, d]$, which implies

$$\int_a^b g \geq \int_c^d g \geq \int_c^d g(x_0) = g(x_0) \cdot (d - c) > 0$$

Therefore $\int_a^b g > 0$.

(c) First, we show that $|f|$ is integrable. We claim that $\forall S \subseteq [a, b]$,

$$M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S)$$

This is true by the "reverse" triangle inequality. This implies

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$$

for all partitions P. So for all $\epsilon > 0$, pick P such that

$$U(f, P) - L(f, P) < \epsilon$$

Therefore

$$U(|f|, P) - L(|f|, P) < \epsilon$$

Therefore $|f|$ is integrable. Then

$$-|f(x)| \leq f(x) \leq |f(x)| \forall x$$

By part (a),

$$\int_a^b -|f| \leq \int_a^b f \leq \int_a^b |f|$$

Therefore

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

$\square$

Math 104: Introduction to Analysis **Summer 2019**

Lecture 20: Integration contd.

Lecturer: Michael Christianson *30 July* *Aditya Sengupta*

Last time, we covered some properties of integrals, which we continue to discuss here.

Remark 20.1. *$|f|$ integrable does not imply f is integrable.*

Corollary 20.2. *If $f(x) < g(x) \forall x$, then $\int_a^b f < \int_a^b g$.*

Proof. Let $h = g - f$. Then $h(x) > 0 \forall x$. This implies

$$\int_a^b h \geq \int_a^b 0 = 0$$

If $\int_a^b h = 0$, then $h(x) = 0 \forall x$. But $h(x) > 0$ at any point implies $\int_a^b h > 0$, therefore $\int_a^b f < \int_a^b g$. □

Lemma 20.3. *Let $f, g : [a, b] \to \mathbb{R}$. Suppose g is integrable at $f(x) = g(x)$ for all but finitely many x. Then f is integrable, and $\int_a^b f = \int_a^b g$.*

Proof. Let $\{x_1, \dots, x_n\}$ be such that $f(x) \neq g(x)$ if $x = x_i$ for some i. The proof is by induction on n.

<u>Base case</u>: Let $h = |g - f|$. We show that h is integrable and that it integrates to 0. Notice that $h(x) = 0$ for all $x \neq x_i$. By swapping h with $-h$, we may assume $h(x_1) \geq 0$. For all partitions P, x_1 is in at most two intervals in P. For all intervals I not containing x_1,

$$M(h, I) = 0.$$

Therefore

$$U(h, P) \leq 2h(x_1)\,\mathrm{mesh}(P)$$

So, picking $\mathrm{mesh}(P) < \frac{\epsilon}{2h(x_1)}$ for any $\epsilon > 0$, we get

$$U(h) < U(h, P) \leq 2 \cdot h(x_1) \cdot \mathrm{mesh}(P) < \epsilon$$

This is true for any $\epsilon > 0$, therefore $U(h)$ must be less than the lowest possible value ϵ can take:

$$U(h) \leq \inf((0, \infty)) = 0$$

But $h(x) \geq 0 \forall x \implies U(h,P) \geq 0 \forall P \implies U(h) \geq 0$

Therefore $U(h) = 0$. Similarly, we can show $L(h) = 0$, and so $U(h) = L(h) = 0 = \int_a^b h$.

Inductive step: Suppose true for some $n \geq 1$. Let

$$g_1(x) = \begin{cases} g(x) & x \neq x_{n+1} \\ f(x) & x = x_{n+1} \end{cases}$$

Then, $g_1(x) \neq f(x)$ iff $x \in \{x_1, \ldots, x_n\}$. Because $g_1(x) = g(x)$ for all $x \neq x_{n+1}$, by the base case, g_1 is integrable, and

$$\int_a^b g = \int_a^b g_1$$

Therefore, by the inductive hypothesis, f is integrable, and $\int_a^b g_1 = \int_a^b f$.

Then, $h + g = f$ is integrable, and

$$\int_a^b f = \int_a^b g + \int_a^b h = \int_a^b g$$

as required. □

Definition 20.1. *A function $f : [a,b] \to \mathbb{R}$ is piecewise monotonic (and respectively piecewise continuous) if there exists a partition P of $[a,b]$ such that $f|_{(t_{k-1},t_k)}$ is monotonic (and respectively uniformly continuous.)*

Recall that on a closed interval, being continuous means the same as being uniformly continuous, but not on an open interval.

Theorem 20.4. *If $f : [a,b] \to \mathbb{R}$ is piecewise continuous, or bounded and piecewise monotonic, then f is integrable.*

Proof. Let P be a partition as in the definition of a function being piecewise monotonic or piecewise continuous.

If f is piecewise continuous, then $f|_{(t_{k-1},t_k)}$ is uniformly continuous, and therefore it extends to a continuous function

$$f_k : [t_{k-1}, t_k] \to \mathbb{R}$$

f_k is continuous, so it is integrable.

If f is bounded and piecewise monotonic, then $f|_{(t_{k-1},t_k)}$ extends to a monotonic function

$$f_k : [t_{k-1}, t_k] \to \mathbb{R}$$

If $f|_{(t_{k-1},t_k)}$ is increasing, set $f_k(t_{k-1}) = \inf\{f(x) \mid x \in (t_{k-1}, t_k)\}$ and $f_k(t_k) = \sup\{f(x) \mid x \in (t_{k-1}, t_k)\}$. If $f|_{(t_{k-1},t_k)}$ is decreasing, then swap the inf and sup. Then, $f_k : [t_{k-1}, t_k] \to \mathbb{R}$ is monotonic, so f_k is integrable.

Now, $f|_{(t_{k-1},t_k)}(x) = f_k(x)$ except possibly at the endpoints. However, there are finitely many endpoints, and if two functions differ at a finite number of points and one is integrable, then the other is also integrable, and they integrate to the same value. Therefore,

$$f|_{(t_{k-1},t_k)} \text{ integrable } \forall k \implies f \text{ integrable on } [a,b]$$

Further, we can say that

$$\int_a^b f = \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} f$$

using the property of splitting up integration intervals and induction ($[a,c] \cup [c,b] = [a,b]$ where you replace c by any t_k.) □

Theorem 20.5. *("IVT" for integrals) Let $f : [a,b] \to \mathbb{R}$ be continuous.*

$$\exists x \in (a,b) \, \text{s. t.} \, f(x) = \frac{1}{b-a} \int_a^b f.$$

Proof. Let $x_0, y_0 \in [a,b]$ such that

$$f(x_0) \leq f(x) \leq f(y_0) \forall x \in [a,b]$$

If $f(x_0) = f(y_0)$, then f is constant, and this is trivially true:

$$\frac{1}{b-a} \int_a^b f = \frac{1}{b-a} f(x)(b-a) = f(x) \forall x \in [a,b]$$

If $f(x_0) < f(y_0)$, then

$$\begin{aligned} f(y_0) - f(x) &\geq 0 \forall x \\ f(y_0) - f(x_0) &> 0 \end{aligned}$$

Therefore

$$\int_a^b f(y_0) - f(x) \geq \int_a^b 0 = 0$$

$$\therefore \int_a^b f(y_0) - \int_a^b f > 0$$

$$f(y_0) \cdot (b-a) > \int_a^b f$$

$$f(y_0) > \frac{1}{b-a} \int_a^b f$$

Likewise,

$$f(x_0) < \frac{1}{b-a} \int_a^b f$$

$\square$

Definition 20.2. *Let $f : (a,b) \to \mathbb{R}$ be a function. We say f is integrable on $[a,b]$ if every extension of f to $[a,b]$ is integrable. In this case, we define*

$$\int_a^b f = \int_a^b \widetilde{f}$$

where $\widetilde{f} : [a,b] \to \mathbb{R}$ is any extension of f. This does not depend on the choice of $\widetilde{f}$.

Theorem 20.6. *(FTC 1) Let $g : [a,b] \to \mathbb{R}$ be continuous. Suppose g is differentiable on (a,b) and that $g' : (a,b) \to \mathbb{R}$ is integrable. Then*

$$\int_a^b g'(x)dx = g(b) - g(a)$$

Proof. The key idea here is to use the mean value theorem to relate g' to g.

Let P be a partition of $[a,b]$. For all k, there exists $x_k \in (t_{k-1}, t_k)$ such that

$$g'(x_k) = \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}$$

This is true by the MVT. This gives us

$$g'(x_k)(t_k - t_{k-1}) = g(t_k) - g(t_{k-1})$$

Summing up over all k, we get

$$\sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}) = \sum_{k=1}^{n}(g(t_k) - g(t_{k-1}))$$

This is a telescoping sum, that ends up with just

$$\sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}) = g(b) - g(a)$$

Then, for all P,

$$\begin{aligned} L(g', P) \leq \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}) \leq U(g', P) \\ L(g', P) \leq g(b) - g(a) \leq U(g', P) \end{aligned}$$

Also,

$$L(g', P) \leq L(g') = \int_a^b g' = U(g') \leq U(g', P)$$

This gives us

$$\left| \int_a^b g' - (g(b) - g(a)) \right| \leq U(g', P) - L(g', P)$$

Because g' is integrable, we know that the difference between $U(g', P)$ and $L(g', P)$ is arbitrarily small:

$$\begin{aligned} \forall \epsilon > 0, \exists P \,\mathrm{s.\,t.}\, U(g', P) - L(g', P) < \epsilon \\ \left| \int_a^b g' - (g(b) - g(a)) \right| < \epsilon \forall \epsilon > 0 \\ \implies \int_a^b g' = g(b) - g(a) \end{aligned}$$

□

Definition 20.3. *Let $f : [a, b] \to \mathbb{R}$ be integrable. We define*

$$\begin{aligned} \int_a^a f = 0 \\ \int_b^a f = \int_a^b f \end{aligned}$$

Theorem 20.7. *if FTC is so great why isn't there an (FTC 2): Let $f : [a,b] \to \mathbb{R}$ be integrable. Define $F : [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(t)dt$$

(a) F is continuous on $[a,b]$

(b) $\forall x_0 \in (a,b)$, if f is continuous at x_0 then F is differentiable at x_0 and $F'(x_0) = f(x_0)$

Proof. (a) f is bounded implies there exists $B \in \mathbb{R}$ such that $\forall x \in [a,b]$, $|f(x)| \le B$. Let $\epsilon > 0$ and define $\delta = \frac{\epsilon}{B}$ Then $\forall x, y \in [a,b]$ such that $|x-y| < \delta$ (we may assume $x < y$),

$$|F(y) - F(x)| = \left|\int_x^y f(t)dt\right|$$
$$\int_a^y f - \int_a^x f \le \int_x^y |f(t)|dt \le \int_x^y Bdt = B(y-x) < \epsilon$$

Therefore F is uniformly continuous, so it is continuous.

