Notes for Math 185: Complex Analysis
UC Berkeley Spring 2021

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Math 185: Complex Analysis **Spring 2021**

Lecture 1: Logistics and Motivation

Lecturer: Di Fang *19 January* *Aditya Sengupta*

Note: *LaTeX format adapted from template for lecture notes from CS 267, Applications of Parallel Computing, UC Berkeley EECS department.*

Why take complex analysis? Because you need it to get a math degree. That's boring.

Let's try again: why is complex analysis useful?

1. It makes things complete: $\mathbb{C}$ is algebraically complete and $\mathbb{R}$ isn't.
2. Even if you're fine only dealing with real numbers, complex analysis can still help you! Suppose you're integrating $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}$. This is a real integral, and we know its antiderivative is $\arctan x$. However, if you change it to $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^4}$, we don't know the antiderivative and the old method breaks down. We can further ruin our lives by changing the integrand to $\frac{x^2}{1+x^4}$. There's no closed-form antiderivative for these. Complex analysis lets us do these integrals easily!
3. You might think "okay, but why are we just studying better integration techniques? I can do that numerically". In fact, there are other applications: machine learning/AI and quantum computing depend heavily on complex analysis. Machine learning has often been helped by Fourier analysis, which is based on complex analysis; quantum mechanics works entirely in the complex world.
4. It's unique and beautiful by itself!

This semester, we're going to relearn calculus (as we've done a few times already): functions, limits, continuity (chapters 1-2), derivatives (chapter 3), integrals (chapter 4), and series (chapter 5). However, we'll also cover residues (chapter 6-7), which are unique to complex analysis.

Consider two cases: a function $f : \mathbb{R} \to \mathbb{R}$ (over real numbers), and a function $f : \mathbb{C} \to C$ (over complex numbers).

True-false questions:

- If f is differentiable everywhere, f is infinitely differentiable everywhere. False: $f(x) = x|x|$ at $x = 0$ (and in fact it's possible to make a function that is differentiable everywhere but second-differentiable nowhere.) But True for complex numbers.
- If f is smooth (infinitely differentiable) everywhere, then its Taylor series is equal to itself. False: consider $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. f is infinitely differentiable at $x = 0$ (the derivative is 0), and so the Taylor series at $x = 0$ is just the constant function 0. But True for complex numbers.
- If f is differentiable everywhere and bounded, then f must be a constant. False: this is bullshit (direct quote from Di) due to $\cos x, \sin x$, but it's True for complex numbers (Liouville's theorem).

We'll start math by reviewing complex numbers. My fridge is getting delivered so this bit might be sparse, but we know what complex numbers are. $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ under the isomorphism $(x, y) \leftrightarrow x + iy$. If $z = x + iy$, we'll denote $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$.

For the sake of rigor, we'll review some basic properties and definitions:

1. For $z_1, z_2 \in \mathbb{C}$, $z_1 = z_2 \iff \operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 = \operatorname{Im} z_2$.
2. Summation works like you would expect: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.
3. The product: $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

We continue (over me dealing with the fridge) to show $\mathbb{C}$ is a field.

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Lecture 2: Complex Numbers, Definitions

Lecturer: Di Fang *21 January* *Aditya Sengupta*

2.1 Complex number properties

Remark 2.1. *Many formulas and results on the reals still hold true on the complex numbers. For example, 0 is still the additive identity and 1 is still the multiplicative identity. However, this does not always hold: we need to check everything. For example, there is no natural comparison operator on* $\mathbb{C}$*.*

Example 2.1. [An incorrect comparison on $\mathbb{C}$] Suppose we said $x_1 + iy_1 < x_2 + iy_2$ if $x_1 < x_2$ or $(x_1 = x_2$ and $y_1 < y_2)$. But this does not have the property that $z_1 > 0, z_2 > 0 \implies z_1 z_2 > 0$. For example, $i > 0$ but $i \cdot i = -1 < 0$. □

Never compare two complex numbers.

Definition 2.1. *The modulus or magnitude of a complex number* $z = x + iy$ *is* $|z| = \sqrt{x^2 + y^2} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$.

Properties are below.

1. $\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$ and $\operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|$, just geometrically. Note that while $z_1 < z_2$ is meaningless, we can say $|z_1| < |z_2|$.
2. The distance between two complex numbers is $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
3. A circle in complex space is the set of z such that for some constant $z_o \in \mathbb{C}, R \in \mathbb{R}$, $|z - z_o| = R$.
4. (Triangle Inequality) $|z_1 + z_2| \leq |z_1| + |z_2|$. Further, $||z_1| - |z_2|| \leq |z_1 + z_2|$ This is easily proven:

   Proof. $|z_1| = |(z_1 + z_2) - z_2| \leq |z_1 + z_2| + |z_2|$, which implies $|z_1| - |z_2| \leq |z_1 + z_2|$. □

2.2 The complex conjugate

Definition 2.2. *Let* $z = x + iy \in \mathbb{C}$*, the complex conjugate of* z *is* $\bar{z} = x - iy$*.*

Properties The properties are below

1. $\bar{\bar{z}} = z$
2. $|\bar{z}| = |z|$
3. $z = \bar{z} \iff z \in \mathbb{R}$
4. $z_1 \bar{\pm} z_2 = \bar{z_1} \pm \bar{z_2}$
5. $z_1 \bar{z}_2 = \bar{z_1}\bar{z_2}$
6. $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$, $\operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$
7. $|z|^2 = z\bar{z} = \bar{z}z$
8. $|z_1 z_2| = |z_1||z_2|$. This is nontrivial to show:

   Proof. $|z_1 z_2|^2 = z_1 z_2 \bar{z_1}\bar{z_2} = z_1\bar{z_1} z_2 \bar{z_2} = |z_1|^2 |z_2|^2$ □

Let's try to re-prove the Triangle Inequality using property 7 above.

WTS: $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$

Proof. $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z_1} + \bar{z_2}) = |z_1|^2 + |z_2|^2 + z_1\bar{z_2} + z_2\bar{z_1}$.

The cross terms look similar: we claim (and can easily show by property 5 and the definition of the complex conjugate above) that they are each other's complex conjugate. Therefore there is no imaginary part, and we get

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z_2}). \tag{2.1}$$

Further, we can say that $2\operatorname{Re}(z_1\bar{z_2}) \leq 2|z_1\bar{z_2}|$. Finally, using properties 8 and 2, we get

$$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|, \tag{2.2}$$

which is precisely the right hand side. □

2.3 Exponential forms

We know that complex numbers are isomorphic to $\mathbb{R}^2$. We've been representing these vectors in $\mathbb{R}^2$ in Cartesian coordinates: is there a polar representation? Yes!

We know that polar coordinates are related to Cartesian coordinates by $x = r\cos\theta, y = r\sin\theta$, and so we get $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ by Euler's formula. This is the exponential form of complex numbers.

Remark 2.2. *For $z = 0$, θ is undefined.*

Remark 2.3. *We call θ the argument of z. θ is not unique, as* cos *and* sin *are periodic functions: for any valid θ we could shift it by $2\pi n$ for any $n \in \mathbb{Z}$ and still have a valid θ. We denote by* $\arg z$ *the set of all arguments. Further, we denote by $Argz$ the principal argument, i.e. the unique argument lying in $(-\pi, \pi]$.*

Example 2.2. $\operatorname{Arg} 1 = 0, \operatorname{Arg}(-1) = \pi, \operatorname{Arg} i = \frac{\pi}{2}, \operatorname{Arg}(-i) = -\frac{\pi}{2}$

□

Properties

1. $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. This is nontrivial: remember we need to prove everything again for $\mathbb{C}$!

   Proof.

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) && (2.3)\\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) && (2.4)\\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) && (2.5)\\ &= e^{i(\theta_1+\theta_2)} && (2.6) \end{aligned}$$

□

2. $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$
3. $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$
4. $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$
5. $\arg(z_1z_2) = \arg z_1 + \arg z_2, \arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$. Note that this does not hold if we consider Arg instead of arg. For example, if $z_1 = -1, z_2 = i$, then $\operatorname{Arg} z_1 = \frac{\pi}{2}, \operatorname{Arg} z_2 = \pi, \operatorname{Arg}(z_1z_2) = -\frac{\pi}{2} \neq \frac{3\pi}{2}$.

2.4 de Moivre's formula

Lemma 2.4 (de Moivre's formula)**.**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, n \in \mathbb{Z} \tag{2.7}$$

Proof.

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta. \tag{2.8}$$

□

Example 2.3. With $n = 2$, we get the double-angle formulas,

$$(\cos\theta + i\sin\theta) = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta) = \cos(2\theta) + i\sin(2\theta) \tag{2.9}$$

□

2.5 Roots of complex numbers

Suppose $z^n = 1$. Which z satisfy this? Consider $z_k = e^{i\frac{2k\pi}{n}}$ where $k = 0, 1, \ldots, n-1$. By raising to the nth power, we get $e^{i(2k\pi)} = e^{i\cdot 0} = 1$. Therefore we were able to find n roots of 1 (in the complex sense).

Define $\omega_n = z_1 = e^{i\frac{2\pi}{n}}$. We see that $z_k = \omega_n^k$, and therefore w_n generates all the n roots: they are $1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}$. ω_n is in some sense the most fundamental root.

The modulus of all of these roots is 1. We could place all of them on the unit circle.

Another way to solve the earlier equation is to factorize:

$$z^n - 1 = 0 \tag{2.10}$$

$$(z-1)(z^{n-1} + z^{n-2} + \cdots + z^2 + z + 1) = 0 \tag{2.11}$$

That is, for any root that is not 1, we must have that $z^{n-1} + z^{n-2} + \cdots + z^2 + z + 1 = 0$. In particular, if we choose $z = \omega_n$, we get

$$1 + \omega_n + \omega_n^2 + \cdots + \omega_n^{n-1} = 0. \tag{2.12}$$

That is, the sum of all the roots of 1 is 0. We can see this geometrically too: if we vector-add all the points on the unit circle, they'll all cancel out.

What about $z^n = z_o$ for any arbitrary (nonzero) complex number?

Let $z_o = r_o e^{i\theta_o}$. This implies $z = r_o^{1/n} e^{i(\theta+(2k\pi))/n}$ for $k \in \mathbb{Z}$. Splitting the exponential term, we get

$$z = r_o^{1/n} e^{i\theta/n} e^{2k\pi/n} \tag{2.13}$$

We already know how to deal with the last term, as we just found the roots of 1. The remaining two terms are just constants: they are referred to as the principal root ($r_o^{1/n}$) and the principal argument ($e^{i\theta_o/n}$) respectively.

Example 2.4. We can find $1^{1/3}$: the three roots are $1, e^{i2\pi/3}, e^{i4\pi/3}$.

□

Example 2.5. Consider $(-16)^{1/4}$

$$-16 = 16e^{i\pi} \tag{2.14}$$

$$(-16)^{1/4} = (16)^{1/4} e^{i\pi/4} e^{i2\pi k/n}, k = 0, 1, 2, 3. \tag{2.15}$$

For concreteness, plug in for k: we get $\sqrt{2}(1+i)$, $\sqrt{2}(-1+i)$, $\sqrt{2}(-1-i)$, $\sqrt{2}(1-i)$.

□

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Lecture 3: "Topology dictionary", functions and mappings

Lecturer: Di Fang *26 January* *Aditya Sengupta*

Warm-up question: if $z_1 = z_2$ where $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$ is it necessarily the case that $r_1 = r_2$ and $\theta_1 = \theta_2$? Not necessarily: θs are only unique up to 2π rotations.

Today we'll start describing the geometric structure of $\mathbb{C}$ using theory from topology. If we have a number line over the reals, intervals and comparisons are a useful tool for making some sense of it: this is bigger than that, or this number and that one define an interval and you can reason about everything in between them. But in complex numbers, we don't have these tools. How do we build them?

3.1 Neighborhoods, interiors, exteriors, boundaries

Our first basic tool will be an $\epsilon-$neighborhood (nbhd).

Definition 3.1. *For some point z_o, an $\epsilon-$nbhd of z_o is the set of all $z \in \mathbb{C}$ such that $|z - z_o| < \epsilon$; the set of all points within a circle of radius ϵ centered at z_o.*

We denote this by $B_\epsilon(z_o)$ (the B is for ball, to allow for higher dimensions.)

Definition 3.2. *A deleted or punctured neighborhood is a neighborhood with the center removed.*

$$B'_\epsilon(z_o) = \{z \in \mathbb{C} \mid 0 < |z - z_o| < \epsilon\} \tag{3.1}$$

Now we define the interior and exterior of a set. It's tempting to let this just be $z \in S, z \notin S$, but this doesn't consider points on the boundary: do you put them in S or not? What does it mean to be on the boundary anyway? The following definitions will leverage ϵ-neighborhoods to explain this.

Definition 3.3. *z is an interior point of a set S if there exists some ϵ such that $B_\epsilon(z) \subset S$.*

Definition 3.4. *z is an exterior point of a set S if there exists some ϵ such that $B_\epsilon(z) \cap S = \varnothing$.*

Definition 3.5. *z is a boundary point of a set S if it is neither an interior nor an exterior point. More rigorously, for any $\epsilon > 0$, the neighborhood $B_\epsilon(z) \cap S \neq \varnothing$ (there is some point in S) and also $B_\epsilon(z) \cap S^c \neq \varnothing$ (there is some point not in S).*

We denote these by $\operatorname{int} S, \operatorname{ext} S, \partial S$. The interior of S is also denoted S^o.

Example 3.1. Let S be the unit circle: $S = B_1(0) = \{z \in C \mid |z| < 1\}$.

The interior, exterior, and boundary can be defined based on this:

$$\operatorname{int} S = \{z \mid |z| < 1\} \operatorname{ext} S = \{z \mid |z| > 1\} \partial S = \{z \mid |z| = 1\}. \tag{3.2}$$

□

We can use this as a starting example to examine the properties of interior/exterior/boundaries. To start with, notice that in this case $\partial S \not\subset S$. Is the boundary always outside of S, i.e. is $\partial S \cap S = \varnothing$? The answer is no. Consider the following example:

Example 3.2. From the unit circle, include only the top half of the boundary, i.e. $S = B_1(0) \cup \{z \mid r = 1, 0 < \theta < \pi\}$.

□

This does not satisfy $\partial S \cap S = \varnothing$. There is no general rule as to whether the boundary is inside or outside S.

It may seem like the formalism is useless and we can just eyeball the interior/exterior/boundary of any set. To challenge this, consider the following.

Example 3.3. Let $S = \{x + iy \mid x, y \in \mathbb{Q}\}$. You can't sketch this and eyeball it, but the formalism is useful!

For any point in S, a neighborhood of that point will always contain an irrational number (density of the irrationals in the reals.) Thus $\operatorname{int} S = \varnothing$. But similarly, for any point *not* in S, a neighborhood of that point will always contain a rational number (density of the rationals in the reals), so $\operatorname{ext} S = \varnothing$ too! In fact, every point in $\mathbb{C}$ satisfies the constraint for the boundary: every possible neighborhood has some points in S and some not in S. Therefore $\partial S = \mathbb{C}$.

□

3.2 Open and closed sets

Definition 3.6. *A set S is open if it contains no boundary points ($S^{\circ} = S$), and is closed if it contains all its boundary points.*

Definition 3.7. *The closure of a set S is defined as $\bar{S} = S \cup \partial S$.*

Clearly $S^{\circ} \underbrace{\subseteq}_{\text{if equal, open}} S \underbrace{\subseteq}_{\text{if equal, closed}} \bar{S}$.

Not all sets are closed or open. Example2 is neither.

Remark 3.1. *1. Some sets can be neither closed nor open.*

2. Some sets are both closed and open: these are $\varnothing$ and $\mathbb{C}$.

Definition 3.8. *An open set $S \subseteq \mathbb{C}$ is called connected if for all $z_1, z_2 \in S$, they can be joined by a polygonal line (a line consisting of a finite number of line segments.)*

Now, let's start thinking about functions.

Definition 3.9. *A set $S \subseteq \mathbb{C}$ is a domain if S is open and connected.*

Remark 3.2. *This doesn't necessarily mean it's the domain of a particular function yet.*

Definition 3.10. *S is a region if $S \setminus \partial S$ is a domain. (check this)*

Definition 3.11. *S is bounded if $\exists R > 0$ such that $S \subset B_R(0)$.*

Example 3.4. Consider the set $\{z \mid \mathrm{Im}(1/z) > 1\}$. Using the Cartesian form, we see that

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}, \tag{3.3}$$

so

$$\mathrm{Im}\,\frac{1}{z} > 1 \iff -\frac{y}{x^2+y^2} > 1. \tag{3.4}$$

Completing the square shows us that this is equivalent to

$$x^2 + \left(y + \frac{1}{2}\right)^2 < \frac{1}{4} \tag{3.5}$$

This defines a circle. This set is a domain and a neighborhood of the point $-\frac{1}{2}i$.

□

Definition 3.12. *z_o is called an accumulation point or a limit point of a set $S \subseteq \mathbb{C}$ if all $B'(z_o)$ (all punctured neighborhoods of z_o) satisfy $B'(z_o) \cap S \neq \varnothing$.*

Less formally, there is always an element of the set arbitrarily close to an accumulation point. The set of accumulation points is often denoted S'.

Example 3.5. Let $S = \{\frac{1-i}{n} \mid n = 1, 2, \ldots\}$. The accumulation point of S is 0: however close you get to the origin, any neighborhood around the origin will have a point in the set (this follows almost exactly from the limit definition of convergence of $1/n \to 0$.)

□

Example 3.6. $S = \{(-1)^n \mid n = 1, 2, \ldots\}$ has no accumulation points as it only consists of $-1, 1$.

□

Remark 3.3. *$\bar{S} = S \cup \partial S = S \cup S'$, but $\partial S \neq S'$ in general. This is proved, e.g. in Rudin, but we'll just take it as a fact.*

3.3 Functions

Consider $S \subseteq \mathbb{C}$ and a function $f : S \to \mathbb{C}$. We can describe this as a rule $z \to f(z)$ or $x + iy \to u + iv$.

Example 3.7.

$$f(z) = z^2 \tag{3.6}$$
$$f(x, y) = (x^2 - y^2) + 2ixy \tag{3.7}$$
$$u = x^2 - y^2, v = 2xy \tag{3.8}$$

□

Example 3.8.

$$f(z) = |z|^2 \tag{3.9}$$
$$f(x, y) = x^2 + y^2 \tag{3.10}$$
$$u = x^2 + y^2, v = 0. \tag{3.11}$$

This is a "real-valued" function because v is identically zero.

□

Example 3.9. We still have polynomials, i.e. functions of the form $p(z) = \sum_{k=0}^{n} a_k z^k$, and we can still take ratios of polynomials to get rational functions.