(b) Suppose f is continuous at x_0; then $\forall x > x_0$,

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0}\int_{x_0}^x f(t)dt$$

We also know that

$$\frac{1}{x - x_0}\int_{x_0}^x f(x_0)dt = f(x_0)$$

Therefore

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x [f(t) - f(x_0)]dt$$

Let $\epsilon > 0$. f is continuous at x_0, so there exists δ such that $\forall t \in (a,b)$ such that $|t - x_0| < \delta$, $|f(t) - f(x_0)| < \epsilon$. Therefore, for all $x > x_0$ such that $|x - x_0| < \delta$, $|t - x_0| < \delta \forall t \in (x_0, x)$. So

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| \le \frac{1}{x - x_0}\int_{x_0}^x |f(t) - f(x_0)|dt \qquad < \frac{1}{x - x_0}\int_{x_0}^x \epsilon dt = \epsilon$$

$\square$

Math 104: Introduction to Analysis **Summer 2019**

Lecture 21: Integration Properties, Riemann-Stieltjes Integrals, Review

Lecturer: Michael Christianson *31 July* *Aditya Sengupta*

21.1 Integral Properties

The second FTC can be used to derive a number of properties of integrals.

1. Antiderivative formulas. For example, let

$$g(x) = \frac{x^{n+1}}{n+1}$$
$$g'(x) = x^n$$

   Then

$$\int_a^b x^n dx = \frac{b^{n+1} - a^{n+1}}{n+1}$$

2. u-substitution (analogous to the chain rule)
3. Integration by parts (analogous to the product rule)

Theorem 21.1. *(u-substitution) Let $J \subseteq \mathbb{R}$ be an open interval, let $u : J \to \mathbb{R}$ be differentiable, let I be an open interval such that $u(J) \subseteq I$, and let $f : I \to \mathbb{R}$ be continuous. If u' is continuous, then for all $a, b \in J$,*

$$\int_a^b f(u(x))u'(x) = \int_{u(a)}^{u(b)} f(u)du$$

Proof. Let $c \in I$ and let

$$F(x) = \int_c^x f(t)dt$$

Let $g = F \circ u$. f is continuous, therefore F is differentiable, so g is differentiable and by the chain rule,

$$g'(x) = F'(u(x)) \cdot u'(x) = f(u(x)) \cdot u'(x)$$

Therefore

$$\begin{aligned}
\int_a^b f(u(x))u'(x)dx &= \int_a^b g'(x)dx = g(b) - g(a) \\
&= F(u(b)) - F(u(a)) \\
&= \int_c^{u(b)} f(t)dt - \int_c^{u(a)} f(t)dt \\
&= \int_{u(a)}^{u(b)} f(t)dt
\end{aligned}$$

□

Theorem 21.2. *(Integration by parts) Let $u, v : [a, b] \to \mathbb{R}$ be differentiable on (a, b) and suppose u', v' are integrable on $[a, b]$. Then*

$$\int_a^b u'(x)v(x)dx + \int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a)$$

Proof. Let $g = u \cdot v$. Then $g' = u'v + uv'$ is integrable, and

$$\int_a^b g'(x)dx = g(b) - g(a) = u(b)v(b) - u(a)v(a)$$

$$\int_a^b u(x)v'(x)dx + \int_a^b u'(x)v(x)dx = g(b) - g(a) = u(b)v(b) - u(a)v(a)$$

□

Example 21.1. Consider some $f : [0, 2] \to \mathbb{R}$ continuous, and

$$G(x) = \int_0^{x^2} f(t)dt$$

We see that $G = F \circ g$ where $F(x) = \int_0^x f(t)dt$ and $g = x^2$. Because f is continuous on $[0, 2]$, F is differentiable, so G is differentiable on $[0, 2]$, and by the chain rule,

$$G'(x) = F'(g(x))g'(x) = 2xf(x^2)$$

□

21.2 Riemann-Stieltjes Integrals

The Riemann-Stieltjes integral is a tool for dealing with expected values in probability theory.

Example 21.2. Suppose you have a weighted die such that $P(2) = \frac{1}{2}, P(5) = \frac{1}{6}, P(1) = P(3) = P(4) = P(6) = \frac{1}{12}$.

The expected average roll is

$$E(x) = \sum_{k=1}^{6} k \cdot P(k) = (1 + 3 + 4 + 6) \cdot \frac{1}{12} + 2 \cdot \frac{1}{2} + 5 \cdot \frac{1}{6} = 3$$

If the die is unweighted,

$$E(x) = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3.5$$

□

Suppose the probability is continuous, not discrete, and it is given by the function (PDF) $P : \mathbb{R} \to [0, 1]$. Let $c(y) = P(X \leq y)$ be the CDF; then if $X \in \mathbb{Z}$,

$$P(X = k) = P(X \leq k) - P(X \leq k - 1) = c(k) - c(k - 1)$$

So in the above example, we can rewrite the expected value as

$$E(x) = \sum_{k=1}^{6} k \cdot P(k) = \sum_{k=1}^{6} k \cdot (c(k) - c(k - 1))$$

This looks like a Riemann sum, with each interval $[t_{k-1}, t_k]$ being weighted by a function c.

Definition 21.1. *Let $f : [a, b] \to \mathbb{R}$ be integrable and let $\alpha : [a, b] \to \mathbb{R}$ be increasing.*

For all partitions P of $[a, b]$, define

$$U_\alpha(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (\alpha(t_k) - \alpha(t_{k-1}))$$
$$L_\alpha(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (\alpha(t_k) - \alpha(t_{k-1}))$$

You could call these Darboux-Stieltjes sums. Or you could not do that.

Similarly to a regular Darboux sum, we define the upper and lower sums,

$$U_\alpha(f) = \inf\{U_\alpha(f, P) \mid P \ \textit{a partition}\ \}$$
$$L_\alpha(f) = \sup\{L_\alpha(f, P) \mid P \ \textit{a partition}\ \}$$

If $L_\alpha(f) = U_\alpha(f)$, f *is Riemann-Stieltjes integrable (*α*-integrable), and*

$$\int_a^b f(x) d\alpha = U_\alpha(f) = L_\alpha(f)$$

Example 21.3. If $\alpha(x) = x \forall x$, this reduces to a normal Riemann sum. □

Example 21.4. If α is differentiable on (a, b) and α' is integrable on $[a, b]$, then

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx$$

□

Example 21.5. If

$$\alpha(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$$

then

$$\int_a^b f(x) dx = f(c)$$

If (c_n) is a sequence such that $c_n \geq 0$, (s_n) is a sequence in $[a, b]$, then

$$\int_a^b f(x)d\alpha = \sum_{k=1}^{n} c_n f(s_n)$$

That is, the integral with respect to a step function ends up summing up a sequence's function values weighted by some other nonnegative sequence.

□

21.3 Applications

If X is a random variable with cdf $c : [a, b] \to [0, 1]$, then

$$E(f(x)) = \int_a^b f dc$$

If $m(x)$ is the mass element at some position x, then the moment of inertia of the object is

$$I = \int_a^b x dm$$

In stochastic calculus, the Ito integral is really just a R-S integral.

Mathematicians see Riemann-Stieltjes integrals as a special case of the Lebesgue integral, an even more general concept in measure theory.

21.4 Review

A list of important things:

- Series
- Continuity
- Derivatives
- Integrals
- Limits of functions

This is a pretty vague list.

21.4.1 Integrals

- Use definition of $U(f,P)$ and $L(f,P)$ and the $U(f,P)-L(f,P)<\epsilon$ theorem (Cauchy integrability)
- See section 32 up to theorem 32.5
- $P \subseteq Q \implies L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$
- Explicitly compute $\int_a^b f$ from the theorems
- Maybe basic proofs: do the section 32 exercises
- Section 33: write a lot of the thms and definitions
- Section 34: use the fundamental theorem on a composition (see hw 6)

21.4.2 Series

- Tests: comparison, root, ratio, integral, alternating
- $\sum a_n$ converges means $\lim\limits_{n\to+\infty} a_n = 0$
- Cauchy criterion
- Partial sum definition
- Use tests or basic proofs (e.g. 14.6 or 14.7)

21.4.3 Limits of Functions

Roughly in order of usefulness:

- $\delta - \epsilon$ definitions of limits
- f is continuous at a if and only if $\lim\limits_{x\to a} f(x) = f(a)$, assuming f is defined on an open interval containing a.
- More generally, if $\lim\limits_{x\to a} g(x) = b$ and f is continuous at b, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)$$

- Limit theorems (the same as for sequences)
- Occasionally, we'll use the fact that a limit exists if the left and right limits exist.
- Occasionally, if $\lim\limits_{x\to a} f(x) > b$, $\exists\delta > 0$ such that $\forall |x-a| < \delta$ we have $f(x) > b$. (Not necessarily worth space on the note sheet, but do it for practice)
- Sequence definition of a limit (can be useful to prove a limit doesn't exist)

21.4.4 $\delta - \epsilon$ definitions

$$\lim_{x \to a} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \,\text{s.t.}\, |x-a| < \delta \implies |f(x) - L| < \epsilon$$

For $x \to a^+$, switch the $|x-a| < \delta$ for $x \in (a, a+\delta)$ and for $x \to a^-$, switch it for $x \in (a-\delta, a)$.

For $x \to \infty$, replace $|x-a| < \delta$ with $\exists \alpha > 0$ such that $\forall x > \alpha$, and for $x \to -\infty, \exists \alpha < 0 \,\text{s.t.}\, \forall x < \alpha$.

If $L = +\infty$, replace $\forall \epsilon > 0$ with $\forall M > 0$, and replace $|f(x) - L| < \epsilon$ with $f(x) > M$. If $L = -\infty$, replace $\forall \epsilon > 0$ with $\forall M < 0$, and replace $|f(x) - L| < \epsilon$ with $f(x) < M$.

21.4.5 Derivatives

- Use the definition of the derivative for doable examples.
- Differentiable implies continuous
- Probable go-to counterexample is $x^n \sin \frac{1}{x}$ for $n = 0, 1, 2$.
- Know the chain rule and product rule but nothing much from there.
- Be able to use L'Hopital's.
- MVT and applications: know results in section 29, and there'll probably be a proof that uses the MVT.
- See exercises in section 29.

21.4.6 Continuity

- $\delta - \epsilon$ definitions of continuity
- Prove discontinuity with tthe sequence definition
- Theorems in section 19 are important

Math 104: Introduction to Analysis **Summer 2019**

Lecture 22: Power series

Lecturer: Michael Christianson *5 August* *Aditya Sengupta*

22.1 Definition of power series

Definition 22.1. *A power series is a series of the form*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

where x is a variable and $a_n \in \mathbb{R} \forall n$.

Remark 22.1. *We take $x^0 = 1$ even when $x = 0$ only in the case of power series.*

Theorem 22.2. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Define*

$$\beta = \limsup |a_n|^{1/n}$$
$$R = \begin{cases} 0 & \beta = +\infty \\ +\infty & \beta = 0 \\ \frac{1}{\beta} & \text{otherwise} \end{cases}$$

(a) For all x such that $|x| < R, \sum a_n x^n$ converges.

(b) For all x such that $|x| > R, \sum a_n x^n$ diverges.

Proof. For all $x \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned}\alpha &= \limsup |a_n x^n|^{1/n} = \limsup |a_n|^{1/n} \cdot |x_n|^{1/n} \\ &= |x| \limsup |a_n|^{1/n}\end{aligned}$$

We can carry out this last step by the following statement that tells us that it is possible to pull out constant factors from lim sups:

For all subsequences $t_{n_k} = \left(|x| \cdot |a_{n_k}|^{1/n_k}\right)_{k \in \mathbb{N}}$,

$$\lim_{n \to +\infty} t_{n_k} = |x| \cdot \lim_{n \to +\infty} |a_{n_k}|^{1/n_k} = |x| \cdot \lim_{n \to +\infty} |a_n|^{1/n}$$

so the suprema of both are also equal, i.e. the lim sups are equal.

Therefore, we see that $\alpha = |x| \cdot \beta$. Now, we take three cases.

Case. *If $0 < R < +\infty$, then $\alpha = |x|\beta = \frac{|x|}{R}$. Therefore*

$$\begin{aligned} |x| < R \implies \alpha < 1 &\implies \textit{convergence by Root Test} \\ |x| > R \implies \alpha > 1 &\implies \textit{divergence by Root Test} \end{aligned}$$

Case. *If $R = +\infty$, then $\alpha = |x|\beta = 0\ \forall x$. Therefore $|x| < R$ for all x and $\sum a_n x^n$ always converges by the Root Test. The second statement is vacuously true.*

Case. *If $R = 0$, then $\beta = +\infty$, so $\alpha = |x|\beta = +\infty$. So $\sum a_n x^n$ diverges for all $x \neq 0$ by the Root Test. Therefore, $\forall |x| > 0 = R \implies \sum a_n x^n$ diverges.*

□

Definition 22.2. *R in the above theorem is called the radius of convergence of $\sum a_n x^n$.*

We have shown that if $R = 0$, the sum diverges except if $x = 0$; if $R = +\infty$, the sum converges for all x; otherwise, if $0 < R < \infty$, the sum converges in some open interval $(-R, R)$, diverges outside that, and may or may not converge for $x = \pm R$.

Definition 22.3. *If $R \neq 0$, the interval of convergence is the interval of x values such that $\sum a_n x^n$ converges. This may be open, half-open, or closed.*

Corollary 22.3. *Let $\sum a_n x^n$ be a power series. Let*

$$\beta = \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and let R be defined as above.

(a) For all x such that $|x| < R, \sum a_n x^n$ converges.

(b) For all x such that $|x| > R, \sum a_n x^n$ diverges.

Proof. If β exists, then $\beta = \lim_{n \to +\infty} |a_n|^{1/n} = \limsup |a_n|^{1/n}$. □

Example 22.1. Consider $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Let $a_n = \frac{1}{n!}$.

$$\begin{aligned} \beta = \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \to +\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1} \\ &= \lim_{n \to +\infty} \frac{1}{n+1} = 0 \therefore R = +\infty \implies \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ converges } \forall x \in \mathbb{R}. \end{aligned}$$

It turns out that this is the definition of e^x.

□

Example 22.2. Consider $\sum_{n=0}^{\infty} x^n$, i.e. $a_n = 1 \forall n$.

$$\beta = \limsup |a_n|^{1/n} = \limsup 1 = 1 \implies R = \frac{1}{\beta} = 1$$

Therefore $\sum_{n=0}^{\infty} x^n$ converges for all $|x| < 1$. Also, it diverges when $x = \pm 1$ because we can analyze this with geometric series. Therefore the interval of convergence is $(-1, 1)$.

□

Example 22.3. Consider

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n} x^n$$
$$\beta = \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to +\infty} \frac{n}{n+1} = 1$$
$$R = \frac{1}{\beta} = 1$$

We check the endpoints,

$$x = 1 : \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \text{ converges by the alt. series test}$$
$$x = -1 : \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (-1)^n}{n} = sum_{n=0}^{\infty} \frac{1}{n} \text{ diverges.}$$

Therefore the interval of convergence is $(-1, 1]$.

□

Example 22.4. Consider

$$\sum_{n=0}^{\infty} 2^{-n} x^{3n}$$

The wrong way to do this is to take $\beta = \limsup (2^{-n})^{1/n} = \frac{1}{2}$. But $a_n \neq 2^{-n}$. Instead,

$$a_n = \begin{cases} 2^{-n/3} & n \mod 3 = 0 \\ 0 & n \mod 3 \neq 0 \end{cases}$$

Therefore,

$$\beta = \limsup |a_n|^{1/n} = \lim_{n\to+\infty} |2^{-n/3}|^{1/n} = 2^{-1/3}$$

Therefore, $R = \frac{1}{\beta} = 2^{1/3}$.

At the endpoints, we get $\sum 2^{-n} \cdot 2^n = \sum 1$ which diverges, and $\sum 2^{-n} \cdot (-2)^n = \sum (-1)^n$ which also diverges. Therefore the interval of convergence is $(-\sqrt[3]{2}, \sqrt[3]{2})$.

$\square$

We can also consider power series of the form $\sum_{n=0}^{\infty} a_n (x-b)^n$. Let $y = x - b$; then $\sum_{n=0}^{\infty} a_n y^n$ converges for $|y| < R$ and diverges for $|y| > R$. Therefore the interval of convergence becomes $(b - R, b + R)$.

Example 22.5.

Consider

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n} (x-1)^n$$

We previously showed that the unshifted version of this has an interval of convergence of $(-1, 1]$. Adding 1, we see the interval shifts to $(0, 2]$.

$\square$

22.2 Differentiability and integrability of power series

Let a power series be defined by $\sum_{n=0}^{\infty} a_n x^n$, and let the power series have radius of convergence R. Then $f : (-R, R) \to \mathbb{R}, f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a function. We want to know whether this function is differentiable or integrable, and what its derivative or integral might be.

Consider the partial sums

$$f_n(x) = \sum_{k=0}^{n} a_k x^k.$$

These are all polynomials, so they are continuous and differentiable, and we can use the power rule. For all $x \in (-R, R)$, $\lim_{n\to+\infty} f_n(x) = f(x)$.

So given a sequence of continuous functions $(g_n)_{n\in\mathbb{N}}$, is the function $g(x) = \lim_{n\to+\infty} g_n(x)$ continuous wherever it is defined? It turns out that this is not the case.

Example 22.6. Let $g_n(x) = (1 - |x|)^n$ for $x \in (-1, 1)$. g_n is continuous for all n, and $\forall x \in (-1, 1), 0 < 1 - |x| \leq 1$. So if $x \neq 0$,

$$\lim_{n\to+\infty} g_n(x) = \lim_{n\to+\infty} (1 - |x|)^n = 0$$

If $x = 0$, $g_n(0) = 1 \forall n$, so $\lim_{n\to+\infty} g_n(0) = 1$. Therefore,

$$g(x) = \lim_{n\to+\infty} g_n(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

which is not continuous.

□

Definition 22.4. *Let (f_n) be a sequence of functions with $\operatorname{dom}(f_n) = S \subseteq \mathbb{R} \forall n$. We say the sequence f_n converges pointwise to a function $f : S \to \mathbb{R}$ if $\lim_{n\to+\infty} f_n(x) = f(x) \forall x \in S$. We denote this by $f_n \to f$ pointwise, or $\lim_{n\to+\infty} f_n = f$ pointwise.*

Example 22.7. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R, and $f_n(x) = \sum_{k=0}^{\infty} a_k x^k$, then $f_n \to f$ pointwise on $(-R, R)$.

□

Notice that $f_n \to f$ pointwise iff

$$\forall x \in S, \forall \epsilon > 0, \exists N \text{ s.t.} \forall n > N, |f(x) - f_n(x)| < \epsilon.$$

Note that N can depend on both x and ϵ.