□

Remark 3.4. *An interesting type of function is the linear fractional transformation, of the form* $\frac{az+b}{cz+d}$.

We consider a generalization of functions that may not just return a single number: *multi-valued* functions.

Example 3.10. $\arg z$ is multi-valued, as it is only unique up to 2π and so we can add factors of $2k\pi$ to get as many values as we like for this.

□

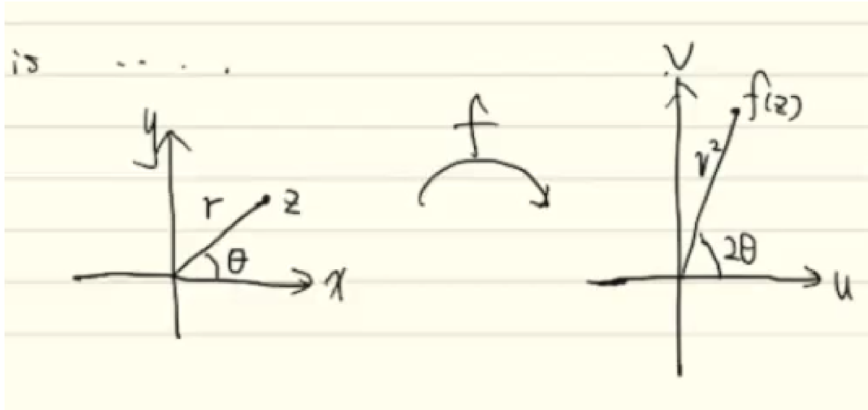

Figure 3.1: Mapping a complex function

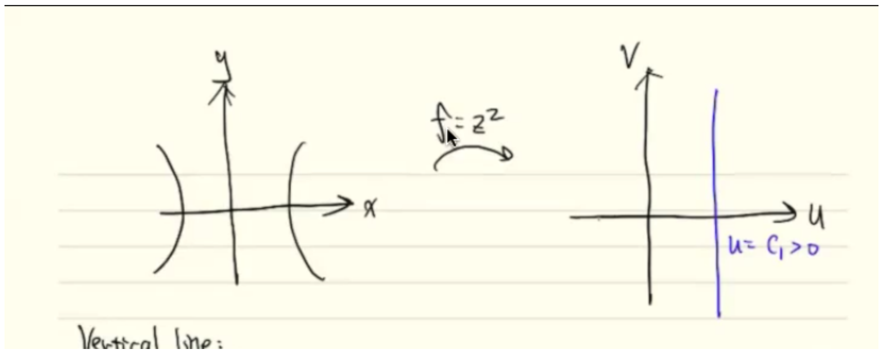

Figure 3.2: Mapping the inpiuts that lead to a certain output

Example 3.11. $z^{1/2}$ is multi-valued. □

If we have the $(x, y) \to (u, v)$ relationship of a function, it's easy to graph it. It may help to plot in polar coordinates though. For example, $f(z) = z^2$ takes $(r, \theta) \to (r^2, 2\theta)$.

Let's also consider the reverse question of how to identify the preimage of a function. We know in the case of $f(z) = z^2$ that $u = x^2 - y^2$, so if we see that $u = c_1$ then we know that $x^2 - y^2 = c_1$ describes a hyperbola.

Which inputs correspond to which outputs, exactly? For example, which direction along the right part of the hyperbola leads to an increase along the constant-u straight line?

From algebra, we get

$$x = \begin{cases} \sqrt{y^2 + c_1} & \text{right} \\ -\sqrt{y^2 + c_1} & \text{left} \end{cases} \tag{3.12}$$

This gives us

$$v = \begin{cases} 2y\sqrt{y^2 + c_1} & \text{right} \\ -2y\sqrt{y^2 + c_1} & \text{left} \end{cases} \tag{3.13}$$

On the right, $y \uparrow \implies v \uparrow$ and on the left, $y \uparrow \implies v \downarrow$.

If we had a horizontal line, we could do the same:

$$v = 2xy = c_1 \implies y = \frac{c_2}{2x} \tag{3.14}$$

and $u = x^2 - \frac{c_2}{4x^2}$. Thus for $x > 0$, $x \uparrow \implies u \uparrow$ and $x < 0$, $x \uparrow \implies u \downarrow$.

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Lecture 4: Limits and continuity

Lecturer: Di Fang *28 January* *Aditya Sengupta*

4.1 Warmup

True/False.

1. A subset of $\mathbb{C}$ is either open or closed. (False: $\mathbb{C}$ or $\varnothing$ are both, and it's possible to be neither.)
2. Since $\bar{S} = S \cup \partial S = S \cup S'$, we have $\partial S = S'$. (False: look at $\{(-1)^n \mid n \in \mathbb{Z}\}$. There are no accumulation points, but the boundary is the set itself.)
3. S is closed if and only if it contains all of its accumulation points. (True: if S is closed, $S = \bar{S} = S \cup S'$).

4.2 Limits

Definition 4.1. $\lim_{z \to z_o} = \omega_o$ *if* $\forall \epsilon > 0, \exists \delta > 0$ *such that* $0 < |z - z_o| < \delta \implies |f(z) - \omega_o| < \epsilon$.

It is important that $0 < |z - z_o|$ (why?) This statement is similar to real analysis, except instead of absolute value of the difference, we have the modulus. This is a similar statement but should be treated as a 2D thing. Therefore a complex limit is a significantly stronger condition than a real one.

We can equivalently say $\forall \epsilon > 0, \exists \delta > 0 \,\text{s.t.}\, z \in B'_\delta(z_o) \implies f(z) \in B_\epsilon(\omega_o)$. This can be interpreted as "there exists a punctured δ-ball centered at z_o such that its image under f is a subset of a punctured ϵ-ball centered at $f(z_o)$".

Theorem 4.1. *If* $\lim_{z \to z_o} f(z)$ *exists, it is unique.*

The proof of this is exactly the same as in real analysis.

Example 4.1. Let $f(z) = \frac{iz}{2}$ in $|z| < 1$. We want to show that $\lim_{z \to 1} f(z) = \frac{i}{2}$.

Proof. $\forall \epsilon > 0 \exists \delta > 0$ such that $0 < |z - 1| < \delta$ implies

$$\left| f(z) - \frac{i}{2} \right| = \frac{|z-1|}{2} < \frac{\delta}{2} = \epsilon, \tag{4.1}$$

so we take $\delta = 2\epsilon$. □

□

Example 4.2. Let $f(z) = \frac{z}{\bar{z}}$ for $z \neq 0$. We want to show that $\lim_{z \to 0} f(z)$ doesn't exist.

Proof. From the real axis, $z = x$ and the function approaches 1 because $f(z) = \frac{x}{x} = 1 \to 1$. From the imaginary axis, $z = iy$ and the function approaches -1 because $f(z) = \frac{iy}{-iy} = -1 \to -1$. Therefore $1 \neq -1$ and the limit does not exist. □

□

4.3 Properties of limits

1. Suppose $\lim_{z \to z_o} = \omega_o$. Then $\lim_{z \to z_o} \operatorname{Re} f(z) = \operatorname{Re} \omega_o$ and $\lim_{z \to z_o} \operatorname{Im} f(z) = \operatorname{Im} \omega_o$. This is an if-and-only-if.

   Proof. (sketch) In the forward direction,

$$0 < |z - z_o| < \delta \implies |\operatorname{Re} f(z) - \operatorname{Re} \omega_o| = |\operatorname{Re}(f(z) - \omega_o)| < |f(z) - \omega_o| < \epsilon, \tag{4.2}$$

   and similarly for the imaginary part: basically, the magnitude of the real/imaginary part is less than that of z so a delta that works for z will work for its constituent parts.

   In the backward direction, let's say

$$\forall \frac{\epsilon}{2} > 0, \exists \delta_1 > 0 \,\text{s.t.}\, 0 < |z - z_o| < \delta_1 \implies |\operatorname{Re} f(z) - \operatorname{Re} \omega_o| < \frac{\epsilon}{2} \tag{4.3}$$

$$\forall \frac{\epsilon}{2} > 0, \exists \delta_2 > 0 \,\text{s.t.}\, 0 < |z - z_o| < \delta_2 \implies |\operatorname{Im} f(z) - \operatorname{Im} \omega_o| < \frac{\epsilon}{2} \tag{4.4}$$

$$\tag{4.5}$$

   Now we want to find some δ that combines these and meets the limit statement, i.e. that $|z - z_o| < \delta \implies |f(z) - \omega_o| < \epsilon$.

$$|f(z) - \omega_o| = |\operatorname{Re} f(z) - \operatorname{Re} \omega_o + i(\operatorname{Im} f(z) - \operatorname{Im} \omega_o)| \tag{4.6}$$

$$\leq |Ref(z) - \operatorname{Re} \omega_o| + |\operatorname{Im} f(z) - \operatorname{Im} \omega_o| \tag{4.7}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{4.8}$$

   □

2. If $\lim_{z \to z_o} f(z) = \omega_1$, $\lim_{z \to z_o} g(z) = \omega_2$ then

   (a) limits and sums commute and linear combination are valid: $\lim_{z \to z_o} af(z) + bg(z) = a\omega_1 + b\omega_2$

(b) products work: $\lim_{z \to z_o} f(z)g(z) = \omega_1 \omega_2$

(c) quotients work: if $\omega_2 \neq 0$, then $\lim_{z \to z_o} \frac{f(z)}{g(z)} = \frac{\omega_1}{\omega_2}$

We can combine the first two properties to show that polynomials have well-defined limits that we can find just by plugging in the point. $\lim_{z \to z_o} P(z) = P(z_o)$

4.4 Limits involving infinity

In real analysis, we had $\pm\infty$: two infinities for the two directions. In our case, however, we can go in infinite directions, so do we have *infinite* infinities? (something something John Green).

We'll introduce a notion of infinity that is direction-independent. We refer to the complex plane with infinity as the "extended complex plane", $\mathbb{C} \cup \{\infty\}$. Imagine this as the surface of a sphere.

Consider a sphere whose projection on the complex plane is the unit circle. For any point in the complex plane z, consider the line between the north pole of the sphere at $(0, 0, 1)$ and the point $z = x + iy = (x, y, 0)$. This will intersect with the sphere at some point. This defines a mapping from the complex plane to the Riemann sphere(conventionally, we also say the origin maps to the south pole). Under this mapping, we say infinity maps to the north pole (in all directions.)

Definition 4.2. *A neighborhood of* ∞*,* $B_R(\infty)$*, is the set* $|z| > R$*.*

This would be a circle around the North Pole on the Riemann sphere.

Definition 4.3. *The three types of limits involving infinity are defined as follows:*

1. $\lim_{z \to z_o} f(z) = \infty$ *if* $\forall R > 0, \exists \delta > 0 \, \text{s.t.} \, z \in B'_\delta(z_o) \implies f(z) \ni B_R(\infty)$.
2. $\lim_{z \to \infty} f(z) = \omega_o$ *if* $\forall \epsilon > 0, \exists R > 0 \, \text{s.t.} \, z \in B_R(\infty) \implies f(z) \in B_\epsilon(\omega_o)$.
3. $\lim_{z \to \infty} f(z) = \infty$ *if* $\forall R > 0, \exists r > 0 \, \text{s.t.} \, z \in B_r(\infty) \implies f(z) \in B_R(\infty)$.

Theorem 4.2.

1. $\lim_{z \to z_o} f(z) = \infty$ *if* $\lim_{z \to z_o} \frac{1}{f(z)} = 0$.
2. $\lim_{z \to \infty} f(z) = \omega_o$ *if* $\lim_{z \to 0} f((\frac{1}{z}) = \omega_o$.
3. $\lim_{z \to \infty} f(z) = \infty$ *if* $\lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$.

Example 4.3. Suppose we want to show that $\lim_{z \to \infty} \frac{1}{z+1} = 0$. It suffices to show that $\lim_{z \to 0} \frac{1}{\frac{1}{z}+1} = 0$ which is equivalent to $\lim_{z \to 0} \frac{z}{1+z} = 0$.

□

4.5 Continuity

Definition 4.4. *f is continuous at z_o if $f(z_o)$ is defined, $\lim_{z \to z_o} f(z)$ is defined, and $\lim_{z \to z_o} f(z) = f(z_o)$. Note that the third condition contains the first two.*

More precisely, $f : S \to \mathbb{C}$ is continuous at z_o if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\forall z \in S$, $|z - z_o| < \delta \implies |f(z) - f(z_o)| < \epsilon$. More compactly, $f(B_\delta(z_o)) \subset B_\epsilon(f(z_o))$.

Definition 4.5. *f is continuous in a region R if it is continuous at each point in R.*

Recall that if f, g are continuous at z_o, then so are $f + g, fg$, and $\frac{f}{g}$ provided that $g \neq 0$. We also find that every polynomial is continuous in the entire plane.

Theorem 4.3. *Let $f : A \to B, g : B \to C$ for $A, B, C \in \mathbb{C}$. If f is continuous at z_o and g i continuous at $f(z_o)$, then $g \circ f : A \to C, z \to g(f(z))$ is continuous at z_o.*

We can prove this roughly using the definition twice. For any ϵ there exists some α such that continuity holds for g, and for that α there exists some δ such that it holds for f.

Theorem 4.4. *If f is continuous at z_o and $f(z_o) \neq 0$, then $f \neq 0$ in some neighborhood of z_o.*

Proof. Let $\epsilon = \frac{|f(z_o)|}{2} > 0$, since $f(z_o) \neq 0$. Then by continuity, $\exists \delta > 0 \,\text{s. t.}\, |z - z_o| < \delta \implies |f(z) - f(z_o)| < \frac{|f(z_o)|}{2}$.

We show this by contradiction. Suppose there exists a point such that $f(z) = 0$. Then we would have it cancel out, and we would have $|f(z_o)| < \frac{|f(z_o)|}{2}$, a contradiction. □

Math 185: Complex Analysis **Spring 2021**

Lecture 5: Complex Differentiation

Lecturer: Di Fang *2 February* *Aditya Sengupta*

We begin with a wrong thing. Convert the following statement to $\epsilon - \delta$.

$$\lim_{z \to 1} z^2 = \infty \tag{5.1}$$

This can be written as

$$\forall R > 0 \exists \delta > 0 \,\text{s.t.}\, 0 < |z - 1| < \delta \implies z^2 > R. \tag{5.2}$$

We could also use neighborhoods:

$$\forall R > 0 \exists \delta > 0 \,\text{s.t.}\, z \in B'_\delta(1) \implies z^2 \in B_R(\infty). \tag{5.3}$$

Theorem 5.1. *Consider a function $f(z) = u(x, y) + iv(x, y)$. Then f is continuous at $z_0 = x_0 + iy_0$ iff u, v are continuous at (x_0, y_0).*

Theorem 5.2. *If $f : R \to \mathbb{C}$ is continuous in R, for R compact (closed and bounded), then there exists a real number $M > 0$ such that $|f(z)| \leq M \; \forall z \in R$, with equality for at least one $z_0 \in R$.*

Proof. (sketch) Since f is continuous, then $\sqrt{u^2(x, y) + v^2(x, y)}$ is continuous over R because each component is continuous over R so this combination must also be continuous. Therefore, since R is compact, f attains a maximum value M somewhere in R. □

5.1 Derivatives

Let f be a function defined on S.

Definition 5.1. *The derivative of f at z_0 is the limit*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{5.4}$$

Equivalently, let $\Delta z = z - z_0$. Then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{5.5}$$

A function is said to be differentiable at z_0 if $f'(z_0)$ exists.

We can check limit existence either by the definition or by composition properties.

If we let $\omega = f(z)$ and analogously define $\Delta\omega = f(z+\Delta z) - f(z)$, then we have

$$\frac{\mathrm{d}\omega}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{\Delta\omega}{\Delta z}. \tag{5.6}$$

Example 5.1. Let $f(z) = \frac{1}{z}$ for $z \neq 0$. Then

$$\frac{\Delta\omega}{\Delta z} = \frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta z} = -\frac{1}{z(z+\Delta z)} \tag{5.7}$$

$$\implies f'(z) = \frac{\mathrm{d}\omega}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{\Delta\omega}{\Delta z} = -\frac{1}{z^2} \tag{5.8}$$

This is the same as what we would do for real numbers, with the exception that this is now a limit in $\mathbb{C}$.

□

Example 5.2. Let $f(z) = \bar{z}$ (this one is specific to complex numbers).

$$\frac{\Delta\omega}{\Delta z} = \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}. \tag{5.9}$$

We claim this limit does not exist. To see why, we split it up into real and imaginary parts:

$$\frac{\Delta\omega}{\Delta z} = \frac{\Delta x + i\Delta y}{\Delta x - i\Delta y}, \tag{5.10}$$

and if we approach this along $\operatorname{Im} z = 0$ we'll get 1 but if we approach it along $\operatorname{Re} z = 0$ we'll get -1. Therefore the limit does not exist. In other words, $f(z)$ is not differentiable for any $z \in \mathbb{C}$.

□

Example 5.3. Let $f(z) = |z|^2 = z\bar{z}$. This is nowhere differentiable except $z = 0$.

$$\frac{\Delta\omega}{\Delta z} = \frac{(z+\Delta z)(z+\bar{\Delta} z) - z\bar{z}}{\Delta z} = \frac{\Delta z\bar{z} + z\bar{\Delta} z + \Delta z\bar{\Delta} z}{\Delta z} \tag{5.11}$$

$$= \bar{z} + z\frac{\bar{\Delta} z}{\Delta z} + \bar{\Delta} z. \tag{5.12}$$

Due to the $\bar{z}$ component, the limit does not exist unless $z = 0$. In the case $z = 0$, this does simplify to $\lim_{\Delta z \to 0} (\bar{z} + \bar{\Delta} z)\big|_{z=0} = 0$, so $f(0) = 0$.

□

Example 5.4. Consider the piecewise function

$$f(z) = \begin{cases} e^{-1/z^2} & z \neq 0 \\ 0 & z = 0 \end{cases}. \tag{5.13}$$

This is interesting because of the following property that is false in $\mathbb{R}$ but true in $\mathbb{C}$: if f is smooth, then its Taylor series is equal to the function itself. In $\mathbb{R}$, the above function is the counterexample, as all of its derivatives at 0 are themselves zero. In $\mathbb{C}$, this function is not smooth, and in fact is not even first-order differentiable.

First, we do this in $\mathbb{R}$:

$$f'(0) = \lim_{x \to 0} \frac{e^{-1/x^2} - 0}{x - 0} = \lim_{x \to 0} e^{-\frac{1}{x^2}} x. \tag{5.14}$$

Let $y = \frac{1}{x}$:

$$f'(0) = \lim_{y \to \infty} \frac{y}{e^{y^2}} = 0. \tag{5.15}$$

Repeating this, we see that $f^{(n)}(0)$.

In $\mathbb{C}$, however, we have

$$\frac{\Delta\omega}{\Delta z} = \frac{e^{-\frac{1}{(\Delta z)^2}}}{z} \tag{5.16}$$

Consider this limit along $\Delta z = i\Delta y$, as $\Delta y \to 0$. This goes to infinity.

$$\frac{\Delta\omega}{\Delta z} = \lim_{\Delta y \to 0} \frac{e^{\frac{1}{(i\Delta y)^2}}}{i\Delta y} = \infty \tag{5.17}$$

However, along $\Delta z = \Delta x$, this is the same as the real case, and

$$\frac{\Delta\omega}{\Delta x} = \lim_{\Delta x \to 0} e^{-\frac{1}{(\Delta x)^2}} \Delta x = 0. \tag{5.18}$$

Therefore the limit does not exist.

□

Remark 5.3. *Even if the real and imaginary components u, v of a function $f(x, y) = u(x, y) + iv(x, y)$ are individually differentiable, f itself may not be (see the example $f(z) = \bar{z}$).*

Remark 5.4. *There exists some function that is differentiable at some z_0, but not differentiable in its neighborhood. This is satisfied by $f(z) = |z|^2$. This shows us that differentiability is local.*

5.2 Rules of Differentiation

These are basically the same as in real analysis.

$$\frac{\mathrm{d}}{\mathrm{d}z} c = 0, \frac{\mathrm{d}}{\mathrm{d}z} z = 1 \tag{5.19}$$

The derivative is linear:

$$\frac{\mathrm{d}}{\mathrm{d}z} c f(z) = c \frac{\mathrm{d}}{\mathrm{d}z} f(z) \tag{5.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}z} (f + g) = \frac{\mathrm{d}}{\mathrm{d}z} f + \frac{\mathrm{d}}{\mathrm{d}z} g \tag{5.21}$$

And we have the power rule, the product rule, the quotient rule, and the chain rule.

However, we do have a unique property in complex differentiation, which is expressed in the Cauchy-Riemann equations.

5.3 Cauchy-Riemann Equations

We want to find a necessary condition for the complex derivative to exist.

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at z_0. Then the partial derivatives of u, v exist. Additionally, the partials satisfy certain equations (the Cauchy-Riemann equations): $u_x = v_y, u_y = -v_x$ at (x_0, y_0). Also, the derivative can be written in terms of these partials, in the form

$$f'(z_0) = u_x + iv_x = v_y - iu_y, \tag{5.22}$$

all evaluated at (x_0, y_0).

Proof. Let f be differentiable at z_0. Then, the following limit exists:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{5.23}$$

Since this limit exists, we get the same result if we go along $\Delta z = \Delta x$ and if we go along $\Delta z = i\Delta y$.

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \tag{5.24}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \tag{5.25}$$

$$= u_x(x_0, y_0) + iv_x(x_0, y_0). \tag{5.26}$$

And in the y direction, we similarly get

$$f'(z_0) = \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y} \tag{5.27}$$

$$= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} \tag{5.28}$$

$$= -iu_y(x_0, y_0) + v_y(x_0, y_0). \tag{5.29}$$

Matching up the real and imaginary parts, we get the desired result. □

Math 185: Complex Analysis **Spring 2021**

Lecture 6: The Cauchy-Riemann condition

Lecturer: Di Fang *4 February* *Aditya Sengupta*

6.1 A sufficient condition for differentiability

Recall that last time we looked at $f(z) = |z|^2$, which we said is not differentiable at $z \neq 0$. In addition to showing this by the definition, we can also show it does not satisfy the C-R condition. To do this, we could show that $u_x \neq v_y$ or $u_y \neq v_x$.

$$f(z) = |z|^2 = x^2 + y^2 = (x^2 + y^2) + i(0). \tag{6.1}$$

Taking derivatives, we have that $u_x = 2x, u_y = 2y, v_x = v_y = 0$. Therefore we see that the only way for the required derivatives to match up is if $x = y = 0$. Therefore when $z \neq 0$, f is not differentiable.

Note that if $z = 0$, we don't know that f *is* differentiable, just that it definitely cannot be differentiable at $z \neq 0$. In other words, the C-R condition is necessary but not sufficient. Now, we'll develop a sufficient condition for complex differentiability.

Example 6.1. Consider the function

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases} \tag{6.2}$$

We can show that at $z = 0$, the C-R condition is satisfied. After lots of simplification, we get

$$f(z) = \frac{(x^3 - 3xy^2) + i(y^3 - 3x^2y)}{x^2 + y^2} \tag{6.3}$$

Therefore we get

$$u = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & x \neq 0 \text{ or } y \neq 0 \\ 0 & x = y = 0 \end{cases} \tag{6.4}$$

We can't take a derivative of this via composition rules, so let's use the definition.

$$u_x = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1 \tag{6.5}$$

$$v_y = \lim_{\Delta y \to 0} \frac{v(0, \Delta y) - v(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y - 0}{\Delta y} = 1 \tag{6.6}$$

$$\tag{6.7}$$

Therefore $u_x = v_y$, and if we did the same again we could confirm that $u_y = -v_x$. So the C-R condition is satisfied.

Despite this (from the homework) we know that this function is *not* differentiable at $z = 0$.

□

So let's develop a *sufficient* condition on top of this.

Theorem 6.1. *Let a function $f(z) = u(x,y) + iv(x,y)$ be defined throughout some $\epsilon-$neighborhood of $z_0 = x_0 + iy_0$ and suppose that*

1. u_x, u_y, v_x, v_y *exist everywhere in the neighborhood.*
2. *these partials are continuous at (x_0, y_0) and satisfy C-R.*

Then $f'(z_0)$ exists and its value is $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Proof. First, we want to show that the limit $\lim_{\Delta z \to 0} \frac{\Delta \omega}{\Delta z}$ exists.

$$\Delta\omega = \Delta u + i\Delta v = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) + i(v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)) \tag{6.8}$$

Since the partials are continuous, we can Taylor expand.

$$u(x_0 + \Delta x, y_0 + \Delta y) = u(x_0, y_0) + u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \tag{6.9}$$

where $\epsilon_1 \to 0$ as $\Delta x \to 0$ and $\epsilon_2 \to 0$ as $\Delta y \to 0$. Proceed similarly for v.

Therefore

$$\Delta\omega = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta x + \epsilon_1 \Delta x + \epsilon_2 \Delta y + i(v_x(x_0, y_0)\Delta x \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y) \tag{6.10}$$

This implies

$$\frac{\Delta\omega}{\Delta z} = \frac{1}{\Delta z}\left(u_x(x_0, y_0)(\Delta x + i\Delta y) + v_x(x_0, y_0)(i\Delta x - \Delta y) + (\epsilon_1 + i\epsilon_3)\Delta x + (\epsilon_2 + i\epsilon_4)\Delta y\right) \tag{6.11}$$

Under the limit $\Delta z = \Delta x + i\Delta y \to 0$, we get

$$\lim_{\Delta z \to 0} \frac{\Delta \omega}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + \lim_{\Delta z \to 0} (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z} \tag{6.12}$$

Note that $\left|\frac{\Delta x}{\Delta z}\right| \leq 1$ and $\left|\frac{\Delta y}{\Delta z}\right| \leq 1$, so the ϵ_is going to zero dominates. Therefore the last terms die, and we get

$$\lim_{\Delta z \to 0} \frac{\Delta \omega}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0). \tag{6.13}$$

□

Example 6.2. Let $f(z) = e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$. The partials exist in $\mathbb{R}$, they are continuous, and they satisfy C-R:

$$u_x = e^x \cos y, v_x = e^x \sin y \tag{6.14}$$
$$u_y = -e^x \sin y, v_y = e^x \cos y \tag{6.15}$$

Therefore $f'(z)$ exists everywhere.

□

Example 6.3. Let $f(z) = |z|^2$. Use C-R to show that $f'(0)$ exists.

We have $u = x^2 + y^2, v = 0$. The partials are

$$u_x = 2x, v_x = 0 \tag{6.16}$$
$$u_y = 2y, v_y = 0. \tag{6.17}$$

The partials exist in $\mathbb{R}$ and are continuous, and therefore wherever C-R is satisfied, the derivative exists. This is true only at 0. Therefore $f'(0)$ exists.

□

6.2 C-R in polar form

Let $z = re^{i\theta}, x = r\cos\theta, y = r\sin\theta$. We convert the partials to polar using the chain rule:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x\cos\theta + u_y\sin\theta \tag{6.18}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -u_x r\sin\theta + u_y r\cos\theta, \tag{6.19}$$

and the same for v_x, v_y (just replace u with v throughout). If we compare u_r and v_θ, we note that they differ only by an r if we know that $u_y = -v_x$ and $u_x = v_y$. Similarly, u_θ and v_r differ only by a $-r$. Therefore we have C-R in polar form:

$$ru_r = v_\theta \tag{6.20}$$

$$u_\theta = -rv_r. \tag{6.21}$$

Then, let $f(z) = u(r,\theta) + iv(r,\theta)$ be defined in some $\epsilon-$neighborhood of a nonzero points $z_0 = r_0 e^{i\theta_0}$ and suppose that

1. $u_r, u_\theta, v_r, v_\theta$ exist everywhere in the neighborhood
2. the partials are continuous at (r_0, θ_0) and satisfy the C-R equations (above) at (r_0, θ_0).

Then $f'(z_0)$ exists and is equal to $f'(z_0) = e^{-i\theta}(u_r + iv_r)$.

Proof. The proof that the derivative exists proceeds basically the same, but we want to prove that $f'(z_0)$ is the aforementioned value.

$$u_r + iv_r = u_x\cos\theta + u_y\sin\theta + i(u_x\sin\theta - u_y\cos\theta) \tag{6.22}$$

$$= u_x(\cos\theta + i\sin\theta) + iu_y(-\cos\theta - i\sin\theta) \tag{6.23}$$

$$= u_x e^{i\theta} + iv_x e^{i\theta}, \tag{6.24}$$

and therefore $f'(z_0) = u_x + iv_x = e^{-i\theta}(u_r + iv_r)$. □

A lot of functions are easier to deal with in polar!

Example 6.4. Let $f(z) = \frac{1}{z^2}$. This is a nightmare in Cartesian, but in polar we have $f(z) = \frac{1}{r^2}e^{-2i\theta} = \frac{1}{r^2}\cos 2\theta - i\frac{1}{r^2}\sin 2\theta$. Therefore

$$ru_r = \frac{-2}{r^2}\cos 2\theta, u_\theta = \frac{-2}{r^2}\sin 2\theta \tag{6.25}$$

$$rv_r = \frac{2}{r^2}\sin 2\theta, v_\theta = \frac{-2}{r^2}\cos 2\theta \tag{6.26}$$

□

Math 185: Complex Analysis **Spring 2021**

Lecture 7: Analytic and harmonic functions

Lecturer: Di Fang *9 February* *Aditya Sengupta*

For a warmup, let $f(x+iy) = x^3 + i(1-y)^3$. We want to find out whether this is differentiable.

We check the C-R conditions:

$$u_x = 3x^2, u_y = 0 \tag{7.1}$$

$$v_x = -3(1-y)^2, v_y = 0 \tag{7.2}$$

For $u_x = v_y$ and $u_y = -v_x$, we need $x = 0$ and $y = 1$. Further, since the partials exist and are continuous everywhere, the function is differentiable at $z = i$.

At this point, the derivative is

$$f'(0+1i) = u_x(x,y) + v_x(x,y) = 0. \tag{7.3}$$

7.1 Analytic functions

Definition 7.1. *1. $f : S \to C$ is analytic (where S is an open set) if $\forall z \in S, f'(z)$ exists.*

2. f is analytic at a point z_0 if it is analytic in some neighborhood of z_0.

3. f is entire if $f : \mathbb{C} \to \mathbb{C}$ is analytic in $\mathbb{C}$.

Analytic functions are also referred to as holomorphic. This is sort of accidental, as it wasn't realized that being holomorphic (as above) was equivalent to being analytic (existence of the Taylor series) until after both properties were defined. More on that later.

Example 7.1. $f(z) = \frac{1}{z}$ is differentiable at $\mathbb{C} \setminus \{0\}$, and therefore it is analytic at $\mathbb{C} \setminus \{0\}$. □

Example 7.2. $f(z) = |z|^2$ is differentiable only at $z = 0$. Since this is only a single point, there's no neighborhood over which f is differentiable, and so f is analytic nowhere. □

Example 7.3. Polynomials are entire. □

Example 7.4. Rational functions of the form $f(z) = \frac{P(z)}{Q(z)}$ where P, Q are both polynomials are differentiable as long as $Q(z) \neq 0$. $Q(z) = 0$ only at n points (where $n = \deg Q$), so only these points need to be excluded. Therefore f is analytic at $\mathbb{C} \setminus \{\text{zeros of } Q\}$. □

7.2 Properties of analytic functions

Let f, g be analytic in S.

1. $f + g, f \cdot g, \frac{f}{g}$ (if $g \neq 0$ in S) are analytic.
2. $g \circ f$ is analytic and the chain rule holds
3. f analytic in a domain D implies f is continuous in D, and the C-R equations are satisfied.
4. If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.

Proof. (of property 4): The derivative is

$$f'(z) = u_x + iv_x = v_y - iu_y = 0, \tag{7.4}$$

i.e. u_x, u_y, v_x, v_y must all be 0 over D.

Next, we show that $u(x, y)$ is constant over any line segment within D. Let L be a line segment from P to P', with unit vector $\vec{v} = \frac{P'-P}{|P'-P|}$. Let s be the distance along the line.

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \nabla u \cdot \vec{v} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \vec{v} = 0, \tag{7.5}$$

so u remains constant along any line segment. Similarly, v is also constant along any line segment.

Now, for our final step, note that since D is a domain, it is connected and any two points can be connected with a finite number of line segments. u and v remain zero along any line segment and therefore f is constant along the whole path. □

Definition 7.2. *z_0 is called a singular point if f is not analytic at z_0, but is analytic at some point in every neighborhood of z_0.*

Example 7.5. $f(z) = \frac{z^2+3}{(z+1)(z^2+5)}$ is analytic in $\mathbb{C} \setminus \{-1, \pm i\sqrt{5}\}$. $\square$

Corollary 7.1. *Suppose $f = u + iv$ and $\overline{f} = u - iv$ are both analytic in D. Then $f(z)$ is constant in D.*

Proof. If f and $\overline{f}$ are both analytic, then $u_x = v_y = -v_y$ and $u_y = -v_x = -u_y$. So all the partials must be 0 and so f is constant. $\square$

Corollary 7.2. *If f is analytic in D and $|f(z)|$ is constant in D, then $f(z)$ is constant in D.*

Proof. First, let $|f(z)| = c \in \mathbb{R}$. If $c = 0$ then $f(z) = 0$ in D and we are done. If $c \neq 0$,

$$|f(z)|^2 = f(z)\overline{f(z)} = c^2 \neq 0, \tag{7.6}$$

so we can divide:

$$\overline{f(z)} = \frac{c^2}{f(z)}, \tag{7.7}$$

and this is analytic in D as it is the quotient of two analytic functions. Therefore as f and $\overline{f}$ are both analytic, f must be a constant. $\square$

Note that if we want to use any of the properties above on homework, we have to prove them.

7.3 Harmonic functions

Definition 7.3. *A function $H(x, y) : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is harmonic if H has partial derivatives up to second order (i.e. $H \in \mathbb{C}^2(D)$) and it satisfies $H_{xx}(x, y) + H_{yy}(x, y) = 0$, i.e. $\Delta H = 0$.*

This is known as Laplace's equation, from E&M etc.

Theorem 7.3. *If $f(x + iy) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its components u, v are harmonic.*

We can't prove that u, v are differentiable up to second order until we reach chapter 4, section 57, but we can prove Laplace's equation is satisfied.

Proof. Since f is analytic, $u_x = v_y$ and $u_y = -v_x$. Taking second partials, we get

$$u_{xx} = v_{yx} = v_{xy} \tag{7.8}$$
$$u_{yy} = -vxy \tag{7.9}$$

and therefore $u_{xx} + u_{yy} = 0$ in D. Similarly we can show $v_{xx} + v_{yy} = 0$. $\square$

Remark 7.4. *The reverse direction is not true: u, v harmonic does not imply $u + iv$ is analytic. An easy way to say this is to interchange them: $v + iu$ may not be analytic, as C-R is not commutative.*

7.4 Elementary functions

Exp The exponential function is defined as $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$. e^z is entire (prove this using the C-R sufficient condition) and $(e^z)' = e^z$, like in $\mathbb{R}$. The modulus is $|e^z| = e^x$ and the argument is $\arg e^z = y + 2n\pi, n \in \mathbb{Z}$. In the complex plane, e^z is periodic with period $2\pi i$. Other properties hold as we might expect: $e^z \neq 0 \forall z \in \mathbb{C}$, $e^{z_1}e^{z_2} = e^{z_1+z_2}$, $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$, $e^0 = 1$ and $\frac{1}{e^z} = e^{-z}$.

Example 7.6. Suppose we want to find $z \in \mathbb{C}$ such that $e^z = 1 + \sqrt{3}i$. Put this in polar form:

$$1 + \sqrt{3}i = 2e^{i\frac{\pi}{3}} = e^x e^{iy}. \tag{7.10}$$

Term-matching gives us

$$z = \ln 2 + i\left(\frac{\pi}{3} + 2n\pi\right). \tag{7.11}$$

□

What we've basically done is define the logarithm!

$$\log z = \ln r + i(\theta + 2n\pi) \tag{7.12}$$

Note that it's multivalued, so we restrict it to its principal argument.

Log The principal value of the logarithm is defined by $\text{Log}(re^{i\theta}) = \ln r + i \arg z$.

Math 185: Complex Analysis **Spring 2021**

Lecture 8: Log, branches, power functions

Lecturer: Di Fang *11 February* *Aditya Sengupta*

Example 8.1. Let's calculate the logs of $-1, 1-i, i$.

$$\log -1 = i(\pi + 2n\pi), n \in \mathbb{Z} \tag{8.1}$$

$$\log(1-i) = \frac{1}{2}\ln 2 + i\left(-\frac{\pi}{4} + 2n\pi\right), n \in \mathbb{Z} \tag{8.2}$$

$$logi = i\left(\frac{\pi}{2} + 2n\pi\right), n \in \mathbb{Z}. \tag{8.3}$$

□

Do log rules still apply? Let $x \in \mathbb{R}$ and consider $\log z^x$. For $z \in \mathbb{R}$ this is equal to $x \log z$. Is this still true?