Definition 22.5. *We say (f_n) converges uniformly to a function $f : S \to \mathbb{R}$ if $\forall \epsilon > 0$, $\exists N$ such that*

$$|f(x) - f_n(x)| < \epsilon \forall x \in S, n > N.$$

We denote this by $f_n \to f$ uniformly, or $\lim\limits_{n \to +\infty} f_n = f$ uniformly.

Example 22.8. Let $f_n(x) = \frac{1}{n}\sin(nx)$. We claim that $f_n \to 0$ uniformly.

$$|0 - f_n(x)| = \left|\frac{1}{n}\sin(nx)\right| = \frac{1}{n}|\sin(nx) \leq \frac{1}{n}$$

Therefore, for all $\epsilon > 0$, let $N = \frac{1}{\epsilon}$. Then for all $n > N$,

$$|0 - f_n(x)| \leq \frac{1}{n} < \frac{1}{N} = \epsilon$$

□

Recall that $f : S \to \mathbb{R}$ is uniformly continuous if $\forall \epsilon > 0, \exists \delta \,\text{s. t.}\, \forall x, y \in S, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

Theorem 22.4. *Let (f_n) be a sequence of functions with domain S. If $f_n \to f$ uniformly on S, and if f_n is continuous at $a \forall n$, then f is continuous at a.*

Proof. The key idea is

$$\begin{aligned}|f(x) - f(a)| &= |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)| \\ &= |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|\end{aligned}$$

By uniform convergence with $\frac{\epsilon}{3}$, there exists some N such that $\forall y \in S, n > N$, we get

$$|f(y) - f_n(y)| < \frac{\epsilon}{3}$$

Also, continuity of $f_{\lceil N \rceil + 1}$ at a gives us δ such that

$$|f_{N+1}(x) - f_{N+1}(a)| < \epsilon$$

when $|x - a| < \delta$. Therefore all three parts can be bounded by $\frac{\epsilon}{3}$, i.e. $|f(x) - f(a)| < \epsilon$ so f is continuous at a. □

Math 104: Introduction to Analysis **Summer 2019**

Lecture 23: Uniform convergence

Lecturer: Michael Christianson *6 August* *Aditya Sengupta*

Proposition 23.1. *A sequence of functions (f_n) on a set S converges uniformly to some function f if and only if*

$$\lim_{n\to+\infty} \sup\{|f(x) - f_n(x)| \mid x \in S\} = 0$$

The proof will be on HW7 (exercise 24.12). This proposition is useful if $f - f_n$ is differentiable.

Note that the above limit is not the lim sup, because it is the limit over all x, not as $n >$some N.

Example 23.1. Let $f_n(x) = \frac{x}{1+nx^2}$. If $x \neq 0$, $\lim\limits_{n\to+\infty} f_n(x) = 0$, and if $x = 0, f_n(0) = 0 \implies \lim\limits_{n\to+\infty} f_n(0) = 0$. So $f_n \to 0$ pointwise on $\mathbb{R}$.

We claim that $f_n \to 0$ uniformly on $\mathbb{R}$. To show this, we want to show that for any $n \in \mathbb{N}$,

$$\lim_{n\to+\infty} \sup\{|f_n(x) \mid x \in \mathbb{R}\} = 0$$

We do this by examining the derivative:

$$f_n'(x) = \frac{(1+nx^2)\cdot 1 - x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

Therefore $f_n'(x)$ is defined for all x. This means f_n can only attain a maximum when $f_n'(x) = 0$, which is at $x = \pm\frac{1}{\sqrt{n}}$. At these points, $f_n\left(\pm\frac{1}{\sqrt{n}}\right) = \pm\frac{1}{2\sqrt{n}}$.

We claim that f_n has a global maximum at these points. This is the case because $\forall x > \frac{1}{\sqrt{n}}, f_n$ is decreasing; $\forall x \in \left(-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right)$, f_n is increasing, and $\forall x < -\frac{1}{\sqrt{n}}, -x > \frac{1}{\sqrt{n}} \implies f_n\left(\frac{1}{\sqrt{n}}\right) \geq f_n(-x) = -f_n(x)$ so $f_n\left(\frac{1}{\sqrt{n}}\right) \geq f_n(x)$.

This gives us

$$\sup\{|f_n(x)| \mid x \in \mathbb{R}\} = f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2\sqrt{n}}$$

and because $\lim_{n\to+\infty} \frac{1}{2\sqrt{n}} = 0, f_n \to 0$ uniformly.

□

Definition 23.1. *A sequence of functions (f_n) on S is uniformly Cauchy if*

$$\begin{aligned}\forall \epsilon > 0, \exists N \,\text{s. t.}\\ \forall m, n > N, |f_n(x) - f_m(x)| < \epsilon \forall x \in S, (f_n(x))_{n\in\mathbb{N}}\end{aligned}$$

Theorem 23.2. *Let (f_n) be a sequence of functions on S. (f_n) is uniformly convergent on S if and only if it is uniformly Cauchy on S.*

Proof. In the forward direction, suppose $f_n \to f$ uniformly for some $f : S \to \mathbb{R}$.

$$\begin{aligned}\forall \epsilon > 0, \exists N \,\text{s. t.}\\ \forall n > N, |f(x) - f_n(x)| < \frac{\epsilon}{2} \forall x \in S\end{aligned}$$

Therefore, $\forall m, n > N$ and $\forall x \in S$,

$$\begin{aligned}|f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)\\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)|\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

Therefore (f_n) is uniformly Cauchy.

In the backward direction, suppose (f_n) is uniformly Cauchy for all $x \in S$¿ Then the sequence $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy so it is convergent. So, define $f : S \to \mathbb{R}$ by

$$f(x) = \lim_{n\to+\infty} f_n(x)$$

$f_n \to f$ pointwise by definition. We show $f_n \to f$ uniformly.

Let $\epsilon > 0$. (f_n) uniformly Cauchy tells us that

$$\begin{aligned}\exists N \,\text{s. t.} \forall m, n > N,\\ |f_n(x) - f_m(x)| < \frac{\epsilon}{2} \forall x \in S\end{aligned}$$

So for a fixed n, $f_m(x) \in \left(f_n(x) - \frac{\epsilon}{2}, f_n(x) + \frac{\epsilon}{2}\right)$. Therefore, $\forall x \in S$,

$$m > N \implies \lim_{m\to\infty} f_m(x) \in \left[f_n(x) - \frac{\epsilon}{2}, f_n(x) + \frac{\epsilon}{2}\right]$$

Therefore

$$|f(x) - f_n(x)| \leq \frac{\epsilon}{2} < \epsilon$$

This works for all $n > N, x \in S$, so $f_n \to f$ uniformly by definition. $\square$

Definition 23.2. *A series of functions on S is a sum*

$$\sum_{n=0}^{\infty} g_n,$$

where $g_n : S \to \mathbb{R}$ is a function. Let $s_n(x) = \sum_{k=0}^{n} g_k(x)$. Then

$$\sum_{n=0}^{\infty} g_n = \lim_{n\to+\infty} s_n.$$

By definition, $s_n \to \sum g_n$ pointwise by definition.

Example 23.2. A power series $\sum a_n x^n$ with radius of convergence R is a series of functions $g_n(x) = a_n x^n \implies \sum a_n x^n = \sum g_n$. $\square$

Definition 23.3. *We say a series of functions $\sum g_n$ converges uniformly if the sequence (s_n) of partial sum functions converges uniformly if the sequence (s_n) of partial sum functions converges uniformly.*

Theorem 23.3. *Let $\sum g_n$ be a series of functions. If $\sum g_n$ converges uniformly and g_n is continuous for all n, then $\sum g_n$ is continuous.*

Proof. Let $s_n(x) = \sum_{k=0}^{n} g_k(x)$. Then $s_n \to \sum g_n$ uniformly by assumption, and s_n is continuous for all n because g_k is continuous for all k. Therefore $\sum g_n$ is continuous. $\square$

Definition 23.4. *A series of functions $\sum g_n$ satisfies the Cauchy criterion uniformly if the sequence of partial sum functions (s_n) is uniformly Cauchy.*

$\sum g_n$ satisfies the Cauchy criterion uniformly on S iff $\forall \epsilon > 0, \exists N \,\text{s.t.}\, \forall n \geq m > N$,

$$\left| \sum_{k=m}^{n} g_k(x) \right| < \epsilon \;\forall x \in S.$$

Theorem 23.4. *(Weierstrass M-Test): Let* $\sum g_n$ *be a series of functions on* S*. For all* n*, let* $M_n \in \mathbb{R}$ *be such that*

$$|g_n(x)| \leq M_n \forall x \in S$$

If $\sum M_n$ *converges, then* $\sum g_n$ *converges uniformly on* S*.*

Proof. $\sum M_n$ satisfies the Cauchy criterion. So $\forall \epsilon > 0, \exists N \text{ s.t. } n \geq m > N$,

$$\sum_{k=m}^{n} M_k = \left| \sum_{k=m}^{n} M_k \right| < \epsilon$$

So $\forall x \in S$ and $\forall n \geq m > N$,

$$\begin{aligned} \left| \sum_{k=m}^{n} g_k(x) \right| &\leq \sum_{k=m}^{n} |g_k(x)| \\ &\leq \sum_{k=m}^{n} M_k < \epsilon \end{aligned}$$

Therefore $\sum g_n$ satisfies the Cauchy criterion uniformly. □

Example 23.3. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

1. $\varphi(x) = |x| \forall x \in [-1, 1]$
2. $\varphi(x + 2) = \varphi(x) \forall x$

and let $g_n(x) = \left(\frac{3}{4}\right)^n \varphi(4^n x)$. Consider $\sum_{n=0}^{\infty} g_n$.

$$\begin{aligned} 0 \leq \varphi(x) &\leq 1 \forall x \in \mathbb{R} \\ 0 \leq g_n(x) &\leq \left(\frac{3}{4}\right)^n \forall x \in \mathbb{R} \end{aligned}$$

We know that $\sum \left(\frac{3}{4}\right)^n$ converges, so $\sum g_n$ converges uniformly on $\mathbb{R}$ by the Weierstrass M-test. Also, φ is continuous, so g_n is continuous on $\mathbb{R} \forall n$. Therefore $\sum_{n=0}^{\infty} g_n$ is continuous. But it turns out that $\sum g_n$ is nowhere differentiable. This is sad.