It turns out no: consider $z = i, x = 2$:

$$\log i^2 = \log -1 = i\pi + i2n\pi \tag{8.4}$$

$$2\log i = i\pi + i4n\pi. \tag{8.5}$$

Now let's consider Log. Is it differentiable? No, it's not even continuous. This is because the principal argument wraps from $-\pi$ to $+\pi$ at the negative real axis.

More rigorously, consider the sequence $z_n = e^{i(-\pi+1/n)} \xrightarrow{n\to\infty} e^{i(-\pi)} = -1$. Apply Log to both sides of this:

$$\operatorname{Log} z_n = i(-\pi + \frac{1}{n}) \tag{8.6}$$

This problem persists no matter where we make the cut for Arg.

Fix $\alpha \in \mathbb{R}$. Suppose we restrict $\theta = \arg z$ to $\alpha < \theta < \alpha + 2\pi$. All rays other than $\theta = \alpha$can be represented with some argument.

Suppose we restrict the logarithm to be single-valued and such that the branch is at α.

$$\log z = \ln r + i\theta(r > 0, \alpha < \theta < \alpha + 2\pi) \tag{8.7}$$

The logarithm is continuous on this domain. We can show it is also analytic on this domain.

Proof. $u = \ln r, v = \theta$; take derivatives,

$$u_r = \frac{1}{r}, v_r = 0 \tag{8.8}$$

$$u_\theta = 0, v_\theta = 1. \tag{8.9}$$

The C-R condition is okay, as $ru_r = v_\theta$ and $rv_r = -u_\theta$. By existence and continuity of partials, $f(z)$ exists. The derivative is therefore

$$f(z) = e^{-i\theta}(u_r + iv_r) = \frac{1}{re^{i\theta}} = \frac{1}{z}. \tag{8.10}$$

□

Definition 8.1. *A branch of a multi-valued function f is any single-valued function F that is analytic in some domain D, and such that $F(z)$ has one of the values of $f(z)$.*

Remark 8.1. $\operatorname{Log} z$ *is called the principal branch of* $\log z$.

Definition 8.2. *A branch cut is a line or curve that is introduced to define a branch.*

Definition 8.3. *Points on the branch cut are singular points, and any point that is shared by all branch cuts is called a branch point.*

8.1 Log properties

1. $\log(z_1 z_2) = \log z_1 + \log z_2$, because arguments add
2. $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$.

   How come you can't just apply the first property to $z_1 = z_2 = z$ and get $\log z^2 = 2\log z$, which we showed was wrong? The answer is that we don't know the two logs have the same n for the choice of argument.

$$\log z^2 = \log(z \cdot z) = \underbrace{\log z}_{\ln r + i(\theta + 2k\pi)} + \underbrace{\log z}_{\ln r + i(\theta + 2n\pi)}, \tag{8.11}$$

   and in general $k \neq n$.

3. The two properties above do not hold for Log; for example, let $z_1 = z_2 = -1$, then $\log 1 = 2n\pi i$ and $\log(-1) = i\pi + i2n\pi$. Therefore $\operatorname{Log} 1 = 0$ and $\operatorname{Log} -1 = i\pi$, and we do not get $\operatorname{Log}(-1 \cdot -1) = 2\operatorname{Log} -1$.
4. $z^n = e^{n \log z}$; this is because the degeneracy in log goes away when you exponentiate it.
5. $z^{1/n} = e^{1/n \log z}$ because of what we found when we were finding the nth roots of complex numbers.

8.2 Power functions

If we have z^c for some $c \in \mathbb{C}$, where $z \neq 0$, we can say $z^c = e^{c \log z}$. This is multivalued, so we need to take a branch cut: $D = \{r > 0, \alpha < \theta < \alpha + 2\pi\}$. This defines a branch of $\log z$ and therefore also a branch of z^c. On D, z^c is single-valued and analytic.

What's the derivative of a power function?

$$\frac{\mathrm{d}}{\mathrm{d}z} z^c = \frac{\mathrm{d}}{\mathrm{d}z} = e^{c \log z} = e^{c \log z} \frac{c}{z}, \tag{8.12}$$

where we can use the chain rule because the exponential function and log (over the branch cut) are entire.

$$\frac{\mathrm{d}}{\mathrm{d}z} z^c = \frac{c}{e^{\log z}} e^{c \log z} = c e^{(c-1) \log z} = c z^{c-1}. \tag{8.13}$$

Example 8.2. Consider i^i: $\log i = i\left(\frac{\pi}{2} + 2n\pi\right)$, so

$$i^i = e^{-\left(\frac{\pi}{2} + 2n\pi\right)}, \tag{8.14}$$

and the principal value is $e^{-\frac{\pi}{2}}$.

□

Math 185: Complex Analysis **Spring 2021**

Lecture 9: Exponential, trig and hyperbolic, functions, conformal mappings

Lecturer: Di Fang *16 February* *Aditya Sengupta*

Definition 9.1. $c^z = e^{z \log c}$. *Once we specify a value of* $\log c$*, this is single-valued and entire, with* $\frac{\mathrm{d}}{\mathrm{d}z} c^z = c^z \log c$.

9.1 Trigonometric functions

Let $|z| = 1$. We know that $z = e^{i\theta} = \cos\theta + i\sin\theta$. From this, we know that we can express sine/cosine in terms of exponentials:

$$\cos\theta = \operatorname{Re} z = \frac{z + \bar{z}}{2} = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{9.1}$$

$$\sin\theta = \operatorname{Im} z = \frac{z - \bar{z}}{2i} = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{9.2}$$

Based on this, we define the complex sine and cosine:

Definition 9.2. $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ *and* $\cos z = \frac{e^{iz} + e^{-iz}}{2}$.

This has the following properties:

1. $\sin z$ and $\cos z$ are entire, with $(\sin z)' = \cos z$ and $(\cos z)' = -\sin z$.
2. $\sin(-z) = -\sin z$ (odd) and $\cos(-z) = \cos z$ (even).
3. The double-angle and angle sum formulas still hold.
4. $\sin^2 z + \cos^2 z = 1$ (from HW 5).
5. They split up into real and imaginary parts as follows:

$$\sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy), \tag{9.3}$$

and $\cos(iy) = \frac{e^y + e{-y}}{2} = \cosh y$, and similarly $\sin(iy) = i \sinh y$. Therefore we get

$$\sin z = \sin x \cosh y + i \cos x \sinh y \tag{9.4}$$

and in the same way

$$\cos z = \cos x \cosh y - i \sin x \sinh y \tag{9.5}$$

6. $|\sin z|^2 = \sin^2 x + \sinh^2 y$ and $|\cos z|^2 = \cos^2 x + \sinh^2 y$

Note that $\sinh y$ blows up as $y \to \infty$, and so $\sin z, \cos z$ are not bounded in $\mathbb{C}$.

9.2 Zeros and singularities of trig functions

Zeros of a function $f(z)$ are values $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.

Theorem 9.1. *The zeros of* $\sin z$ *and* $\cos z$ *in* $\mathbb{C}$ *are the same as* $\sin x$ *and* $\cos x$ *in* $\mathbb{R}$.

We could also state this as: $z_0 = x_0 + iy_0 \in \mathbb{C}$ is a zero of $\sin z$ (resp. cos) iff $x_0 \in \mathbb{R}$ is a zero of $\sin x$ (resp. cos). The backward direction is trivial, but for the forward direction we need to show there are no other zeros. This is easily done by taking $\sin z = 0 \implies |\sin z|^2 = \sin^2 x + \sinh^2 y = 0$. Therefore $\sin x = 0$ and $\sinh y = 0$, i.e. $\frac{e^y - e{-y}}{2} = 0$ and so $y = 0$.

Definition 9.3. $\tan, \sec, \cot, \csc$ *are defined as ratios of* $\sin, \cos, 1$ *as in the reals.* $\tan, \sec$ *are not defined at* $z = \frac{\pi}{2} + n\pi$ *and* $\cot, \csc$ *are not defined at* $z = n\pi$ *for* $n \in \mathbb{Z}$.

9.3 Hyperbolic functions

We can extend these to the complex plane too!

Definition 9.4. $\sinh z = \frac{e^z - e^{-z}}{2}, \cosh z = \frac{e^z + e^{-z}}{2}$

Properties:

1. $(\sinh z)' = \cosh z, (\cosh z)' = \sinh z$
2. $\cosh^2 z = 1 + \sinh^2 z$
3. $\sinh(iz) = i \sin z, \cosh(iz) = \cos z$.
4. By property 3, the zeros of $\sinh z$ in $\mathbb{C}$ are $z = n\pi i, n \in \mathbb{Z}$ and the zeros of $\cosh z$ in $\mathbb{C}$ are $z = \left(\frac{\pi}{2} + n\pi\right), n \in \mathbb{Z}$.

9.4 Conformal mappings

Definition 9.5. *A function or mapping is conformal if it preserves angles locally.*

More precisely,

Definition 9.6. *An analytic complex-valued function is conformal if whenever* r_1, r_2 *are two smooth curves passing through* z_0 *by nonzero tangents, then the curves* $f \circ r_1$ *and* $f \circ r_2$ *have nonzero tangents at* $f(z_0)$ *and the angle from* $r_1'(0)$ *to* $r_2'(0)$ *is the same as the angle from* $(f \circ r_1)'(0)$ *to* $(f \circ r_2)'(0)$.

Definition 9.7. *A conformal mapping* $f : D \to V$ *is a bijective analytic function that is conformal at each point of* D.

Remark 9.2. *We say* D *and* V *are conformally equivalent.*

Checking the derivative sounds like a lot, so let's come up with an algebraic criterion for checking whether a function is conformal.

Theorem 9.3. f *is conformal in* D *if* f *is analytic in* D *and* $f' \neq 0$ *in* D.

Example 9.1. $f(z) = e^z$ implies $f'(z) = e^z \neq 0$ for all $z \in \mathbb{C}$

□

Example 9.2. If we have $f(z) = \frac{az+b}{cz+d}$ for $z \neq -\frac{d}{c}$, we can take a derivative:

$$f'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0. \tag{9.6}$$

Therefore f is conformal in $\mathbb{C} \setminus \{-d/c\}$ if $ad - bc \neq 0$.

□

Example 9.3. $f(z) = \bar{z}$ is not conformal, as only the magnitude of the angle is preserved, not necessarily the sense. This is an example of an *isogonal mapping*: the angle is preserved, but the orientation isn't always preserved.

□

If we have $f(z)$ conformal, then $f(\bar{z})$ is an isogonal mapping.

Example 9.4. $f(z) = 1 + z^2 \implies f'(z) = 2z \neq 0$, so this is conformal on $\mathbb{C} \setminus \{0\}$.

□

If we look at this mapping at 0, we find that the angle doubles. In general, if z_0 is a critical point of $w = f(z)$ there is an integer $m \geq 2$ such that the angle between two smooth curves passing through z_0 is multiplied by m by going through f. This integer m will be the smallest integer such that $f^{(m)}(z_0) \neq 0$.

If $f(z)$ is conformal at z_0, there exists a local inverse there. That is, $w_0 = f(z_0)$ and f conformal at z_0 implies there exists a unique transform $z = g(w)$ defined and analytic in a neighborhood, denoted as N, such that $g(w_0) = z_0$ and $f[g(w)] = w$ for all $w \in N$. Further, $g'(w) = \frac{1}{f'(z)}$.

We can show all this through the multivalued implicit function theorem: if $f = u + iv$ then $f' = u_x + iv_x$ and the Jacobian is $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$. The Jacobian is nonzero, and so(?) by the IFT there is a continuous and differentiable inverse.

The derivatives of the local inverse are

$$x_u = \frac{1}{J} v_y, y_u = -\frac{1}{J} v_x \tag{9.7}$$

$$x_v = -\frac{1}{J} u_y, y_v = \frac{1}{J} u_x. \tag{9.8}$$

Math 185: Complex Analysis		**Spring 2021**
	Lecture 10: Integrals	
Lecturer: Di Fang	*18 February*	*Aditya Sengupta*

A brand new journey! Today, we'll discuss definite integrals and contours; in the near future, we'll start talking about contour integrals.

10.1 Setup for integration

Consider a parameterized curve $\omega(t) : [a, b] \to \mathbb{C}$ where $\omega(t) = u(t) + iv(t)$.

Definition 10.1.

$$\omega'(t) = u'(t) + iv'(t) \tag{10.1}$$

This is a derivative in the real sense, not the complex sense.

Many, but not all, rules carry over from $\mathbb{R}$. For example, the chain rule.

Theorem 10.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and let u, v be differentiable at a point $t \in \mathbb{R}$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\omega(t)) = f'(\omega(t))\omega'(t) \tag{10.2}$$

Proof. Let $f(x + iy) \triangleq g(x, y) + ih(x, y)$ where $g, h : \mathbb{R}^2 \to \mathbb{R}$. Then by C-R, g and f are differentiable.

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\omega(t)) = g'(u(t), v(t)) + ih'(u(t), v(t)) \tag{10.3}$$

$$\underbrace{=}_{\text{multivariate chain rule}} g_x(u, v)v' + g_y(u, v)v' + ih_x(u, v)u' + ih_y(u, v)v' \tag{10.4}$$

$$\underbrace{=}_{\text{CR}} g_x v' - h_x v' + ih_x u' + ig_x v' \tag{10.5}$$

$$= (u' + iv')(g_x + ih_x) \tag{10.6}$$

$$= \omega'(t) \cdot f'(\omega(t)). \tag{10.7}$$

□

However, the mean value theorem no longer holds. For example, let $\omega(t) = e^{it}$. $\omega(0) = \omega(2\pi) = 1$ so the MVT should say there exists some $t \in [0, 2\pi]$ such that the slope of the tangent line is $\frac{\omega(2\pi) - \omega(0)}{2\pi - 0} = 1$, but in fact $\omega'(t) = ie^{it} \neq 0 \ \forall t \in \mathbb{R}$.

10.2 Definite Integrals

We integrate (in the R-to-C sense) just by integrating components.

Definition 10.2.

$$\int_a^b \omega(t)\mathrm{d}t \triangleq \int_a^b u(t)\mathrm{d}t + i\int_a^b v(t)\mathrm{d}t \tag{10.8}$$

Key properties:

1. Real and imaginary parts carry through integrals:
$$\operatorname{Re}\int_a^b \omega(t)\mathrm{d}t = \int_a^b \operatorname{Re}\omega(t)\mathrm{d}t \tag{10.9}$$
$$\operatorname{Im}\int_a^b \omega(t)\mathrm{d}t = \int_a^b \operatorname{Im}\omega(t)\mathrm{d}t \tag{10.10}$$

2. FTC still holds: let the integral of $\omega(t) = u(t) + iv(t)$ be $W(t) = U(t) + iV(t)$. Then
$$\int_a^b \omega(t)\mathrm{d}t = U(t)|_a^b + iV(t)|_a^b = W(t)|_a^b\,. \tag{10.11}$$

3. $\left|\int_a^b \omega(t)\mathrm{d}t\right| \le \int_a^b |\omega(t)|\mathrm{d}t$. The proof isn't hard but has an interesting idea.

Proof. If $\int_a^b \omega(t)\mathrm{d}t = 0$ this is trivially true because the integral of the modulus must be at least 0. Otherwise, let $\int_a^b \omega(t)\mathrm{d}t = re^{i\theta}$ for r, θ constant. Then

$$\left|\int_a^b \omega(t)\mathrm{d}t\right| = r = \int_a^b e^{-i\theta}\omega(t)\mathrm{d}t = \operatorname{Re}\int_a^b e^{-i\theta}\omega(t)\mathrm{d}t \tag{10.12}$$
$$= \int_a^b \operatorname{Re}(e^{-i\theta}\omega(t))\mathrm{d}t \tag{10.13}$$
$$\le \int_a^b \left|e^{-i\theta}\omega(t)\right|\mathrm{d}t \tag{10.14}$$
$$= \int_a^b |\omega(t)|\mathrm{d}t. \tag{10.15}$$

□

Example 10.1. Consider $\int_0^{\pi/4} e^{it}\mathrm{d}t$. We can compute this either from the definition,

$$\int_0^{\pi/4} \cos t + i\sin t\mathrm{d}t = \sin t|_0^{\pi/4} - i\cos t|_0^{\pi/4} = \frac{\sqrt{2}}{2} - i\left(\frac{\sqrt{2}}{2} - 1\right), \tag{10.16}$$

or from FTC,

$$\frac{\mathrm{d}-ie^{it}}{\mathrm{d}t} = e^{it} \implies \int_0^{\pi/4} e^{it}\mathrm{d}t = -ie^{it}\Big|_0^{\pi/4} = \frac{\sqrt{2}}{2} - i\left(\frac{\sqrt{2}}{2} - 1\right). \tag{10.17}$$

□

Example 10.2. The MVT for integrals ($\int_a^b \omega(t)\mathrm{d}t = \omega(c)(b-a)$ for some $c \in [a,b]$) fails, as $\int_0^{2\pi} e^{it}\mathrm{d}t = 0$ but $e^{it}(2\pi - 0) \neq 0\ \forall t$.

□

10.3 Contour integrals

Contours let us define a C-to-C sense integral; in $\mathbb{C}$, we want to compute integrals along certain curves.

Let $x(t)$ and $y(t)$ be the real and imaginary curve components, i.e. functions $[a,b] \subseteq \mathbb{R} \to \mathbb{R}$ such that the curve can be defined as

$$C : z(t) = x(t) + iy(t), t \in [a,b]. \tag{10.18}$$

Definition 10.3. *C is an arc if x, y are continuous.*

Definition 10.4. *An arc C is a simple arc if it does not cross itself.*

Definition 10.5. *An arc C is a simple closed arc if it is simple except at the endpoints, i.e. $z(a) = z(b)$.*

Definition 10.6. *Closed curves that are counterclockwise (have the enclosed region to their left) are called positively oriented, and those that are clockwise are called negatively oriented.*

Definition 10.7. *If x, y are differentiable on $[a,b]$ and x', y' are continuous on $[a,b]$, we call C a differentiable arc.*

Definition 10.8. *We call C a smooth arc if it is a differentiable arc and $z'(t) \neq 0$ on (a,b).*

Useful math objects are the unit tangent vector, $T = \frac{z'(t)}{|z'(t)|}$, and the arclength, $L = \int_a^b |z'(t)|\mathrm{d}t$.

Definition 10.9. *A contour is a piecewise smooth arc, consisting of a finite number of smooth arcs joined end to end.*

Definition 10.10. *A simple closed contour is a contour made of simple closed arcs.*

The Jordan curve theorem says something completely obvious, the proof of which is a bit out of our scope:

Theorem 10.2. *The points on any Jordan curve (simple closed curve in the plane) C are the boundary points of two distinct domains. One of these is the interior of C and is bounded, and the other is the exterior of C and unbounded.*

Example 10.3. Let $z(t) = \begin{cases} t + it & t \in [0,1] \\ t + i & t \in [1,2] \end{cases}$. This is an arc, but it is not differentiable.