□

Example 23.4. Consider

$$\sum_{n=0}^{\infty} 2^{-n}x^n$$
$$\beta = \limsup |2^{-n}|^{1/n} = \limsup 2^{-1} = \frac{1}{2}$$
$$R = \frac{1}{\beta} = 2$$

Therefore $\sum 2^{-n}x^n$ converges for all $x \in (-2, 2)$. We can check the endpoints and see that it diverges at both.

We claim that $\sum 2^{-n}x^n$ is continuous on $(-2, 2)$.

Proof. For all $a \in (-2, 2), \exists b > 0 \,\text{s.t.}\, a \in [-b, b] \subseteq (-2, 2)$ (take $|a| < b < 2$). For all $x \in [-b, b]$,

$$|2^{-n}x^n| = 2^{-n}x^n \leq 2^{-n}b^n = \left(\frac{b}{2}\right)^n$$

Because $b < 2$, the series is geometric with ratio less than 1, so it converges.

Therefore $\sum 2^{-n}x^n$ is continuous on $[-b, b]$, so it is continuous at $a \in [-b, b]$. Therefore $\sum 2^{-n}x^n$ is continuous on $(-2, 2)$. □

However, $\sum 2^{-n}x^n$ does not converge uniformly on $(-2, 2)$, because $\sup\{|2^{-n}x^n| \mid x \in (-2, 2)\} = 2^{-n}2^n = 1$.

□

Lemma 23.5. *If $\sum g_n$ converges uniformly on a set S, then*

$$\lim_{n \to +\infty} \sup\{|g_n(x)| \mid x \in S\} = 0$$

Uniform convergence is a sufficient but not a necessary condition for continuity.

Theorem 23.6. *Let $\sum g_n$ be a series of functions on $[a, b]$. If*

1. *g_n is continuous on $[a, b] \forall n$,*
2. *$\sum g_n$ is continuous on $[a, b]$, and*
3. *$g_n(x) \geq 0 \forall x \in [a, b], n \in \mathbb{N}$*

then $\sum g_n$ converges uniformly.

23.1 Uniform Convergence and Swapping Limits

Let f_n be continuous at a for all n, and let $f_n \to f$ uniformly. Then f is continuous at a, so we can say

$$\lim_{x\to a} f(x) = f(a)$$

$$\lim_{n\to+\infty}\left(\lim_{x\to a} f_n(x)\right) = \lim_{n\to+\infty} f_n(a) = f(a) = \lim_{x\to a} f(x) = \lim_{x\to a}\left(\lim_{n\to+\infty} f_n(x)\right)$$

Uniform convergence implies we can swap limits.

Math 104: Introduction to Analysis **Summer 2019**

Lecture 24: Uniform convergence

Lecturer: Michael Christianson *7 August* *Aditya Sengupta*

Last time, we saw that limits could be swapped in the case of uniformly convergent sequences of functions; we can see that this is a useful property because it doesn't necessarily hold when uniform convergence does not hold. For example, consider

$$\lim_{m\to\infty}\lim_{n\to+\infty}\frac{m}{m+n} = \lim_{m\to\infty} 0 = 0$$
$$\lim_{n\to+\infty}\lim_{m\to\infty}\frac{m}{m+n} = \lim_{n\to+\infty} 1 = 1$$

Often, uniform convergence allows us to swap limits and derivatives/integrals.

Theorem 24.1. *Let (f_n) be a sequence of functions on $[a,b]$. If $f_n \to f$ uniformly on $[a,b]$ and f_n is integrable for all n, then f is integrable, and*

$$\lim_{n\to+\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to+\infty} f_n(x)dx = \int_a^b f(x)dx$$

Proof. $\forall n$, let

$$s_n = \sup\{|f(x) - f_n(x)| \mid x \in [a,b]\}$$

Because $f_n \to f$ uniformly, we can say $\lim\limits_{n\to+\infty} s_n = 0$. Further, for all $x \in [a,b]$,

$$|f(x) - f_n(x)| \leq s_n \; \forall n$$

Therefore

$$f_n(x) - s_n \leq f(x) \leq f_n(x) + s_n$$

Now, for all $S \subseteq [a,b]$,

$$\begin{aligned} m(f_n - s_n, S) \leq f_n(x) - s_n \leq f(x) \; \forall x \in S \\ m(f_n - s_n, S) \leq m(f, S) \\ L(f_n - s_n, P) \leq L(f, P) \leq L(f) \; \forall \text{ partitions } P \\ \int_a^b f_n(x) - s_n dx = L(f_n - s_n) \leq L(f) \end{aligned}$$

Likewise, $M(f, S) \leq M(f_n + s_n, S)$, so

$$\begin{aligned} U(f, P) &\leq U(f_n + s_n, P) \forall P \\ U(f) \leq U(f_n + s_n) &= \int_a^b f_n(x) + s_n dx \end{aligned}$$

Therefore

$$\int_a^b f_n(x) - s_n dx \leq L(f) \leq U(f) \leq \int_a^b f_n(x) + s_n dx$$

We can rearrange this to get a bound on the difference between Darboux sums,

$$U(f) - L(f) \leq \int_a^b (f_n + s_n) - \int_a^b (f_n - s_n) = 2\int_a^b s_n = 2s_n(b-a)$$

So $s_n \to 0$ as $n \to \infty$ implies

$$\begin{aligned} &\forall \epsilon > 0, \exists N \,\text{s. t.} \\ s_n &< \frac{\epsilon}{2(b-a)} \forall n > N \\ \implies U(f) - L(f) \leq 2s_n(b-a) &< \epsilon \forall n > N \end{aligned}$$

So $U(f) - L(f)$ is less than every positive number, so $U(f) = L(f)$. Therefore f is integrable.

This tells us that

$$\begin{aligned} \left(\int_a^b f_n\right) - s_n(b-a) \leq L(f) = U(f) = \int_a^b f &\leq \left(\int_a^b f_n\right) + s_n(b-a) \\ -s_n(b-a) \leq \int_a^b f - \int_a^b f_n &\leq s_n(b-a) \end{aligned}$$

Because $s_n \to 0$, by the squeeze lemma we can say

$$\lim_{n \to +\infty} \int_a^b f_n = \int_a^b f$$

as desired. □

The Dominated Convergence Theorem gives us another case in which we can swap integrals and limits.

Theorem 24.2. *(Dominated Convergence Theorem) Let (f_n) be a sequence of functions on $[a, b]$ Suppose*

1. $f_n \to f$ *pointwise on* $[a, b]$,
2. f *is integrable and* f_n *integrable* $\forall n$,
3. $\exists M > 0 \,\text{s.t.}\, |f_n(x)| \leq M \forall n \in \mathbb{N}, x \in [a, b]$.

Then

$$\lim_{n\to+\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to+\infty} f_n(x)dx$$

To swap integrals, the required condition ends up just being continuity.

Theorem 24.3. *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be continuous. Then*

$$\int_c^d \left(\int_a^b f(x, y)dx \right) dy = \int_a^b \left(\int_c^d f(x, y)dy \right) dx$$

Intuitively, we think about the result of the integral over dx as being a function $F(y)$:

$$F(y) = \int_a^\infty f(x, y)dx = \lim_{n\to+\infty} \int_a^n f(x, y)dx$$
$$F_n(y) = \int_a^n f(x, y)dx$$

Then $F_n(y) \to F(y)$ pointwise.

If f is continuous and $F_n \to F$ uniformly on $[c, d]$, then

$$\int_c^d \left(\int_a^\infty f(x, y)dx \right) dy = \int_a^\infty \left(\int_c^d f(x, y)dy \right) dx$$

If both integrals are improper, then uniform convergence is not enough.

Example 24.1.

$$\int_1^\infty \int_1^\infty \frac{x^2 - y^2}{(x^2 + y^2)^2} dxdy = \frac{\pi}{4}$$
$$\int_1^\infty \int_1^\infty \frac{x^2 - y^2}{(x^2 + y^2)^2} dydx = -\frac{\pi}{4}$$

□

Similarly, we can consider derivatives.

Example 24.2.

$$f_n(x) = \frac{1}{n}\sin(nx)$$

$f_n \to f$ uniformly on $\mathbb{R}$ where f is identically zero.

$$f_n'(x) = \frac{1}{n}\cos(nx)\cdot n = \cos(nx) \forall x \in \mathbb{R}$$

But $\lim_{n\to+\infty} f_n'(x)$ does not exist in general for $x \in \mathbb{R}$. So we can't have $\lim_{n\to+\infty} f_n'(x) = f'(x)$.

□

The condition for derivative-limit swapping is given by the following theorem:

Theorem 24.4. *Let $f_n \to f$ pointwise on $[a,b]$. If f_n is differentiable on (a,b) for all n, and the sequence (f_n') converges uniformly on (a,b), then*

(a) $f_n \to f$ uniformly

(b) f is differentiable on $[a,b]$, and

(c) $\lim_{n\to+\infty} f_n'(x) = f'(x) = \left(\lim_{n\to+\infty} f_n(x)\right)'$

24.1 Power Series

Theorem 24.5. *Let $\sum a_n x^n$ be a power series with radius of convergence $R > 0$.*

(a) $\forall R_1 \,\text{s.t.}\, 0 < R_1 < R$, the series $\sum a_n x^n$ converges uniformly on $[-R_1, R_1]$.