□

Example 10.4. $z = e^{i\theta}, \theta \in [0, 2\pi]$, $z = e^{-i\theta}, \theta \in [0, 2\pi]$ and $z = e^{i2\theta}, \theta \in [0, \pi]$ are all different parameterizations of the same curve. The second one is negatively oriented, and the first and third are positively oriented. This shows the parameterization of a curve is not unique.

□

Example 10.5. Consider the curve $z(t) = t^3 + i, t \in [-1, 1]$. This is a differentiable arc, but for smoothness we require that the derivative is nowhere zero; here, $z'(t) = 3t^2$ and therefore $z'(0) = 0$. This is a contour.

□

Math 185: Complex Analysis **Spring 2021**

Lecture 11: Contour integrals

Lecturer: Di Fang *2 March* *Aditya Sengupta*

This is an exciting new thing that's specific to complex analysis.

Consider $f : \mathbb{C} \to C$, let C be a contour parameterized by $z(t) = x(t) + iy(t), t \in [a, b]$. Let f be piecewise continuous along C.

Definition 11.1. *The contour integral of f along C is*

$$\int_C f(z)\mathrm{d}z \triangleq \int_a^b f(z(t))z'(t)\mathrm{d}t. \tag{11.1}$$

Splitting this up into components, we get

$$\int_C f(z)\mathrm{d}z = \int_a^b (u + iv)(x' + iy')\mathrm{d}t \tag{11.2}$$

$$= \int_C (u\mathrm{d}x - v\mathrm{d}y) + i \int_C (v\mathrm{d}x + u\mathrm{d}y) \tag{11.3}$$

$$= \int_C (u + iv)(\mathrm{d}x + i\mathrm{d}y). \tag{11.4}$$

Note that since C is a contour and z' is piecewise continuous, the integrand $f(z(t))z'(t)$ is piecewise continuous on $[a, b]$.

We can define a sort of algebra of contours:

1. if $C : z(t), t \in [a, b]$ is a contour, $-C : z(-t), t \in [-b, -a]$ is its negative, and $\int_{-C} f(z)\mathrm{d}z = -\int_C f(z)\mathrm{d}z$
2. We can add contours: if C_1, C_2 are contours, then $C_1 + C_2$ is the combined path traced out by both of them. $\int_{C_1+C_2} f(z)\mathrm{d}z = int_{C_1} f(z)\mathrm{d}z + int_{C_2} f(z)\mathrm{d}z$
3. A contour integral over C is independent of the parameterization of C.

We can make this last property more precise.

Theorem 11.1. *Let C be a contour of finite length L. Then for all $M \geq 0$ such that $|f(z)\mathrm{d}z| \leq M\ \forall z \in C$, we can say that $\left|\int_C f(z)\mathrm{d}z\right| \leq ML$.*

Example 11.1. Let C be the upper semicircle $|z| = 2, 0 \leq \theta < \pi$, and let $f(z) = \frac{z-2}{z^4+1}$. This is a complicated integral, but we can upper-bound it:

$$\left|\int_C \frac{z-2}{z^4+1}\mathrm{d}z\right| \leq 2\pi M \tag{11.5}$$

where we can find an M that works using the triangle inequality:

$$\left|\frac{z-2}{z^4+1}\right| \leq \frac{|z|+2}{||z^4|-1|} = \frac{4}{15}, \tag{11.6}$$

i.e. the integral is upper-bounded by $\frac{8\pi}{15}$.

□

Proof. Start with the absolute value of the integral,

$$\left|\int_C f(z)\mathrm{d}z\right| \underbrace{=}_{def} \left|\int_a^b f(z(t))z'(t)\mathrm{d}t\right| \leq \int_a^b |f(z(t))|\,|z'(t)|\,\mathrm{d}t \tag{11.7}$$

$$\leq M\int_a^b |z'(t)|\mathrm{d}t = M \cdot L. \tag{11.8}$$

□

Example 11.2. Consider $\int_C \frac{\mathrm{d}z}{z}$ over $z = e^{i\theta}, \theta \in [0, \pi]$. The definition gives us

$$\int_C \frac{\mathrm{d}z}{z} = \int_0^\pi e^{-i\theta} i e^{i\theta} \mathrm{d}\theta = i\pi. \tag{11.9}$$

Note that if we go over the $[\pi, 2\pi]$ range, we get another $i\pi$ making a total of $2i\pi$. This is despite these two together forming a closed curve.

□

Remark 11.2. *The contour integral depends on the contour, not just the end points.*

Example 11.3. Consider the integral

$$\int_C \bar{z}\mathrm{d}z = -\int_{-C} \bar{z}\mathrm{d}z \tag{11.10}$$

on a contour $-C : z = e^{i\theta}, \theta \in [0, 2\pi]$. This is

$$\int_C \bar{z}\mathrm{d}z = -\int_0^{2\pi} e^{-i\theta} ie^{i\theta}\mathrm{d}\theta = -2\pi i. \tag{11.11}$$

□

Example 11.4. Let $z = x + iy$ and $f = (y - x) - i3x^2$. Consider the contour going from O (0) to A (i) to B($1 + i$), and directly O to B, in line segments.

Along OA: $\int_{OA} f\mathrm{d}z = \int_0^1 yi\mathrm{d}y = \frac{i}{2}$.

Along AB: $\int_{AB} \int f\mathrm{d}z = \int_0^1 (1 - x) - i3x^2\mathrm{d}x = \frac{1}{2} - i$.

Along OB: Di and I are both not going to show it, but $\int_{OB} f\mathrm{d}z = -i$.

Note that $\int_{OA+AB} \neq \int_{OB}$.

□

11.1 Contour integrals involving branches

Consider $f(z) = z^{1/2}$. Since this really depends on log, because $z^{1/2} = e^{\frac{1}{2}\log z}$ and log is multivalued, we have to pick a branch cut.

Example 11.5. Let $C : z = 3e^{i\theta}, \theta \in (0, \pi)$. The integral of $z^{1/2}$ is most easily done in polar coordinates, which requires taking a branch cut: $|z| > 0, 0 < \arg z < 2\pi$.

$$I = \int_0^\pi \sqrt{3}e^{\frac{i\theta}{2}} 3re^{i\theta}\mathrm{d}\theta \tag{11.12}$$

$$= 3\sqrt{3}\int_0^\pi e^{\frac{3i\theta}{2}}\mathrm{d}\theta = 3\sqrt{3} \cdot \frac{2}{3}\, e^{3i\theta}2\Big|_0^\pi = 2\sqrt{3}(-i - 1). \tag{11.13}$$

Now, if we change the branch cut to $|z| > 0, \frac{3\pi}{2} < \operatorname{Arg} z < \frac{7\pi}{2}$, the old parameterization doesn't work: instead, we need to take $\theta \in (2\pi, 3\pi)$.

□

Example 11.6. Consider $f(z) = z^{-1+i}$ where C is the positively oriented unit circle, on the branch $|z| > 0, -\pi < \operatorname{Arg} z < \pi$.

$$\int_C f(z)\mathrm{d}z = \int_{-\pi}^{\pi} e^{i\theta-\theta} ie^{i\theta}\mathrm{d}\theta \tag{11.14}$$

$$= \int_{-\pi}^{\pi} ie^{-\theta}\mathrm{d}\theta \tag{11.15}$$

$$= -i(e^{-\pi} - e^{\pi}). \tag{11.16}$$

□

Example 11.7. Let C be the positively-oriented upper half circle with radius 3, i.e. $3e^{i\theta}, 0 \leq \theta \leq \pi$. We can see that $L = 3\pi$. We want to estimate $\int_C \frac{z^{1/2}}{z^2+1}\mathrm{d}z$. We upper-bound the integrand,

$$\left|\frac{z^{1/2}}{z^2+1}\right| = \left|\frac{e^{\frac{1}{2}\operatorname{Log} z}}{z^2+1}\right| \tag{11.17}$$

$$= \frac{|e^{\frac{1}{2}(\ln r + i\theta)}|}{z^2+1} \tag{11.18}$$

$$\leq \frac{\sqrt{r}}{||z|^2-1|} = \frac{\sqrt{3}}{8} \tag{11.19}$$

and therefore we can say

$$\left|\int_C \frac{z^{1/2}}{z^2+1}\mathrm{d}z\right| \leq \frac{3\sqrt{3}}{8}\pi \tag{11.20}$$

□

Example 11.8. Let C be the closed path $(1) \to (1+i) \to (i) \to 1$ in line segments.

$$I = \int_C e^{i\bar{z}} + \bar{z}\mathrm{d}z \tag{11.21}$$

The length of this path is $L = 2 + \sqrt{2}$, so we look for an upper bound on the integrand:

$$|e^{i\bar{z}} + \bar{z}| \leq |e^{i\bar{z}} + |\bar{z}| \tag{11.22}$$
$$= e^{y} + |z|, \tag{11.23}$$

and the first term is maximized with $y = 1$ and the second is maximized with $z = 1 + i$, so

$$|e^{i\bar{z}} + \bar{z}| \leq e + \sqrt{2}. \tag{11.24}$$

Therefore $|I| \leq (2 + \sqrt{2})(e + \sqrt{2})$.

□

Math 185: Complex Analysis **Spring 2021**

Lecture 12: Antiderivatives, fundamental theorem of contour integrals

Lecturer: Di Fang *4 March* *Aditya Sengupta*

Warmup: let C be the positively oriented unit circle, and let $f(z) = \bar{z}$.

$$\int_C f(z)\mathrm{d}z = \int_0^{2\pi} ie^{i\theta}e^{-i\theta}\mathrm{d}\theta = 2i\pi \tag{12.1}$$

Now on to anti-derivatives. We covered how in general, contour integrals are path-dependent, and we might remember from multivariable calculus that some vector fields are conservative or path-independent, meaning they could be expressed in terms of some scalar potential. Let's try and define the analogous concept on the complex plane.

Let $f : D \to \mathbb{C}$ be continuous.

Definition 12.1. *F on D is an antiderivative of f on D if $F' = f$ on D.*

F is analytic on D (we have to be able to differentiate it), and anti-derivatives differ up to a constant on D.

Proof. Let $F_1' = F_2' = f$. Then $(F_1 - F_2) = 0$ on D, so $F_1 - F_2 = C$. □

Theorem 12.1 (Fundamental theorem of contour integrals)**.** *Let f be continuous on D. Then the following statements are equivalent:*

1. *$f(z)$ has an antiderivative $F(z)$ throughout D.*
2. *Integrals of $f(z)$ along contours lying entirely in D, extending between fixed points $z_1, z_2 \in D$, have the same value. That is, for all $z_1, z_2 \in D$ and for all contours C_1, C_2 in D lying between them, $\int_{C_1} f(z)\mathrm{d}z = \int_{C_2} f(z)\mathrm{d}z \triangleq \int_{z_1}^{z_2} f(z)\mathrm{d}z = F(z_2) - F(z_1)$.*
3. *$\oint_C f(z)\mathrm{d}z = 0$ for all closed contours C in D.*

Example 12.1. Let $f(z) = \bar{z}$. This does not have an antiderivative as it does not integrate to 0 over a closed contour (unit circle). □

Example 12.2. Let $f(z) = e^{\pi z}$. We can see that if you take a derivative of $\frac{1}{\pi}e^{\pi z}$, you get f back. Therefore, this is path-independent. □

Example 12.3. $f(z) = \frac{1}{z^2}$ is continuous on $\mathbb{C} \setminus \{0\}$, and it has an antiderivative $F(z) = -\frac{1}{z}$ in $\mathbb{C} \setminus \{0\}$.

Therefore, $\int_C \frac{1}{z^2} \mathrm{d}z = 0$ where C is the positively oriented unit circle.

□

Example 12.4. But *now*, and this is the classic Math 53 example, let's try the same thing for $\frac{1}{z}$.

$$\int_C \frac{1}{z} \mathrm{d}z = \int_0^{2\pi} e^{-i\theta} i e^{i\theta} \mathrm{d}\theta = 2\pi i \neq 0. \tag{12.2}$$

Why doesn't this act the same way as $\frac{1}{z^2}$? This is because $\frac{1}{z}$ has no antiderivative on $\mathbb{C} \setminus \{0\}$; $\log z$ would work but it isn't differentiable unless we take a branch cut.

□

Example 12.5. Let $f(z) = \frac{1}{z}$ and let C be some closed curve that excludes the origin. Then we can pick a branch cut that does not intersect the curve, so the conditions of the FTCI are satisfied and so the integral is 0.

□

Proof. We show each statement implies the next one.

First, we show 1 implies 2. We start with the statement that $F' = f$ on D, and we claim that $\int_C f(z) \mathrm{d}z = F(z_2) - F(z_1)$. We know that C is made up of a finite number of smooth arcs, i.e. $C = C_1 + \cdots + C_n$ where $C_i : z^i \to z^{i+1}$, $z^1 = z_1$ and $z^{n+1} = z_2$.

Each C_k is smooth with parameterization $z = z_k(t), t \in [a, b]$. We can apply the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t} F(z_k(t)) = F'(z_k(t)) z'(t) = f(z_k(t)) z'(t), \tag{12.3}$$

and therefore

$$\int_{C_k} f \mathrm{d}z = \int_a^b f(z_k(t)) z_k'(t) \mathrm{d}t \tag{12.4}$$

$$= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F(z_k(t)) \mathrm{d}t \tag{12.5}$$

$$= F(z_k(b)) - F(z_k(a)) \tag{12.6}$$

$$= F(z^{k+1}) - F(z^k). \tag{12.7}$$

To show the claim, we add all of these smooth arc components together:

$$\int_C f\mathrm{d}z = \sum_{k=1}^{n} f\mathrm{d}z = F(z^{n+1}) - F(z_1) = F(z_2) - F(z_1). \tag{12.8}$$

Therefore 1 implies 2.

Next, we show that 2 implies 3. Consider an arc C_1 going $z_1 \to z_2$ and another arc C_2 going $z_2 \to z_1$. By path-independence,

$$\int_{C_1} f\mathrm{d}z = \int_{-C_2} f\mathrm{d}z = -\int_{C_2} f\mathrm{d}z, \tag{12.9}$$

and so

$$\int_C f\mathrm{d}z = \int_{C_1} f\mathrm{d}z + \int_{C_2} f\mathrm{d}z = -\int_{C_2} f\mathrm{d}z + \int_{C_2} f\mathrm{d}z = 0. \tag{12.10}$$

Finally, we show that 3 implies 1. To do this, we'll need to show 3 implies 2, but this is easy: for any two points z_1, z_2, make a contour going $z_1 \to z_2 \to z_1$ and split it at z_2. The two components must have equal and opposite contributions because the closed loop has a contour integral of 0. Therefore, flipping the second component, the two must be equal.

Now, we show 3 and 2 imply 1.

Let $z_0 \in D$ and consider any $z \in D$. Let C be a contour $z_0 \to z$, and path independence holds by 2. Define

$$F(z) = \int_{z_0}^{z} f(s)\mathrm{d}s, \tag{12.11}$$

and we want to show that $F'(z) = f(z)\ \forall z \in D$. We proceed by the definition:

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = f(z)\ \forall z \in D. \tag{12.12}$$

This quantity is

$$\Delta w = F(z + \Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)\mathrm{d}s - \int_{z_0}^{z} f(s)\mathrm{d}s \tag{12.13}$$

and so

$$\frac{\Delta w}{\Delta z} = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(s) - f(z)\mathrm{d}s \xrightarrow{\text{WTS}} 0 \tag{12.14}$$

Because f is continuous at z, we can use the definition of continuity:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t.} |s - z| < \delta \implies |f(s) - f(z)| < \epsilon. \tag{12.15}$$

Fix $|\Delta z| < \delta$. This implies $|s - z| < \delta$, and by continuity $|f(s) - f(z)| < \epsilon$. To extend this to the integral, we use the $M \cdot L$ property, choosing the straight line $z \to z + \Delta z$.

$$\left| \frac{1}{\Delta z} \int_z^{z+\Delta z} f(s) - f(z) \mathrm{d}s \right| < \frac{\epsilon}{|\Delta z|} \int_z^{z+\Delta z} |\mathrm{d}s| = \frac{\epsilon}{|\Delta z|} |\Delta z| = \epsilon, \tag{12.16}$$

and so the limit definition is satisfied. □

Example 12.6. Consider $\int_{C_1} \frac{1}{z} \mathrm{d}z$ on an arc from $-i$ to i along the unit circle. Taking a branch cut that does not intersect (e.g. $r > 0, -\pi < \operatorname{Arg} z < \pi$) we can use the fact that the antiderivative is $\operatorname{Log} z$. Evaluating at $\pm i$ and taking the difference, we get that $\int_{C_1} \frac{1}{z} \mathrm{d}z = \operatorname{Log} i - \operatorname{Log}(-i) = \pi i$.

□

Example 12.7. The same example, but now on the left semicircle. The branch is now $r > 0, 0 < \theta < 2\pi$. We get the same result by now taking $\log(-i) - \log i = \frac{3\pi}{2} i - \frac{\pi}{2} i = \pi i$.

□

Therefore $\int_{C_1 + C_2} \frac{1}{z} \mathrm{d}z = 2\pi i$, the same result as with the parameterization definition.

Example 12.8. Consider $\int_C f(z) \mathrm{d}z$ where $f(z) = z^{\frac{1}{2}} = \sqrt{r} e^{i\theta/2}$ on $r > 0, 0 < \theta < 2\pi$ and where $C = 3e^{i\theta}, 0 < \theta < \pi$. This branch cut has a point that doesn't work at $\theta = 0$. We claim that this is equal to the integral of $g(z) = z^{\frac{1}{2}} = \sqrt{r} e^{i\theta/2}$ on $r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

But aren't these different functions? In fact, f and g share the same values on the contour, because the contour lies in their intersection. We can then easily integrate g.

□

Math 185: Complex Analysis **Spring 2021**

Lecture 13: Cauchy-Goursat Theorem

Lecturer: Di Fang *9 March* *Aditya Sengupta*

We just learned what the fundamental theorem of contour integrals, that having a complex antiderivative is equivalent to path-independence, which is also equivalent to all integrals on closed contours being zero.

Let $\oint_C f(z)\mathrm{d}z = 0$. We know this is true whenever f has an antiderivative in D, where C lies in D. However, how would we do the reverse - if you're given a function, how do you know if it has an antiderivative? We want to find a condition that's easier to check.

Theorem 13.1 (Cauchy-Goursat Theorem v0)**.** *If f is analytic and f' is continuous at all points interior to and on a simple closed contour C, then $\oint_C f(z)\mathrm{d}z = 0$.*

Proof. Let $f = u + iv$.

$$\oint_C f(z)\mathrm{d}z = \int_C (u + iv)(\mathrm{d}x + i\mathrm{d}y) \tag{13.1}$$

$$= \int_C u\mathrm{d}x - v\mathrm{d}y + i\int v\mathrm{d}x + u\mathrm{d}y. \tag{13.2}$$

This looks a lot like Green's theorem: $\int_C P\mathrm{d}x + Q\mathrm{d}y = \iint_D Q_x - P_y\mathrm{d}x\mathrm{d}y$ for $P, Q \in C^1(D)$ and $\partial D = C$.

$$\oint_C f(z)\mathrm{d}z = \iint_D -v_x - u_y\mathrm{d}x\mathrm{d}y + i\iint_D u_x - v_y\mathrm{d}x\mathrm{d}y, \tag{13.3}$$

and by the C-R conditions (which hold because f is analytic), both of these are 0. Therefore $\oint_C f(z)\mathrm{d}z = 0$. □

Can we do even better? We can actually drop continuity of the derivative. The *true* version of the Cauchy-Goursat theorem does this:

Theorem 13.2 (Cauchy-Goursat Theorem v1)**.** *If f is analytic at all points interior to and on a simple closed contour C, then $\oint_C f(z)\mathrm{d}z = 0$.*

This is proved in the textbook, but we'll just take it on faith.

Let's prove instead that the Cauchy-Goursat theorem holds for a triangle, then generalize to other contours.

Proof. Let our triangle be $T^{(0)}$. We want to show that $\int_{\partial T^{(0)}} f(z)\mathrm{d}z = 0$.

We can subdivide the triangle into four smaller triangles, $T_i^{(1)}$ for $i = 1, 2, 3, 4$.

The contour integral can be split into a sum of contour integrals over these four:

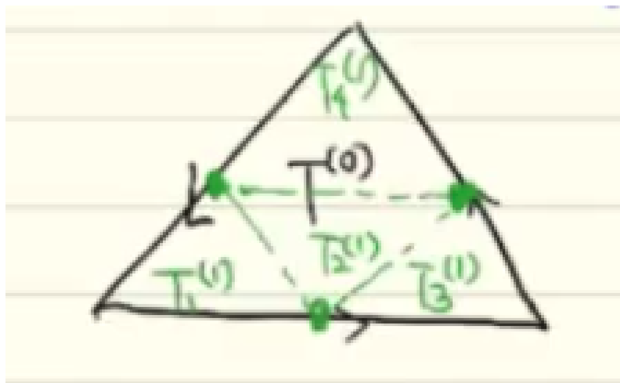


Figure 13.3: Triangle decomposition

$$\int_{\partial T^{(0)}} f(z)\mathrm{d}z = \sum_{i=1}^{4} \int_{\partial T_i^{(1)}} f(z)\mathrm{d}z \tag{13.4}$$

Pick the term with the maximal absolute value: there exists some $T_j^{(1)} \triangleq T^{(1)}$ such that

$$\left| \int_{\partial T^{(0)}} f(z)\mathrm{d}z \right| \leq 4 \left| \int_{\partial T^{(1)}} f(z)\mathrm{d}z \right| \tag{13.5}$$

Repeating this procedure, we get that

$$\left| \int_{\partial T^{(1)}} f(z)\mathrm{d}z \right| \leq 4 \left| \int_{\partial T^{(2)}} f(z)\mathrm{d}z \right| \tag{13.6}$$

and if we take it out infinitely many times, we get a sequence of triangles, $T^{(0)} \supset T^{(1)} \supset T^{(2)} \supset \cdots \supset T^{(n)} \supset \ldots$, such that

$$\left| \int_{\partial T^{(0)}} f(z)\mathrm{d}z \right| \leq 4^n \left| \int_{\partial T^{(n)}} f(z)\mathrm{d}z \right| \tag{13.7}$$

For the second part of this, we'll use what's called a *compactness argument.* $\cap_n T^{(n)} \neq \varnothing$: if you intersect all the triangles, you're guaranteed that at least one point is common to all of them. Therefore, let $z^* \in \cap_n T^{(n)}$. Since $f(z)$ is analytic at z^*, we can say that

$$\lim_{z \to z^*} \frac{f(z) - f(z^*)}{z - z^*} = f'(z^*). \tag{13.8}$$

Expanding this out using the limit definition, we can say

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |z - z^*| < \delta \implies \left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \epsilon \tag{13.9}$$

which in turn implies $|f(z) - f(z^*) - f'(z^*)(z - z^*)| < \epsilon |z - z^*|$.

Changing this to a condition on the number of triangle splittings n:

$$\forall \epsilon > 0, \exists N \in \mathbb{R} \,\text{s.t.}\, n \geq N \implies \left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \epsilon \;\forall z \in T^{(n)} \tag{13.10}$$

Therefore, we can say

$$\left| \int_{\partial T^{(n)}} f(z) \mathrm{d}z \right| = \left| \int_{\partial T^{(n)}} f(z) - f(z^*) - f'(z^*)(z - z^*) \mathrm{d}z \right| \tag{13.11}$$

This is the case because $\int_{\partial T^{(n)}} f(z^*) \mathrm{d}z = 0$ and $\int_{\partial T^{(n)}} f(z^*)(z - z^*) \mathrm{d}z = 0$, as a result of the fundamental theorem of contour integrals and as a result of both integrands having antiderivatives.

Therefore, we can use the M-L estimate to bound the right hand side. Say $T^{(0)}$ has diameter d and perimeter P. Then $T^{(n)}$ will have diameter $d_n = \frac{d}{2^n}$ and perimeter $P_n = \frac{P}{2^n}$. Therefore our L can be $\frac{P}{2^n}$. For M, we use the upper bound of $\epsilon_n |z - z^*| \leq \epsilon_n \frac{d}{2^n}$, where we can say

$$\epsilon_n \triangleq \sup_{z \in T^{(n)}} \frac{|f(z) - f(z^*) - f'(z^*)(z - z^*)}{|z - z^*|} \xrightarrow{n \to \infty} 0 \tag{13.12}$$

Therefore, applying the M-L estimate, we can finally say

$$\left| \int_{\partial T^{(0)}} f(z) \mathrm{d}z \right| \leq 4^n \left| \int_{\partial T^{(n)}} f(z) \mathrm{d}z \right| \leq 4^n \epsilon_n \frac{dP}{2^n 2^n} = \epsilon_n dP \xrightarrow{n \to \infty} 0. \tag{13.13}$$

Therefore, $0 \leq \left| \int_{\partial T^{(0)}} f(z) \mathrm{d}z \right| \leq 0$, so $\left| \int_{\partial T^{(0)}} f(z) \mathrm{d}z \right| = 0$, so $\int_{\partial T^{(0)}} f(z) \mathrm{d}z = 0$ as desired. □

Corollary 13.3. *The Cauchy-Goursat theorem applies for all polygons.*

Proof. We can decompose any polygon into many triangles, and apply the Cauchy-Goursat theorem for triangles on each one. □

The most general case of the Cauchy-Goursat theorem considers the case of any simple closed contour C enclosing a domain D, such that $\partial D = C$. Tile the domain D with squares: the contour will go over several partial squares, so if we have an upper-bound estimate for d and P, the behaviour of $\epsilon_n \to 0$ will still dominate and the theorem will still hold.

Math 185: Complex Analysis **Spring 2021**

Lecture 14: Cauchy-Goursat Theorem contd., Cauchy integral formula

Lecturer: Di Fang *11 March* *Aditya Sengupta*

14.1 Cauchy-Goursat Theorem

Definition 14.1. *A simply connected domain is a domain such that every simple closed contour within it encloses only points of D.*

Definition 14.2. *A multiply connected domain is a domain that is not simply connected.*

Theorem 14.1 (Cauchy-Goursat Theorem v2)**.** *Let D be a simply connected domain, and let f be analytic in D. Then $\oint_C f(z)\mathrm{d}z = 0$ for every closed contour C lying in D.*

Proof. If C is simple, this is the same as the Cauchy-Goursat theorem v1.

If C is not simple, decompose it into a sum of simple contours and apply Cauchy-Goursat v1 to each. □

Corollary 14.2. *If f is analytic throughout a simply connected domain D, then f must have an antiderivative in D.*

Corollary 14.3. *Entire functions always possess antiderivatives.*

Theorem 14.4 (Cauchy-Goursat Theorem v3)**.** *Suppose that C is a positively oriented simple closed contour, and that $C_k (k = 1, \ldots, n)$ are negatively-oriented simple closed contours interior to C that are disjoint and whose interiors have no common points.*

If f is analytic on all the contours, and on the multiply connected domain consisting of points inside C and exterior to each C_k, then

$$\int_C f(z)\mathrm{d}z + \sum_{k=1}^{n} \int_{C_k} f(z)\mathrm{d}z = 0. \tag{14.1}$$

Proof. Draw "bridges" to each of the C_is from and to C, and apply Cauchy-Goursat v1.

□

Corollary 14.5 (rinciple of path deformation)**.** *Let C_1, C_2 be positively-oriented simple closed contours, with C_1 interior to C_2. Let R be a closed region consisting of these contours and the points between them. If f is analytic on R then*

$$\int_{C_1} f(z)\mathrm{d}z = \int_{C_2} f(z)\mathrm{d}z. \tag{14.2}$$

Proof. Consider C_2 and $-C_1$ and apply C-G 3. □

Figure 14.4: Cauchy-Goursat for a multiply connected domain

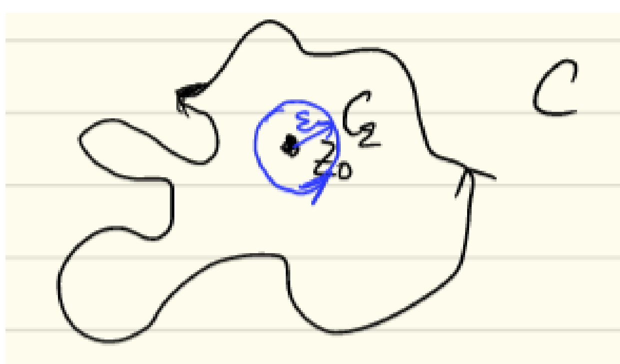


Figure 14.5: A weird squiggle being converted to a nice circle

This lets us skip parameterizations of difficult curves!

Consider the integral $\int_C \frac{\mathrm{d}z}{z-z_0}$ where C is shown in 14.5. We get the same value if we integrate over the circle centered at z_0 instead:

$$\int_C \frac{\mathrm{d}z}{z-z_0} = \int_{C_2} \frac{\mathrm{d}z}{z-z_0} = \int_0^{2\pi} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}} \mathrm{d}\theta = \int_0^{2\pi} i\mathrm{d}\theta = 2\pi i. \tag{14.3}$$

14.2 Cauchy Integral Formula

Theorem 14.6. *Let f be analytic everywhere inside and on a positively-oriented simple closed contour C. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} \mathrm{d}z \tag{14.4}$$

Remark 14.7. *If f is analytic in R (the region enclosed by C), then the values of f interior to C are completely determined only by the values of f on C.*

This is useful because it gives us a third way of calculating contour integrals!

Example 14.1. $\int_C \frac{\mathrm{d}z}{z-z_0} = 2\pi i f(z_0) = 2\pi i$, if we take $f(z) = 1$.

□

Example 14.2. $\int_C \frac{\cos z}{z(z^2+9)} \mathrm{d}z$. Choose $z_0 = 0, f(z) = \frac{\cos z}{z^2+9}$, and $C : |z| = 1$. Then the integral becomes just $2\pi i f(0) = \frac{2\pi i}{9}$.

□

Example 14.3. $\int_{|z|=2} \frac{z^2+z+1}{(z^2-9)(z+i)} \mathrm{d}z$: choose $f(z) = \frac{z^2+z+1}{z^2-9}$ and $z_0 = -i$. The integral is $2\pi i f(-i) = 2\pi i \frac{i}{-10} = \frac{\pi}{5}$.

□

Proof. Let $C_\rho : |z - z_0| = \rho$. By the principle of path deformation, consider the following:

$$\int_{C_\rho} \frac{f(z)}{z-z_0} \mathrm{d}z - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)}{z-z_0} \mathrm{d}z - f(z_0) \int_{C_\rho} \frac{1}{z-z_0} \mathrm{d}z \tag{14.5}$$

$$= \int_{C_\rho} \frac{f(z) - f(z_0)}{z-z_0} \mathrm{d}z. \tag{14.6}$$

Since f is analytic at z_0, f is continous at z_0.

For all $\epsilon > 0$ there exists $\delta > 0$ such that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \epsilon$. Choose $\rho < \delta$ and apply the M-L bound:

$$\left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} \mathrm{d}z \right| \leq \underbrace{2\pi\rho}_{L} \underbrace{\frac{\epsilon}{\rho}}_{M} = 2\pi\epsilon, \tag{14.7}$$

which goes to zero as we let $\epsilon \to 0$. Therefore

$$\int_{C_\rho} \frac{f(z)}{z - z_0} \mathrm{d}z - 2\pi i f(z_0) = 0. \tag{14.8}$$

□

14.3 Cauchy integral formula extensions

Theorem 14.8. *Let f be analytic inside and on a positively-oriented simple closed contour C. If z_0 is any point interior to C, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\mathrm{d}z}{(z - z_0)^{n+1}}, n \in \mathbb{N}. \tag{14.9}$$

Intuitively, this is true by differentiation under the integral sign with respect to z_0.

Example 14.4. If we want to find $\int_C \frac{e^{2z}}{z^4} \mathrm{d}z$, then let $C : |z| = 1$ pos, let $n = 3$, let $f(z) = e^{2z}$ and let $z_0 = 0$. Then

$$\int_C \frac{e^{2z}}{z^4} \mathrm{d}z = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{2\pi i}{3!} 2^3 e^0 = \frac{8\pi i}{3}. \tag{14.10}$$

□

Example 14.5. Consider the integral $\oint_{|z|=2\pi} \frac{z^2 \sin z}{(z-\pi)^3} \mathrm{d}z$. This can be solved by $z_0 = \pi, n = 2, f(z) = z^2 \sin z$.

$$\oint_{|z|=2\pi} \frac{z^2 \sin z}{(z-\pi)^3} \mathrm{d}z = \frac{2\pi i}{2} \left. (z^2 \sin z)'' \right|_{z=\pi} \tag{14.11}$$

$$= \frac{2\pi i}{2} 4\pi \cos \pi = 4\pi^2 i. \tag{14.12}$$

□

Math 185: Complex Analysis **Spring 2021**

Lecture 15: Many theorems

Lecturer: Di Fang *18 March* *Aditya Sengupta*

The first theorem we'll cover today is an extension to the Cauchy integral formula.

Theorem 15.1 (Cauchy formula extension theorem)**.** *Let f be analytic inside and on a simple positively-oriented closed contour C. If z_0 is any point interior to C, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\mathrm{d}z}{(z-z_0)^{n+1}}, n \in \mathbb{N} \tag{15.1}$$

This is an incredible result: we get infinite differentiability!

Proof. It suffices to show the first derivative exists; we can repeat the proof for any higher-order derivative. We want to show that at z_0,

$$\lim_{\Delta z \to 0} \frac{\Delta\omega}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(z)\mathrm{d}z}{(z-z_0)^2} \tag{15.2}$$

The finite difference is

$$\begin{aligned}\frac{\Delta\omega}{\Delta z} &= \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{\frac{1}{2\pi i}\int_C \frac{f(z)\mathrm{d}z}{z-z_0-\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\mathrm{d}z}{\Delta z} & (15.3)\\ &= \frac{1}{2\pi i}\frac{1}{\Delta z}\int_C f(z)\left(\frac{1}{z-z_0-z} - \frac{1}{z-z_0}\right) & (15.4)\\ &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\mathrm{d}z. & (15.5)\end{aligned}$$

Further, we can write

$$\frac{1}{(z-z_0-\Delta z)(z-z_0)} = \frac{1}{(z-z_0)^2} + \frac{\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}, \tag{15.6}$$

and therefore

$$\left|\frac{\Delta\omega}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(z)\mathrm{d}z}{(z-z_0)^2}\right| = \frac{1}{2\pi i}\left|\int_C \frac{\Delta z}{(z-z_0-\Delta z)(z-z_0)^2} f(z)\mathrm{d}z\right| \tag{15.7}$$

which we bound by the ML estimate,

$$\left|\frac{\Delta\omega}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(z)\mathrm{d}z}{(z-z_0)^2}\right| \leq \frac{1}{2\pi i}\underbrace{L}_{\text{length of } C} M. \tag{15.8}$$

To fill in M, define a distance $d \triangleq$ the smallest distance from z_0 to points on C. Then, the minimal value of $z - z_0$ is d. This lets us write the upper bound

$$\left|\frac{\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}\right| \leq \frac{|\Delta z|}{(d-|\Delta z|)d^2}, \tag{15.9}$$

which goes to zero as $\Delta z \to 0$, as desired. This completes the proof. □

This result gives us three consequences, or "miracles"!

Theorem 15.2. *If f is analytic at z_0, then its derivatives of all orders are analytic at z_0.*

Proof. If f is analytic at z_0, there is some neighborhood of z_0 in which f is differentiable. Suppose this neighborhood is an ϵ ball. We want differentiability on the boundary as well in order to apply the Cauchy extension, so take an $\frac{\epsilon}{2}$ ball, i.e. $C : |z - z_0| = \frac{\epsilon}{2}$. Then f is analytic inside and on C. Therefore, $f^{(n)}$ exists at all points inside of C by the Cauchy formula extension, for $n \geq 1$. □

Corollary 15.3. *Let $f = u + iv$. If f is analytic at z_0, then u, v have continuous partial derivatives of all orders at $z_0 = x_0 + iy_0$.*

This tells us that u, v are harmonic, i.e. they are in C^2 and satisfy Laplace's equation.

Now for the second miracle!

Theorem 15.4 (Morera's theorem)**.** *Let f be continuous on a domain D. If $\int_C f(z)\mathrm{d}z = 0$ for any closed contour C in D, then f is analytic in D.*

If D is simply connected, this is the inverse of the Cauchy-Goursat theorem.