(b) $\sum a_n x^n$ is continuous on $(-R, R)$.

Proof. (a) If $0 < R_1 < R$, then $\forall x \in [-R_1, R_1]$,

$$|a_n x^n| = |a_n||x|^n \le |a_n| R_1^n$$

$\sum |a_n| x^n$ has the same radius of convergence as $\sum a_n x^n (\beta = \limsup |a_n|^{1/n}$ and R is defined using $\beta)$. So $R_1 \in (-R, R)$, which implies $\sum |a_n| R_1^n$ converges. Therefore $\sum a_n x^n$ converges uniformly on $[-R, R]$.

(b) Let $x_0 \in (-R, R)$. Pick R such that

$$0 \leq |x_0| < R_1 < R \implies x_0 \in [-R_1, R_1].$$

By part (a), $\sum a_n x^n$ converges uniformly on $[-R_1, R_1] \implies \sum a_n x^n$ continuous on $[-R_1, R_1]$. Therefore $\sum a_n x^n$ is continuous at x_0.

$\square$

We would hope that integrating and differentiating power series works by integrating and differentiating term-by-term, i.e. that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \implies f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Lemma 24.6. *Let $\sum a_n x^n$ be a power series with radius of convergence R. Then*

$$\sum_{n=1}^{\infty} n a_n x^n, \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

both have radius of convergence R.

Proof. Let $\beta = \limsup |a_n|^{1/n}$. Then

$$\begin{aligned} \limsup |(n+1)a_{n+1}|^{1/n} &= \limsup (n+1)^{1/n} |a_{n+1}|^{1/n} \\ \forall n, n^{1/n} &\leq (n+1)^{1/n} \leq (2n)^{1/n} = 2^{1/n} \cdot n^{1/n} \\ \therefore \text{squeeze lemma:} \quad &\lim_{n \to +\infty} (n+1)^{1/n} = 1 \end{aligned}$$

Now, consider the derivative power series:

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} n a_n x^n$$

Therefore the derivative power series has the same radius of convergence as $\sum n a_n x^n$. So,

$$\begin{aligned} \limsup |n a_n|^{1/n} &= \limsup (n^{1/n} |a_n|^{1/n}) \\ &= \lim_{n \to +\infty} (n^{1/n}) \cdot \limsup |a_n|^{1/n} \\ &= \limsup |a_n|^{1/n} = \beta \end{aligned}$$

Therefore the radius of convergence of $\sum na_nx^n$ is R.

Similarly, the integral power series has the same radius of convergence as $\sum \frac{a_n}{n+1}x^n$. So

$$\begin{aligned}
\limsup \left|\frac{a_n}{n+1}\right|^{1/n} &= \limsup \left((\frac{1}{(n+1)^{1/n}}) \cdot |a_n|^{1/n} \right) \\
&= \lim_{n\to+\infty} \left(\frac{1}{(n+1)^{1/n}} \right) \limsup |a_n|^{1/n} \\
&= \limsup |a_n|^{1/n} = \beta
\end{aligned}$$

□

Theorem 24.7. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$.*

(a) $\forall x \in (-R, R)$, f is integrable on $[0, x]$, or $[x, 0]$ if $x < 0$, and

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

(b) f is differentiable on $(-R, R)$ and $\forall x \in (-R, R)$,

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Proof. (a) $\forall x \in (-R, R)$, assume $x > 0$, then f is continuous on $[0, x]$. Therefore f is integrable on $[0, x]$. Also, $\sum a_n x^n$ converges uniformly on $[0, x]$. So we get

$$\int_0^x f(t)dt = \int_0^\infty \left(\lim_{n\to+\infty} \sum_{k=0}^{n} a_k t^k \right) dt = \lim_{n\to+\infty} \sum_{k=0}^{n} \int_0^x a_k t^k dt$$

The two swaps in the last step were possible because of uniform convergence. Then, we can integrate term-by-term to get

$$\lim_{n\to+\infty} \sum_{k=0}^{n} a_k \frac{x^{k+1} - 0^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}.$$

(b) Let $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Then

$$G(x) = \int_0^x g(t)dt = \sum_{n=1}^{\infty} \frac{n a_n}{n} x^n = f(x) - a_0$$

$g(x)$ is continuous on $(-R, R)$, so $G(x)$ is differentiable on $(-R, R)$ and $G'(x) = g(x)$ for all x by FTC. Therefore, $G(x) + a_0 = f(x)$ is differentiable, and

$$f'(x) = G'(x) + 0 = g(x)$$

□

The above does not discuss the continuity of $\sum a_n x^n$ at $x = \pm R$. We can handle that separately by Abel's theorem:

Theorem 24.8. *(Abel) Let $f(x) = \sum a_n x^n$ have a radius of convergence $R > 0$. If $\sum a_n x^n$ converges at $x = R$, then f is continuous at R. If $\sum a_n x^n$ converges at $x = -R$, then f is continuous at $-R$.*

We know that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \forall x \in (-1, 1)$. We can use the derivative and integral properties above to say that

$$\sum_{n=1}^{\infty} n x^{n-1} = -\frac{1}{(1-x)^2} \ \forall x \in (-1, 1).$$

and similarly,

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \int_0^x \frac{1}{1-t} dt = -\ln(1-t)$$

If we replace x by $-x$, we can get

$$\ln(1+x) = -\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} x^{n+1} \ \forall x \in (-1, 1)$$

At $x = 1$, we get an alternating series that converges, so the power series is continuous at 1 by Abel's theorem. Since $\ln(1+x)$ is also continuous at $x = 1$, we can say that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln 2$$

Math 104: Introduction to Analysis **Summer 2019**

Lecture 25: Taylor's Theorem

Lecturer: Michael Christianson *8 August* *Aditya Sengupta*

25.1 Taylor series

We have seen that power series are infinitely differentiable, which suggests that we can construct expansions of any infinitely differentiable function into a power series.

Suppose we have $f : (-b, b) \to \mathbb{R}$ such that $f(x) = \sum a_n x^n \ \forall x \in (-b, b)$. We successively take derivatives and evaluate them at 0,

$$\begin{aligned} f(0) &= a_0 \\ f'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ f'(0) &= 1 \cdot a_1 \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\ f''(0) = 2 \cdot 1 \cdot a_2 \implies a_2 &= \frac{f''(0)}{2} \end{aligned}$$

In general,

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1) \cdot (n-k+1) a_n x^{n-k} \\ f^{(k)}(0) = k! a_k \implies a_k &= \frac{f^{(k)}(0)}{k!} \end{aligned}$$

Therefore

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

In general, if we have $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$ and f has a power series representation

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-c)^n \\ f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n \end{aligned}$$

This is the *Taylor series* of f at c. When f is given by a power series, this power series is uniquely specified by the Taylor series. We are left with the question of when we are allowed to write f as a power series.

Definition 25.1. *Let $f : (a, b) \to \mathbb{R}$ be a function.*

(a) We write $f \in C^k(a, b)$ if $f', f'', \ldots, f^{(k)}$ exist and are continuous for $k \geq 1$.

(b) We write $f \in C^0(a, b)$ if $f^{(0)} = f$, i.e. if f is continuous.

(c) We write $f \in C^\infty(a, b)$ (or f is smooth) if f has derivatives of all orders. We have seen that all power series are in C^∞.

(d) f is analytic at $c \in (a, b)$ if f is equal to its Taylor series at c for all x "near" c, i.e. there exists an open interval $I \ni c$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n \ \forall x \in I.$$

(e) f is analytic if f is analytic $\forall c \in (a, b)$.

(f) $\forall n \geq 1, c \in (a, b)$, we define the nth remainder or partial sum of the Taylor series centered at c,

$$R_{c,n}(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

Note that f is analytic at c if and only if $\lim_{n \to +\infty} R_{c,n}(x) = 0$ $\forall x$ near c.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n = \lim_{n \to +\infty}(f(x) + R_{c,n}(x)) = f(x)$$

Theorem 25.1. *Let $f : (a, b) \to \mathbb{R}$ be such that $f^{(n)}$ exists on (a, b) for some $n \geq 1$. Let $a < c < b$ (we allow $a = -\infty$ and/or $b = +\infty$). $\forall x \in (a, b), x \neq c, \exists y$ between c and x such that*

$$R_{c,n}(x) = \frac{f^{(n)}(y)}{n!}(x - c)^n$$

(y may depend on x.)

Proof. Let M be implicitly defined by

$$R_{c,n}(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x - c)^k = \frac{M(x - c)^n}{n!}$$

We want to show that $M = f^{(n)}(y)$ for some y.

Let $g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(t-c)^k - \frac{M(t-c)^n}{n!}$; $g(x) = 0$ by definition of M, and $g(c) = 0$. So by the Mean Value Theorem on the interval $[c, x]$ or $[x, c]$, $\exists x_1$ between x and c such that

$$g'(x_1) = \frac{g(x) - g(c)}{x - c} = 0$$

This holds for the kth derivative if $k \leq n$ by construction of g, so we can extend this to say

$$\forall k < n, \exists x_k \,\text{s. t.}\, g^{(k)}(x_k) = 0$$

and for each x_k we can say by the mean value theorem that x_k is between x_{k-1} and c. If we take $k = n$, we get

$$g^{(n)}(t) = f^{(n)}(t) - M$$

because the other terms die under the n derivatives. So

$$g^{(n)}(x_n) = 0 \implies f^{(n)}(x_n) = M$$

Therefore, if we take $x_n = y$, the proof is complete. $\square$

Corollary 25.2. *Let $f : (a, b) \to \mathbb{R}$ be $C^\infty(a, b)$. Suppose there exists $C \in \mathbb{R}$ such that*

$$|f^{(n)}(x)| \leq C \ \forall x \in (a, b), n \geq 1.$$

Then $\forall c \in (a, b)$, $\lim_{n\to+\infty} R_{c,n}(x) = 0 \forall x$, i.e. f is analytic on (a, b).