Proof. By the fundamental theorem of contour integrals, if $\int_C f(z)\mathrm{d}z = 0$ then f has an antiderivative in D, i.e. there exists a function F such that $F' = f$ in D. Applying Miracle 1, we get that f is analytic in D as it is the first derivative of F. □

Theorem 15.5 (Cauchy Inequality/Estimate)**.** *Let f be analytic inside and on a positively-oriented circle $C_R : |z - z_0| = R$. If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M_R}{R^n}, n \geq 1. \tag{15.10}$$

Proof. From the Cauchy formula extension,

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z\right| \tag{15.11}$$

The integrand can be upper-bounded by $\left|\frac{f(z)}{(z-z_0)^{n+1}}\right| \leq \frac{M_R}{R^{n+1}}$, as the denominator is always R and the numerator is upper-bounded by M_R by assumption. Therefore the ML estimate gives us

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi} \underbrace{\frac{M_R}{R^{n+1}}}_{M} \underbrace{2\pi R}_{L} = \frac{n! M_R}{R^n}. \tag{15.12}$$

□

Theorem 15.6 (Liouville's theorem (Miracle 2))**.** *If f is entire and bounded in $\mathbb{C}$, then $f(z)$ must be constant in $\mathbb{C}$.*

This is a really strong result! How do we get there?

Proof. For a function to be constant, its first derivative should be zero, so we use the Cauchy estimate with $n = 1$.

$$|f'(z_0)| \leq \frac{M_R}{R} \leq \frac{M}{R} \tag{15.13}$$

where $M = \max_{z\in\mathbb{C}} |f(z)|$, which exists as f is bounded.: the bound over all of $\mathbb{C}$ is greater than or equal to that on the circle. Let $R \to \infty$. Then $|f'(z_0)| \leq 0$, i.e. $f'(z_0) = 0$ for any $z_0 \in \mathbb{C}$. Therefore $f(z)$ is a constant. □

We're about to use this to do something amazing:

Theorem 15.7 (Fundamental theorem of algebra)**.** *Any polynomial of degree $n \geq$,*

$$P(z) = \sum_{i=0}^{n} a_i z^i, a_n \neq 0, \tag{15.14}$$

has at least one zero.

Corollary 15.8. *Every polynomial P of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. If these roots are denoted by $\omega_1, \omega_2, \ldots, \omega_n$, then*

$$P = a_n(z - \omega_1)(z - \omega_2)\ldots(z - \omega_n). \tag{15.15}$$

of the theorem. Towards a contradiction, suppose $P(z)$ has no zero. Then $\frac{1}{P(z)}$ is entire, as $P(z)$ is nonzero everywhere.

We show that $\frac{1}{P(z)}$ is bounded. Consider $\frac{P(z)}{z^n} = a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}$. Take $|z| \to \infty$; all terms other than a_n go to zero. More rigorously, if we take $\epsilon = \frac{|a_n|}{2}$, there exists $R > 0$ such that $|B| < \epsilon = \frac{|a_n|}{2}$ for $|z| > R$. Then

$$\left|\frac{P(z)}{z^n}\right| \geq ||a_n| - |B|| > \left|a_n - \frac{|a_n|}{2}\right| = \frac{|a_n|}{2}, \tag{15.16}$$

for $|z| > R$. This implies

$$|P(z)| \geq |z^n|\frac{|a_n|}{2}, |z| > R \tag{15.17}$$

$$\left|\frac{1}{P(z)}\right| \leq \frac{2}{|z^n||a_n|}, |z| > R \tag{15.18}$$

$$< \frac{2}{R^n|a_n|}. \tag{15.19}$$

When $|z| \leq R$, we can say $\frac{1}{P(z)}$ is analytic in $\mathbb{C}$, therefore it is continuous in $\mathbb{C}$, so its modulus is bounded as the region is bounded.

Therefore, $\frac{1}{P(z)}$ is bounded by the maximum of the bound within $|z| \leq R$ and that over $|z| > R$. Since $\frac{1}{P(z)}$ is both bounded and entire, it must be a constant. This contradicts the fact that P is a degree ≥ 1 polynomial and not a constant. Therefore, $P(z)$ has at least one zero. □

of the corollary. By the theorem, we know P has at least one root; call this ω_1. We can rewrite $z = (z - \omega_1) + \omega_1$, plug into P, and expand.

$$P(z) = (z - \omega_1) \underbrace{Q(z)}_{\text{polynomial of degree } n-1} \tag{15.20}$$

Repeat the theorem on Q. Do this n times (induct on the polynomial degree); we get that $P(z)$ has precisely n roots and $P(z) = C(z - \omega_1) \ldots (z - \omega_n)$. □

The third miracle is Taylor's theorem, which we'll see soon. We'll also see analytic continuation (arguably a fourth miracle) soon.

Theorem 15.9 (Maximum modulus principle)**.** *If f is analytic and not constant in a domain D, then $|f(z)|$ has no maximum value in D.*

Corollary 15.10. *If f is continuous in the closure $\bar{D}$ of a domain, and f is analytic and not constant in D, then $|f(z)|$ reaches its maximum somewhere on the boundary ∂D.*

A corollary to this corollary is that if $f = g$ on a boundary ∂D, and both functions are analytic, then $f \equiv g$ in D.

Moreover, if $f = u + iv$ then the maximum value of $u(x, y)$ is attained on ∂D.

Proof. Let $g(z) = e^{f(z)}$. Then $|g(z)| = |e^{f(z)}| = e^u$. Since g satisfies the corollary condition, its maximum can only be achieved on the boundary, and therefore u is maximized on the boundary as well. □

We first describe intuitively why a maximum value is not likely to be reached on the interior of a domain, then we prove the maximum modulus principle.

Consider the case of a disk enclosed by $C_\rho : |z - z_0| = \rho$. By the Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0} \mathrm{d}z. \tag{15.21}$$

Use the parameterization $z = z_0 + \rho e^{i\theta}, \theta \in [0, 2\pi]$. We get

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} \rho e^{i\theta} \mathrm{d}\theta && (15.22) \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) \mathrm{d}\theta. && (15.23) \end{aligned}$$

This is the mean of a function over the boundary, so this specific case is called the Gauss mean value theorem.

Lemma 15.11. *If $f(z)$ is analytic in $B_\epsilon(z_0)$ and $|f(z)| \leq |f(z_0)|$ for all $z \in B_\epsilon$. Then $f(z) \equiv f(z_0)$ throughout $B_\epsilon(z_0)$.*

Proof. By Gauss MVT, for $0 < \rho < \epsilon$,

$$\begin{aligned} |f(z_0)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) \mathrm{d}\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + \rho e^{i\theta}) \right| \mathrm{d}\theta && (15.24) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| \mathrm{d}z = |f(z_0)|, && (15.25) \end{aligned}$$

since we went through a chain of $\leq$s and got back the original result, all terms along the way must be equal. Therefore

$$\int_0^{2\pi} |f(z_0)| - |f(z_0 + \rho e^{i\theta})| \mathrm{d}\theta = 0, \tag{15.26}$$

which is the case only if the integrand is zero.

$$\begin{aligned} |f(z_0)| &= |f(z_0 + \rho e^{i\theta})|, \theta \in [0, 2\pi] && (15.27) \\ &= |f(z)|, z \in \partial B_\rho(z_0) && (15.28) \end{aligned}$$

and varying ρ we cover all $z \in B_\epsilon(z_0)$. Since f is analytic and its modulus is constant, we have $f(z) = f(z_0)$. ☐

Now, we prove the maximum modulus principle over general domains D.

Proof. Towards a contradiction, suppose there exists z_0 interior to D such that $|f(z)| \leq |f(z_0)|\ \forall z \in D$. Take any point in D and denote it by $\tilde{z}$. z_0 and $\tilde{z}$ can be linked by a polygonal line L. Let d be the shortest distance from the points on L to ∂D. Describe L by a sequence of line segments: there exist points $z_0, z_1, \ldots, z_n = \tilde{z}$ such that $|z_k - z_{k-1}| < d$. If the points do not satisfy this, we subdivide line segments and introduce additional points until they do. Then, we can say that $|z_0 - z_1| < d$, i.e. $z_1 \in B_d(z_0)$. Applying the lemma, we get that $f(z_1) = f(z_0)$. Iterating this, we get that $f(\tilde{z}) = f(z_0)$ as desired. Thus, $f(z) \equiv f(z_0)\ \forall z \in D$. □

Math 185: Complex Analysis **Spring 2021**

Lecture 16: Taylor series and convergence

Lecturer: Di Fang *20 March* *Aditya Sengupta*

Consider an infinite sequence $\{z_n\}_{n=1}^{\infty}$.

Definition 16.1. *$\{z_n\}_{n=1}^{\infty}$ has a limit z if $\forall \epsilon > 0, \exists n_0 > 0\,\text{s.t.}$ whenever $n > n_0$, we have $|z_n - z| < \epsilon$.*

We say $\{z_n\}_{n=1}^{\infty}$ converges to z and $\lim_{n\to\infty} z_n = z$, and we say it diverges if it does not converge.

Theorem 16.1. *$\lim_{n\to\infty} z_n = z$ if and only if the real and imaginary parts separately converge (as in the reals).*

Proof. In the forward direction, if $z_n \to z$, then $|\operatorname{Re} z_n - \operatorname{Re} z| \leq |z_n - z|$ implies that the same n_0 will work, i.e. for all $\epsilon > 0, \exists n_0 > 0\,\text{s.t.}\, |z_n - z| < \epsilon$ whenever $n > n_0$ and the same n_0 implies $|\operatorname{Re} z_n - \operatorname{Re} z| < \epsilon$, and similarly for Im.

In the backward direction, applying the definition: for all $\frac{\epsilon}{2} > 0, \exists n_1 > 0$ such that $|\operatorname{Re} z_n - \operatorname{Re} z| < \frac{\epsilon}{2}$, and $\exists n_2 > 0$ such that $|\operatorname{Im} z_n - \operatorname{Im} z| < \frac{\epsilon}{2}$. Then, choose $n_0 = \max\{n_1, n_2\}$. Therefore

$$|z_n - z| \leq |\operatorname{Re} z_n - \operatorname{Re} z| + |\operatorname{Im} z_n - \operatorname{Im} z| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{16.1}$$

□

Remark 16.2. *If a complex limit exists, $\lim_{n\to\infty} x_n + iy_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} iy_n$.*

Remark 16.3. *Note that if $\lim z_n = z$, it isn't necessarily the case that $\lim \operatorname{Arg} z_n = \operatorname{Arg} z$.*

Example 16.1. Let $z_n = -1 + i\frac{(-1)^n}{n^2}$. We see that $\lim z_n = -1$.

□

Example 16.2. Let $z_n = e^{i\left(-\pi + \frac{1}{n}\right)}$. Then, $\lim_{n\to\infty} z_n = e^{i(-\pi)} = -1$, but note that $\operatorname{Arg} z_n = -\pi + \frac{1}{n} \to -\pi$ but $\operatorname{Arg} \lim_{n\to\infty} z_n = \pi$.

□

Next, we consider convergence of series. We say $\sum_{n=1}^{\infty} z_n \to S$ if the limit of partial sums $S_N = z_1 + z_2 + \cdots + z_N$ converges to S, i.e. $\lim_{N\to\infty} S_N = S$. The series diverges if it does not converge.

Theorem 16.4. *$\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} \operatorname{Re} z_n = \operatorname{Re} S$ and $\sum_{n=1}^{\infty} \operatorname{Im} z_n = \operatorname{Im} S$.*

There's many ways to check series convergence, which I remember from Math 1B so I won't rewrite them. We'll need the individual term test (individual terms of the sum need to go to zero for the sum to converge), absolute convergence implying convergence, and the geometric series.

In $\mathbb{R}$, we say that a sequence of functions converges pointwise on E if $S_n(x) \to S(x)$ as a sequence of real numbers for any $x \in E$. We can't necessarily interchange integrals and limits, i.e. it is not always the case that $\lim \int_a^b S_n(x)\mathrm{d}x = \int_a^b \lim S_n(x)\mathrm{d}x$.

Example 16.3. Consider $S_n(x) = \frac{2n^2 x}{(1+n^2x^2)^2}$. This integrates to $1 - \frac{1}{1+n^2} \xrightarrow{n\to\infty} 1$, but the integrand converges pointwise to 0, and $\int_0^1 0\mathrm{d}x = 0 \neq 1$.

□

To fix this interchangeability issue, we introduce uniform convergence.

Definition 16.2. *$S_n(x) \to S(x)$ uniformly if for all $\epsilon > 0$, there exists $n_0 > 0$ independent of x such that whenever $n > n_0$, $|S_n(x) - S(x)| < \epsilon$.*

Certain

Certain properties from $\mathbb{R}$ hold in $\mathbb{C}$ as well:

1. Uniform continuity implies continuity.
2. Uniform convergence allows us to interchange derivatives and limits.
3. Uniform convergence allows us to interchange integrals and limits.

Theorem 16.5 (Weierstrass M-test)**.** *Suppose $|a_n(x)| \leq (M_n \geq 0)$ for $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} a_n(x)$ converges uniformly in x.*

Now, we're ready for Miracle 3: Taylor's theorem!

Theorem 16.6 (Taylor's theorem)**.** *Suppose f is analytic in a disk $D = \{|z - z_0| < R_0\}$. Then $f(z)$ has a Taylor series around z_0, i.e. there exists a power series such that*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \ \forall z \in D \tag{16.2}$$

Note that this is not true in $\mathbb{R}$; for example, $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ has no Taylor series in $\mathbb{R}$ as all the derivatives at zero are 0.

Proof. Without loss of generality, let $z_0 = 0$. If it is not, we may apply the change of variables $\tilde{z} = z - z_0$ such that it is true.

We apply the Cauchy integral formula, setting $z_0 \to z, z \to s$ and integrating over s:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\mathrm{d}s \tag{16.3}$$

Pick as the contour $C = \{s : |s| = r_0\}$, where $r_0 < R_0$ so that the contour lies entirely in the domain. Next, we rewrite the integrand as follows:

$$\frac{1}{s-z} = \frac{1}{s\left(1+\frac{z}{s}\right)} = \frac{1}{s}\frac{1}{1-\frac{z}{s}} = \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{z}{s}\right)^n, \tag{16.4}$$

which is a valid geometric series as $\left|\frac{z}{s}\right| = \frac{r}{r_0} < 1$. This convergence is actually uniform in s by the M test. Therefore, we can say

$$f(z) = \frac{1}{2\pi i}\int_C \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{z}{s}\right)^n f(s)\mathrm{d}s \tag{16.5}$$

$$\underbrace{=}_{\text{unif conv}} \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_C \frac{1}{s}\frac{z^n}{s^n}f(s)\mathrm{d}s \tag{16.6}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(s)}{s^{n+1}}\mathrm{d}s\right)z^n, \tag{16.7}$$

and by the Cauchy extension formula, we recognize the bracketed term as $\frac{f^{(n)}(0)}{n!}$, and therefore we get

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n. \tag{16.8}$$

□

We can also prove Taylor's theorem in a more quantitative way, using partial geometric sums.

Proof. Expand as follows:

$$\frac{1}{s-z} = \frac{1}{s}\frac{1}{1-\frac{z}{s}}, \tag{16.9}$$

and

$$\frac{1}{1-z} = \underbrace{1+z+\cdots+z^{N-1}}_{n\text{ terms}} + \underbrace{\frac{z^N}{1-z}}_{\text{remainder}} \tag{16.10}$$

and therefore

$$\frac{1}{s-z} = \underbrace{\frac{1}{s}\sum_{n=0}^{N-1}\left(\frac{z}{s}\right)^n}_{n\text{ terms}} + \underbrace{\frac{1}{s}\frac{(z/s)^N}{1-z/s}}_{\text{remainder}} \tag{16.11}$$

$$= \sum_{n=0}^{N-1}\frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N}. \tag{16.12}$$

Therefore, applying the Cauchy integral formula as in the first proof,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\mathrm{d}s \tag{16.13}$$

$$= \frac{1}{2\pi i}\int_C \sum_{n=0}^{N-1}\frac{z^n}{s^{n+1}}f(s)\mathrm{d}s + \frac{1}{2\pi i}\int \frac{z^N}{(s-z)s^N}f(s)\mathrm{d}s \tag{16.14}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{N-1}\left(\int_C \frac{f(s)}{s^{n+1}}\mathrm{d}s\right)z^n + \rho_N \tag{16.15}$$

$$= \sum_{n=0}^{N-1}\left(\frac{f^{(n)}(0)}{n!}\right)z^n + \rho_N \tag{16.16}$$

$$\tag{16.17}$$

where ρ_N is an error term. We want to show that the error ρ_N goes to zero as $N \to \infty$, so that the Taylor series is exactly the function. To do this, we use the ML estimate.

$$|\rho_N| = \left|\frac{1}{2\pi i}\int_C \frac{f(s)z^N}{(s-z)s^N}\mathrm{d}s\right| \tag{16.18}$$

$L = 2\pi r_0$, and for M we can upper bound as follows:

$$\left|\frac{f(s)z^N}{(s-z)s^N}\right| \le \frac{|f(s)|r^N}{(|s|-|z|)r_0^N} \le \frac{Mr^N}{(r_0-r)r_0^N} \tag{16.19}$$

and therefore

$$|\rho_N| = \left|\frac{1}{2\pi i}\int_C \frac{f(s)z^N}{(s-z)s^N}\mathrm{d}s\right| \le \frac{1}{2\pi}\frac{Mr^N}{(r-r_0)r_0^N}2\pi r_0 \tag{16.20}$$

and as $N \to \infty$, this goes to 0 because $\left(\frac{r}{r_0}\right)^N \xrightarrow{N\to\infty} 0$. Therefore $\rho_N \to 0$ as desired. $\square$

Math 185: Complex Analysis **Spring 2021**

Lecture 17: Laurent series and Laurent theorem

Lecturer: Di Fang *30 March* *Aditya Sengupta*

17.1 Examples of Taylor series

If we have a Taylor series around $z = 0$, we call it a Maclaurin series.

Based on known derivative results, we can show that the following well-known formulas from $\mathbb{R}$ still hold in $\mathbb{C}$:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots |z| < 1 \tag{17.1}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots |z| < \infty \tag{17.2}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \dots |z| < \infty \tag{17.3}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots |z| < \infty \tag{17.4}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \tag{17.5}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \tag{17.6}$$

$$\tag{17.7}$$

We can potentially shift these without redoing all the derivatives at the new center!