Proof. By the above theorem, $\forall c, x \in (a, b), n \geq 1$,

$$|R_{c,n}(x)| = \left| \frac{f^{(n)}(y)}{n!}(x - c)^n \right| \leq \frac{c}{n!}|x - c|^n$$

Let $s_n = fraccn!|x - c|^n$. Then by the Ratio Test,

$$\begin{aligned} \lim_{n\to+\infty} \left| \frac{s_{n+1}}{s_n} \right| &= \lim_{n\to+\infty} \left(\frac{|x - c|^{n+1}}{(n + 1)!} \cdot \frac{n!}{|x - c|^n} \right) \\ &= \lim_{n\to+\infty} \frac{|x - c|}{n + 1} = 0 \end{aligned}$$

Therefore $\lim_{n\to+\infty} s_n = 0 \implies \lim_{n\to+\infty} R_{c,n}(x) = 0 \forall x, c.$ $\square$

Example 25.1. Consider $f(x) = e^x$, which has $f^{(n)}(x) = e^x$. The derivative evaluated at 0 is 1 for all orders. Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

For all $x \in \mathbb{R}$, $|f^{(n)}(t)| \leq e^{x+1} \forall t \in (-x-1, x+1)$. Therefore by the corollary,

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \ \forall t \in (-x-1, x+1)$$

Take $t = x$; this proves that e^x is analytic and equal everywhere to its Taylor series. □

Example 25.2.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

These hold for all $x \in \mathbb{R}$ because $|\sin x|, |\cos x| \leq 1 \ \forall x \in \mathbb{R}$. □

Theorem 25.3. *The remainder of a Taylor series can be expressed as the following integral:*

$$R_{c,n}(x) = \int_c^x \frac{(x-c)^{n-1}}{(n-1)!} f^{(n)}(t) dt$$

25.2 Non-analytic functions

A smooth function is not equal to its Taylor series at c if the Taylor series has radius of convergence 0. Additionally, the Taylor series could converge, but not to $f(x)$.

Example 25.3. Let

$$f(x) = \sum_{n=0}^{\infty} e^{-n} \cos\left(n^2 x\right)$$

Let $g_n(x) = e^{-n} \cos\left(n^2 x\right)$. For all $x \in \mathbb{R}, |g_n(x)| \leq e^{-n}$. Therefore $f(x)$ converges uniformly by the Weierstrass-M test.

For all $k \geq 1, |g_n^{(k)}(x)| \leq e^{-n} n^{2k}\ \forall x \in \mathbb{R}$. We claim that $\forall k \geq 1, \sum_{n=0}^{\infty} e^{-n} n^{2k}$ converges. Suppose there exists an N such that $n^{2k} \leq e^{n/2}\ \forall n \geq N$. Then

$$|e^{-n} n^{2k}| = e^{-n} n^{2k} \leq e^{-n} \cdot e^{n/2} = e^{-n/2}$$

and $e^{-n/2}$ converges. Therefore $\sum_{n=N}^{\infty} e^{-n} n^{2k}$ converges by the comparison test.

Let $h(x) = e^{x/2} - x^{2k}$. We want to show that $h(x) \geq 0$ for large x. By applying L'Hopital's Rule $2k$ times, we get that the limit is $+\infty$:

$$\lim_{x \to +\infty} \frac{e^{x/2}}{x^{2k}} = \lim_{x \to +\infty} \frac{\frac{1}{2^{2k}} \cdot e^{x/2}}{(2k)!} = +\infty$$

Therefore, there exists some $x \geq 0$ such that $h(x) > 1$, and we can take $N = \lceil x \rceil$. Therefore, the comparison test argument from before holds and we get $\sum g_n^{(k)}(x)$ converges uniformly for all $x \in \mathbb{R}$. Therefore

$$f^{(k)}(x) = \sum_{n=0}^{\infty} g_n^{(k)}(x)$$

Take $c = 0$; then

$$f^{(k)}(0) = \sum_{n=0}^{\infty} g_n^{(k)}(0) = \sum_{n=0}^{\infty} e^{-2n} (2n)^{2k}$$

So the kth coefficient of the Taylor series is $\frac{f^{(k)}(0)}{k!} x^k$, which is equal to

$$\begin{aligned}
&\frac{x^k}{k!}\sum_{n=0}^{\infty} e^{-2n}(2n)^{2k} \\
\geq &\frac{x^k}{k!} \cdot e^{-2k} \cdot (2k)^{2k} \\
\geq &\frac{x^k e^{-2k}(2k)^{2k}}{k^k} \\
= &\left(\frac{xe^{-2}(2k)^2}{k}\right)^k = (xe^{-2}4k)^k \geq 1 \ \forall k \text{ large}
\end{aligned}$$

Therefore,

$$\lim_{k\to+\infty} \frac{f^{(k)}(0)}{k!} x^k \geq 1 \ \forall x \neq 0$$

Therefore the Taylor series of f diverges for all $x \neq 0$.

$\square$

Example 25.4. Consider the function

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

We claim that $f^{(n)}(0) = 0 \ \forall n$.

For all $x > 0$, we have

$$f^{(n)}(x) = e^{-1/x} \cdot p_n\left(\frac{1}{x}\right) > 0$$

where p_n is some polynomial of degree n. We can show that $f^{(0)}(0) = f(0) = 0$, and by induction we can show that all the other derivatives are 0:

$$\begin{aligned}
f^{(n)}(0) &= 0 \\
f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} &= 0
\end{aligned}$$

Therefore the Taylor series of f at 0 is 0. But $f(x) \neq 0 \forall x > 0$.

$\square$

25.3 Weierstrass's approximation theorem

We previously showed that if $f(x)$ is given by a power series, i.e. if there exists a sequence of partial sums $s_n(x) = \sum_{k=0}^{n} a_n x^n$ converges to f uniformly on $[-R_1, R_1] \forall 0 < R_1 < R$, then f can be "approximated uniformly" by polynomials. Now, we consider the case where f is not analytic. For example, $f(x) = |x|$ is not differentiable at $x = 0$. We want to approximate this function by polynomials, even close to zero.

Theorem 25.4. *(Weierstrass-Stone) Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists a sequence of polynomials (p_n) such that $p_n \to f$ uniformly on $[a, b]$.*

Example 25.5. Consider the Bernstein polynomials,

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k}$$

Then, $B_n f \to f$ uniformly on $[0, 1]$ for all $f : [0, 1] \to \mathbb{R}$.

$\square$

Math 104: Introduction to Analysis **Summer 2019**

Lecture 26: Metric Spaces

Lecturer: Michael Christianson *12 August* *Aditya Sengupta*

Definition 26.1. *1. A metric d on a set X is a function $d : X \times X \to \mathbb{R}$ such that*

(a) $\forall x, y \in X, d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$.

(b) $\forall x, y \in X, d(x, y) = d(y, x)$

(c) the metric obeys the triangle inequality:

$$\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$$

2. A metric space is a pair (X, d), with X a set and d a metric on X.

Example 26.1. $(\mathbb{R}, d)$ is a metric space, where $d(x, y) = |x - y|$. From the triangle inequality that we are familiar with, we can see that d obeys the third requirement to be a metric (the first and second are clear):

$$|x - z| = |x - y + y - z| \leq |x - y| + |y - z|$$

□

Example 26.2. $(\mathbb{R}^k, d)$ $\forall k \geq 1$ is a metric space, with the Euclidean distance as the metric d:

$$d(x, y) = \sqrt{\sum_{j=1}^{k} (x_j - y_j)^2}$$

The case $k = 1$ is the real-line distance in the previous example, and the case $k = 2$ is the distance between two points on a plane, $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

□

Example 26.3. It is easy to define a trivial metric that is valid but not very useful, e.g.

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

□

Example 26.4. Consider the set of infinitely differentiable functions,

$$C^\infty(a,b) = \{f : [a,b] \to \mathbb{R} \text{ s.t. } f \text{ is smooth.}\}$$

We can define the following metric on this set:

$$d(f,g) = \sup\{|f(x) - g(x)| \mid x \in [a,b]\}$$

□

Lemma 26.1. *A subset Y of a metric space (X,d) is also a metric space $(Y, d|_{Y\times Y})$.*

The key idea with metric spaces is to think of everything in terms of open balls.

Definition 26.2. *Let (X,d) be a metric space and let $r \geq 0$. Then*

$$B(x,r) = \{y \in X \mid d(x,y) < r\}$$

Example 26.5. With $X = \mathbb{R}$, $B(x,r)$ is the open interval $(x-r, x+r)$. We can therefore say that x is arbitrarily close to a in more generality by saying $x \in B(a,\delta)$.

□

Definition 26.3. *Let (X,d) be a metric space and let $S \subseteq X$.*

1. *We say $x \in S$ is interior to S if $\exists r > 0$ s.t. $B(x,r) \subseteq S$.*
2. *The interior of S is*

$$S^\circ = \{x \in S \mid x \text{ is interior to } S\}$$

3. *We say S is open if $S^\circ = S$.*

Example 26.6. $B(x,r)$ is always open $\forall x \in X, r \geq 0$. In particular, (a,b) is open in $\mathbb{R}$ for all $a < b$. A set is always open in itself. So $[a,b]$ is open in $[a,b]$, but is not open in $\mathbb{R}$.

□

Definition 26.4. *Let (X, d) be a metric space, $S \subseteq X$.*

(a) The complement of S in X is $S^c = X \setminus S$.

(b) S is closed if S^c is open.

(c) The boundary of S is $S \setminus S^\circ$.

(d) The closure of S is $\overline{S} = \bigcap_{X \supset V \supset S closed} V$.

(e) The boundary of S is $\overline{S} \setminus S^\circ$.

Example 26.7.

$$\begin{aligned}[a, b]^c = (-\infty, a) \cup (b, \infty) \text{ open} \\ \implies [a, b] \text{ closed.}\end{aligned}$$

$\square$

In general, we can say that $\{y \mid d(x, y) \leq r\}$ is closed for all $X \in X, r \geq 0$.

The closure of a boundary is the original set of which the boundary is a subset, and the boundary of a boundary is $D \setminus B(x, r) = \{y \mid d(x, y) = r\}$.

Definition 26.5. *Let (X, d) be a metric space and let $S \subseteq X$. $x \in X$ is a limit point of S if $\forall r > 0, B(x, r) \cap S \neq \varnothing$, i.e. if $\forall r > 0, B(x, r) \cap S$ contains a point other than x.*

Proposition 26.2. *Let (X, d) be a metric space.*

1. *X and $\varnothing$ are both open and closed. (Openness and closeness, and not-openness and not-closeness, are all not mutually exclusive.)*
2. *For any collection $\{U_\alpha\}_{\alpha \in I}$ of open sets U_α in I, $\cup_{\alpha \in I} U_\alpha$ is open.*
3. *For all collections $\{V_\alpha\}_{\alpha \in I}$ of closed sets V_α in I, $\cap_{\alpha \in I} V_\alpha$ is closed.*
4. *For any finite collection of open sets $\{U_1, \ldots, U_r\}, \cap_{i=1}^r U_i$ is open.*
5. *For any finite collection of closed sets $\{V_1, \ldots, V_r\}, \cup_{i=1}^r V_i$ is closed.*

Example 26.8. Consider $X = [0, 1)$ and let $U_n = [0, \frac{1}{n})\ \forall n \geq 1$. Then the intersection $\cap_{n=1}^\infty U_n = \{0\}$ is closed and not open.

$\square$

Proposition 26.3. *Let (X, d) be a metric space, $S \subseteq X$.*

(a) $\overline{S}$ *is closed, and* $S \subseteq \overline{S}$.

(b) S *is closed iff* $S = \overline{S}$.

(c) *If* S' *is the set of limit points of* S, *then* $\overline{S} = S \cup S'$.

Definition 26.6. *Let* (X, d) *be a metric space and let* $S \subseteq X$ *be a subset.*

(a) *An open cover of* S *is a collection of open sets* $\{U_\alpha\}_{\alpha \in I}$ *such that* $S \subseteq U_{\alpha \in I} U_\alpha$.

(b) *We say* S *is compact if* $\forall$ *open covers* $\{U_\alpha\}_{\alpha \in I}$, $\exists$ *a finite subcover, i.e. a finite subset* $J \subseteq I$ *such that* $\{U_\alpha\}_{\alpha \in J}$ *is an open cover for* S.

(c) *We say* S *is bounded if there exists some* $x \in X, r > 0 \,\text{s.t.}\, S \subseteq B(x, r)$. *This is the case if and only if* $\exists M \in \mathbb{R} \,\text{s.t.}\, d(x, y) \leq M \forall x, y \in S$.

Proposition 26.4. *Let* (X, d) *be a metric space and* $K \subset X$ *compact. Then* K *is closed and bounded.*

Proof. First, K is bounded. If $K = \varnothing$, then we're done. If $K \neq \varnothing$, let $x \in K$. Then for all $n \in N$, let $U_n = B(x, n)$. For all $y \in K, \exists n > d(x, y) \implies y \in U_n$, so $\{U_n\}_{n \in \mathbb{N}}$ is an open cover of K. Therefore there exists a finite subcover $\{U_{n_i}\}_{1 \leq i \leq r}$. Let $R = \max\{n_i\}_{1 \leq i \leq r}$; then

$$
\begin{aligned}
U_i \subseteq B(x, R) = U_R \;\forall i \\
&\implies \cup_{i=1}^r U_i \subseteq B(x, R) \\
K \subseteq B(x, R).
\end{aligned}
$$

Therefore K is bounded.

Next, we show that K is closed. Let $x \in K^c$. We show $\exists r \,\text{s.t.}\, x \in B(x, r) \subseteq K^c$. For all $y \in K$, let

$$
\begin{aligned}
U_y &= B\left(x, \frac{d(x, y)}{2}\right) \\
V_y &= B\left(y, \frac{d(x, y)}{2}\right) \\
U_y \cap V_y &= \varnothing \;\forall y
\end{aligned}
$$

Then $\{V_y\}_{y \in K}$ is an open cover of K. Therefore there exists a finite subcover $\{V_{y_i}\}_{1 \leq i \leq n}$. Let $r = \min\{\frac{d(x, y_i)}{2} \mid 1 \leq i \leq n\}$. Then

$$
\begin{aligned}
B(x, r) &\subseteq U_{y_i} \\
B(x, r) \cap V_{y_i} &= \varnothing \\
B(x, r) \cap (\cup_{i=1}^n V_{y_i}) &= \varphi \\
B(x, r) \cap K &= \varnothing \\
B(x, r) &\subseteq K^c.
\end{aligned}
$$

$\square$

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Lecture 27: Metric Spaces continued

Lecturer: Michael Christianson *13 August* *Aditya Sengupta*

27.1 Review of definitions

Definition 27.1. *Let (X, d) be a metric space and $E \subset X$ be a subset.*

1. *E is open if $\forall x \in E, x$ is interior to E, i.e. $\exists r > 0\,\text{s.t.}\, B(x, r) \subseteq E$.*
2. *E is closed if E^c is open.*
3. *The closure $\overline{E}$ is the set $\overline{E} = \cap_{V \supset E \text{ closed}}$*

Example 27.1. We claim that $(a, b) \subset \mathbb{R}$ is open.

$\forall x \in (a, b)$, let $r = \min\{x - a, b - x\}$. Then

$$x - r \geq x - (x - a) = a$$
$$x + r \leq x + (b - x) = b$$

Therefore

$$B(x, r) = (x - r, x + r) \subseteq (a, b)$$

□

Definition 27.2. *$x \in X$ is a limit point of E if $\forall r > 0, B(x, r) \cap E$ contains a point other than x.*

$\overline{E} = E \cup E'$, where E' is the set of all limit points of E.

A set is closed if it contains all of its limit points.

The boundary of any point in a metric space is open in that metric space; $\forall y \in B(x, r)$ for some $r \geq 0, x \in X$, let $r' = r - d(x, y) > 0$. We want to show that $B(y, r') \subseteq B(x, r)$. For all $z \in B(y, r')$,

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r' = d(x, y) + r - d(x, y) = r$$
$$\therefore z \in B(x, r) \implies B(y, r') \subseteq B(x, r).$$

Theorem 27.1. *(Heine-Borel) $\forall E \subset \mathbb{R}^k$, E is compact if and only if E is closed and bounded.*

Suppose there does not exist a finite subcover of $\{U_\alpha\}$ for $[a,b]$. Either $[a,c]$ or $[c,b]$ has no finite subcover, call this interval I. Do this repeatedly and define a sequence of these itnervals I_n; then $\exists x \,\text{s.t.}\; x \in I_n \forall n$. Then $x \subseteq B(x,r) \subseteq U_\alpha \implies I_n \subseteq B(x,r)$ for n large.

Proposition 27.2. *Let (X,d) be a metric space.*

1. *If $K \subset X$ is compact and $E \subset K$ is closed in X, then E is compact.*
2. *If $K_1, K_2, \ldots, K_n \subset X$ are compact, then $\cup_{i=1}^n K_i$ is compact.*

Proof.

1. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of E. Then E^c is open, therefore $\{E^c\} \cup \{U_\alpha\}_{\alpha \in I}$ is an open cover for K. Since K is compact, there exists a finite subcover $\{U_{\alpha_1}, \ldots, U_{\alpha_r}\}$ plus possibly E^c. Exclude E^c if present; then $\{U_{\alpha_1}, \ldots, U_{\alpha_r}\}$ is a finite subcover of E.
2. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of K. For all i, $\{U_\alpha\}_{\alpha \in I}$ is an open cover of K_i, so there exists a finite subcover $U_i \subseteq \{U_\alpha\}_{\alpha \in I}$. Then $U = \cup_{i=1}^n U_i$ is a finite subcover of K.

□

27.2 Convergence in metric spaces

Definition 27.3. *Let (X,d) be a metric space and let (x_n) be a sequence of points in X.*

1. *We say (x_n) converges to $x \in X$ if $\lim\limits_{n \to +\infty} d(x_n, x) = 0$.*
2. *(x_n) is Cauchy if $\forall \epsilon > 0, \exists N \;\text{s.t.}\, \forall m,n > N$, we have*

$$d(x_m, x_n) < \epsilon$$

If $X = \mathbb{R}$, this reduces to the usual Cauchy criterion.

Theorem 27.3. *Let (X,d) be a metric space, $K \subset X$ be compact, and (x_n) be a sequence in K. There exists a convergent subsequence of (x_n).*

Corollary 27.4. *(Bolzano-Weierstrass) Any bounded sequence (x_n) in $\mathbb{R}^k$ has a convergent subsequence.*

Proof. (x_n) bounded implies that $\{x_n\}_{n \in \mathbb{N}} \subseteq B(x,r)$ for some x, r, therefore $B(x,r)$ is compact by Heine-Borel. □

27.3 Completeness and denseness

Definition 27.4. *(X,d) is complete if every Cauchy sequence in X converges.*

Example 27.2. $\mathbb{R}$ is complete, and $\mathbb{R}^k$ is complete for all k.

□

Example 27.3. $\mathbb{Q}$ is not complete, because there exist sequences in $\mathbb{Q}$ converging to $\sqrt{2}$. These are Cauchy, but not convergent in $\mathbb{Q}$.

□

Definition 27.5. *$E \subset X$ is dense if $\overline{E} = X$.*

Example 27.4. $\mathbb{Q}$ is dense in $\mathbb{R}$.

□

Theorem 27.5. *Let (X, d) be a metric space. Then $\exists$ a metric space $(\overline{X}, \bar{d})$ such that*

(a) X is a dense subset of $\overline{X}$.

(b) $\bar{d}\big|_X = d$.