Example 17.1. Suppose we're interested in the Taylor series of $f(z) = \frac{1}{1-z}$ around $z = i$. Let $D : |z - i| < \sqrt{2}$, so that 1 is on the boundary. We want to introduce $(z-i)^n$ terms to the Maclaurin expansion of f:

$$f(z) = \frac{1}{1-z} = \frac{1}{1-i-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-\frac{z-i}{1-i}}, \tag{17.8}$$

which we can expand as long as $\left|\frac{z-i}{1-i}\right| < 1$:

$$f(z) = \frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n = \sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}}. \tag{17.9}$$

□

Example 17.2. Suppose we're interested in the Taylor series of $f(z) = \frac{1}{1-z}$ around $z = 5$. Let $D : |z-5| < 4$.

$$f(z) = \frac{1}{1-5-(z-5)} = \frac{1}{-4}\cdot\frac{1}{1-\frac{z-5}{-4}} \tag{17.10}$$

$$= -\frac{1}{4}\sum_{n=0}^{\infty}\frac{(z-5)^n}{(-4)^n} \tag{17.11}$$

$$= \sum_{n=0}^{\infty}\frac{(z-5)^n}{(-4)^{n+1}}, |z-5| < 4. \tag{17.12}$$

□

Example 17.3. Consider the Taylor series of $f(z) = e^z$ around $z = 1+i$.

$$f(z) = e^z = e^{z-(1+i)}e^{1+i} = e^{1+i}\sum_{n=0}^{\infty}\frac{(z-1-i)^n}{n!} \tag{17.13}$$

□

17.2 Power series with negative powers

If we divide by a power of z, or we plug in $\frac{1}{z}$ as the argument to a function, we get negative powers:

Example 17.4.

$$\frac{e^z}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \dots \tag{17.14}$$

□

Example 17.5.

$$z^3 \cosh \frac{1}{z} = z^3 \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{1}{z^{2n}}. \tag{17.15}$$

□

A specific power series expansion with powers in $\mathbb{Z}$ is referred to as a Laurent series. While Taylor's theorem tells us we can expand f if it is analytic in a disk $|z - z_0| < r$, Laurent's theorem tells us this is doable on an annular domain $D = \{R_1 < |z - z_0| < R_2\}$. This relaxes the need for analyticity at z_0 or in a neighborhood of z_0.

Theorem 17.1 (Laurent's theorem)**.** *Let f be analytic in an annular domain $D = \{R_1 < |z - z_0| < R_2\}$ and let C be any positively oriented simple closed contour around z_0 in D. Then we can expand f as follows: for all $z \in D$,*

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{analytic} + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \tag{17.16}$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z, n = 0, 1, \dots \tag{17.17}$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} \mathrm{d}z, n = 1, 2, \dots. \tag{17.18}$$

Remark 17.2.

1. *We can write this as a single power series where $f(z) = \sum_{n \in \mathbb{Z}} c_n (z - z_0)^n$, where $c_n = a_n$ for $n \geq 0$ and $c_n = b_n$ otherwise. This can be unified into $c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z$.*

2. *If f is analytic, the Laurent series is just the Taylor series, for the following reason. We can say $R_1 = 0$. The negative coefficients are then*

$$b_n = \frac{1}{2\pi i} \int_C f(z)(z - z_0)^{n-1} \mathrm{d}z, \tag{17.19}$$

and as the integrand is analytic in a simply connected domain, Cauchy-Goursat v2 tells us that the contour integral must be 0 and so only the positive powers remain.

of Laurent's theorem. Recall that to prove Taylor's theorem, we first took $z_0 = 0$, then we applied the Cauchy integral formula to f and expanded the denominator using a geometric series. Then, we used uniform continuity to interchange the limit and sum to get a sum over coefficients and powers of z, and these powers matched up with the derivatives of f by the Cauchy integral extension.

We proceed similarly for Laurent's theorem. Let the region be $D = \{r_1 < |z - z_0| < r_2\}$. Once again we set $z_0 = 0$ without loss of generality. Consider a contour γ within D and around any arbitrary z.

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{s-z} \mathrm{d}s \tag{17.20}$$

Due to the inner part of the annulus, we now have a multiply connected domain, so we apply Cauchy-Goursat v3. Draw an outer contour C_2 bounding the inner part of the annulus and γ, and draw an inner contour C_1 bounding just the annulus. Cauchy-Goursat v3 (taking C_2 positively oriented and γ, C_1 negatively oriented) then tells us that

$$\underbrace{\int_{C_2} \frac{f(s)}{s-z} \mathrm{d}s}_{I_2} - \underbrace{\int_\gamma \frac{f(s)}{s-z} \mathrm{d}s}_{2\pi i f(z)} - \underbrace{\int_{C_1} \frac{f(s)}{s-z} \mathrm{d}s}_{I_1} = 0 \tag{17.21}$$

Looking at I_2, we note that for $s \in C_2$, $\left|\frac{z}{s}\right| < 1$, so we can use the geometric expansion.

$$\frac{1}{s-z} = \frac{1}{s} \frac{1}{1-\frac{z}{s}} = \frac{1}{s} \sum_{n=0}^{\infty} \left(\frac{z}{s}\right)^n \underbrace{=}_{\text{unif in } S \text{ by } M \text{ test}} \sum_{n=0}^{\infty} \frac{1}{s^{n+1}} z^n \tag{17.22}$$

Therefore

$$\frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z} \mathrm{d}s = \frac{1}{2\pi i} \int_{C_2} f(s) \sum_{n=0}^{\infty} \frac{1}{s^{n+1}} z^n \mathrm{d}s \tag{17.23}$$

$$\underbrace{\text{unif conv}} \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\int_{C_2} \frac{f(s)}{s^{n+1}} \mathrm{d}s \right) z^n \tag{17.24}$$

$$= \sum_{n=0}^{\infty} a_n z^n, \tag{17.25}$$

where in the last step we change $C_2 \to C$ by the principle of path deformation.

For I_1, we see that $\left|\frac{s}{z}\right| < 1$, and so

$$\frac{1}{s-z} = -\sum_{n=0}^{\infty} \frac{s^n}{z^{n+1}}. \tag{17.26}$$

Therefore

$$-\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s-z}\mathrm{d}s = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{C_1}\frac{f(s)s^{n-1}}{z^n}\mathrm{d}s \tag{17.27}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}\mathrm{d}s\right)\frac{1}{z^n} \tag{17.28}$$

$$= \sum_{n=1}^{\infty} b_n z^{-n}. \tag{17.29}$$

So finally, we get the desired result,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}. \tag{17.30}$$

□

17.3 Laurent series examples

Example 17.6. Let $f(z) = \frac{1}{z(1+z^2)}$, for $0 < |z| < 1$. We're interested in the Laurent series around $z = 0$. This is not possible with a Taylor series, as we have issues at 0.

$$f(z) = \frac{1}{z}\frac{1}{1+z^2} = \frac{1}{z}\frac{1}{1-(-z^2)} \tag{17.31}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n-1} \tag{17.32}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n z^{2n-1} \tag{17.33}$$

□

Example 17.7. Let $f(z) = \frac{z+1}{z-1}$, and consider the power series around $z = 0$. We split this into the two regions $D_1 : |z| < 1$ (for a Taylor series) and $D_2 : 1 < |z| < \infty$ (for a Laurent series).

On D_1, we use the geometric series trick, which only holds for $|z| < 1$, to find that

$$D_1 : f(z) = -1 - 2\sum_{n=1}^{\infty} z^n \tag{17.34}$$

and on D_2, we divide by z on the numerator and denominator to get $f(z) = \frac{1+\frac{1}{z}}{1-\frac{1}{z}}$, and apply the geometric series to both terms to get

$$D_2 : f(z) = 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}. \tag{17.35}$$

□

Example 17.8. Let $f(z) = \frac{1}{(z-1)(z-2)}$ on $D : \{1 < |z| < 2\}$. f is analytic in D, so by partial fraction decomposition we can say

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}. \tag{17.36}$$

But this isn't a Laurent series, so we expand out each term individually into its own Laurent series. We apply the geometric trick to get

$$-\frac{1}{z-1} = -\sum_{n=1}^{\infty} \frac{1}{z^n} \tag{17.37}$$

and

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \tag{17.38}$$

and therefore

$$f(z) = -\sum_{n=1}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}. \tag{17.39}$$

□

Math 185: Complex Analysis **Spring 2021**

Lecture 18: Series convergence

Lecturer: Di Fang *1 April* *Aditya Sengupta*

Today, we'll explore the notion of convergence in more depth. We have three ideas of convergence of series: the usual $\epsilon - N$ convergence, absolute convergence (i.e. the convergence of $\sum_{n=0}^{\infty} |a_n||z-z_0|^n)$ and uniform convergence, i.e. $|S_n(z) - S(z)| < \epsilon$ for all $z \in E$.

Theorem 18.1. *If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at $z = z_1 \neq z_0$, then it converges absolutely at each point z in the open disk $|z - z_0| < R_1 \triangleq |z_1 - z_0|$.*

Proof. At $z = z_1$, we use the following upper bound:

$$|a_n(z_1 - z_0)^n| \leq M > 0 \tag{18.1}$$

$$|a_n(z-z_0)^n| = |a_n||(z_1 - z_0)^n| \left|\frac{z-z_0}{z_1-z_0}\right|^n \tag{18.2}$$

The first two terms can be bound by M, and the second is equal to some $\rho < 1$.

$$|a_n(z-z_0)^n| \leq M\rho^n, \tag{18.3}$$

and $\sum_{n=0}^{\infty} M\rho^n$ converges as it is a geometric series. □

The largest disk centered at z_0 such that the series converges is called the disk of convergence. It is defined by $D = \{z \in C \mid |z - z_0| < R\}$ where R is the convergence radius.

Theorem 18.2. *If z_1 is a point inside of the disk of convergence D of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, then a series converges uniformly in the closed disk $|z - z_0| \leq R_1 \triangleq |z_1 - z_0|$.*

Rather than proving this, we will state some more general results.

The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is $R \triangleq \frac{1}{\limsup \sqrt[n]{|a_n|}}$.

Remark 18.3. *Suppose $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ exists. Then we can replace* $\limsup$ *by the limit.*

The intuition for this is the root test from calculus: if $\lim\limits_{n\to\infty} \sqrt[n]{|c_n|} < 1$, then the series $\sum_{n=0}^{\infty} c_n$ converges. Applying this to $c_n = a_n(z-z_0)^n$, we get

$$\lim_{n\to\infty} \sqrt[n]{|a_n||z-z_0|^n} = \lim_{n\to\infty} \sqrt[n]{|a_n|} \cdot |z-z_0| < 1, \tag{18.4}$$

and therefore

$$|z - z_0| < \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}} = R. \tag{18.5}$$

So why have a lim sup? This is the supremum of the limits of all the subsequences. This is necessary for the following reason. Suppose we have $a_n = \begin{cases} 0 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$, i.e. the series $\sum a_n z^n = z + z^3 + z^5 + \dots$. While $\sqrt[n]{|a_n|}$ does not have a limit, we can still talk about $\limsup \sqrt[n]{|a_n|} = 1$, and this tells us $R = 1$.

We could also use the ratio test: if $\lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists, then it is equal to R as well.

Theorem 18.4 (General results about series convergence)**.** *Let $0 \leq R \leq \infty$ be the convergence radius of $\sum_{n=0}^{\infty} a_n(z - z_0)^n$.*

1. *If $|z - z_0| < R$, the series converges absolutely.*
2. *If $|z - z_0| > R$, the series diverges.*
3. *For any fixed $r < R$, the series converges uniformly for $|z - z_0| < r$.*

(sketch). Without loss of generality let $z_0 = 0$. Then, if $|a_n|r^n$ is bounded for some $r = r_0$, then it must be bounded for any $0 \leq r < r_0$. So, define R to be the supremum of r such that $|a_n|r^n$ is bounded. (This does not necessarily imply that $|a_n|R^n$ is bounded.) Then, for $r > R$, we know that $|a_n|r^n$ is unbounded, which specifically means there exists some sequence a_{k_j} such that $|a_{k_j}|r^{k_j} \to \infty$.

Consider the following cases:

1. $|z| > R$: this tells us that $a_k z^k$ does not converge to 0, bu the divergence test.
2. $|z| \leq r < R$: choose some s such that $r < s < R$. Then

$$|a_n z^n| = \underbrace{|a_n| s^n}_{\leq M} \underbrace{\left|\frac{z}{s}\right|^n}_{\leq \frac{r}{s}}, \tag{18.6}$$

   so we can upper-bound the sum by the geometric series $\sum_{n=0}^{\infty} M \left(\frac{r}{s}\right)^n$, and therefore it converges.

3. $|z| < R$: choose r such that $|z| \leq r < R$. Use the above point: we can show that $\sum |a_n z^n|$.

□

Remark 18.5. *To check the convergence of Laurent series, let $\omega = \frac{1}{z-z_0}$. Then the additional term in a Laurent series has the form $\sum_{n=1}^{\infty} b_n \omega^n$. Analogous to what we derived above, we can say that for $R_- \triangleq \frac{1}{\limsup \sqrt[n]{|b_n|}}$, the Laurent series converges absolutely if $|\omega| < R_-$ and diverges if $|\omega| > R_-$. Changing this back into a condition on $z$and applying this in conjunction with the usual condition, we can say that a Laurent series $\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converges if $\frac{1}{R_-} < |z - z_0| < R$ (where R is the radius of convergence of the a_n term).*

Example 18.1. Consider the Laurent series $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=1}^{\infty} \frac{1}{z^n}$. For the $a_n z^n$ component, we have $R = \frac{1}{\limsup \sqrt[n]{\frac{1}{2^n}}} = \frac{1}{\frac{1}{2}} = 2$ and $R_- = \frac{1}{\limsup \sqrt[n]{1}} = 1$. Therefore the convergence domain is $1 < |z| < 2$.

□

18.1 Continuity of power series

Theorem 18.6. $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ *represents a continuous function* $S(z)$ *at each point inside its disk of convergence* $|z-z_0| < R$.

Proof. Fix any $z_1 \in \{z \mid |z-z_0| < R\}$ and let $r \triangleq |z_1 - z_0|$. We know the series converges uniformly on the closed disk $|z-z_0| \leq r$. Therefore, from the definition, for all $\frac{\epsilon}{3} > 0$ there exists some N_ϵ such that $N > N_\epsilon$ implies $|S_N(z) - S(z)| < \frac{\epsilon}{3}$ for all z such that $|z - z_0| \leq r$.

Fix one such ϵ and choose N accordingly.

$$|S(z) - S(z_1)| \leq |S(z) - S_N(z)| + |S_N(z) - S_N(z_1)| + |S_N(z_1) - S(z_1)| \tag{18.7}$$

The first and third terms converge because of the uniform convergence of $S_n \to S$; the second term converges because the $S_N(z)$ are polynomials and are therefore continuous. Therefore, each term individually is bounded by $\frac{\epsilon}{3}$, so

$$|S(z) - S(z_1)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \tag{18.8}$$

□

Math 185: Complex Analysis **Spring 2021**

Lecture 19: Calculus with power series, uniqueness of Taylor/Laurent series

Lecturer: Di Fang *13 April* *Aditya Sengupta*

Today, we end chapter 5.

19.1 Integration by terms

Theorem 19.1 (Integration by terms)**.** *Let $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ on a disk of convergence $D = \{z \in C \mid |z-z_0| < R\}$. Let C be any closed contour in D and let $g : C \to \mathbb{C}$ be continuous. Then*

$$\int_C g(z)S(z)\mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n \mathrm{d}z. \tag{19.1}$$

This says we can interchange a sum and integral under these conditions.

Proof. Rewrite S in terms of a partial sum and a remainder:

$$S(z) = \sum_{n=0}^{N-1} a_n(z-z_0)^n + \rho_N(z) \tag{19.2}$$

Then,

$$\int_C g(z)S(z)\mathrm{d}z = \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n \mathrm{d}z + \underbrace{\int_C g(z)\rho_N(z)\mathrm{d}z}_{I_N}, \tag{19.3}$$

and it suffices to show that the remainder integral goes to 0. We use the ML estimate. Since g is continuous in C, it has a maximum $M_0 \triangleq \max_{z\in\mathbb{C}} |g(z)|$, and the contour has a known length.

Write $|\rho_N(z)| = |S(z) - S_N(z)|$. $S_N(z)$ converges uniformly to $S(z)$, and so for all $\epsilon > 0$ there exists $N_\epsilon > 0$ such that whenever $N > N_\epsilon$, we have $|S_N(z) - S(z)| < \epsilon$, and so by the ML estimate, $|I_N| \le M_0 \epsilon L$ for $N > N_\epsilon$. This is exactly the statement that $\lim_{N\to\infty} I_N = 0$. ☐

Remark 19.2. *If we take $g(z) \equiv 1$ in C, then we get*

$$\int_C \rho(z)\mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n \mathrm{d}z = 0 \tag{19.4}$$

where the expression on the right is 0 by the Cauchy-Goursat theorem and because polynomials are entire. Therefore, Morera's theorem tells us that $S(z)$ is analytic in D.

Corollary 19.3. *The power series $S(z)$ is analytic within its disk of convergence.*

Therefore, we can conclude that if we have a function $f : D \to \mathbb{C}$ where D is a disk of convergence, f being analytic in D is equivalent to f having a Taylor series in D.

Example 19.1. Let $f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$. Show $f(z)$ is entire.

We could check this using the definition of the derivative at $z = 0$, using L'Hopital's rule (which holds in the complex case although we haven't proved that).

$$\lim_{z\to 0} \frac{\frac{\sin z}{z} - 1}{z - 0} = \lim_{z\to 0} \frac{\sin z - z}{z^2} \tag{19.5}$$

$$= \lim_{z\to 0} \frac{\cos z - 1}{2z} \tag{19.6}$$

$$= \lim_{z\to 0} \frac{-\sin z}{2} = 0 \tag{19.7}$$

However, in light of this new result, we can write out the Taylor series:

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{19.8}$$

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} \tag{19.9}$$

$$\tag{19.10}$$

and if we plug in $z = 0$ to this Taylor series, we get 1, and so our final function does satisfy this Taylor series. We want to show that the radius of convergence of this series is infinity, so that the domain of analyticity is all of $\mathbb{C}$.

let $\omega = z^2$. Then we get

$$\sum_{n=0}^{\infty} (-1)^n \frac{\omega^n}{(2n+1)!}, \tag{19.11}$$

which has $a_n = \frac{(-1)^n}{(2n+1)!}$. We can show by the ratio test that this converges:

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(2n+3)!}}{\frac{1}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+3)} \right| = 0, \tag{19.12}$$

and so the convergence radius is "$\frac{1}{0}$" $= \infty$.

□

19.1.1 Differentiation by terms

We have a similar result for differentiating term by term!

Theorem 19.4. *Let $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ on a convergent disk $D = \{|z-z_0| < R\}$. Then $S'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$ in D.*

Proof. Fix any $z_1 \in D$. Then S is analytic at $z-1$. We want to show that $S'(z_1)$ is the target power series evaluated at z_1.

Start with the Cauchy integral formula extension:

$$S'(z_1) = \frac{1}{2\pi i}\int_C \frac{S(z)}{(z-z_1)^2}\mathrm{d}z, \tag{19.13}$$

and integrating term by term and expanding out S, we can say this is equal to

$$S'(z_1) = \frac{1}{2\pi i}\sum_{n=0}^{\infty} a_n \int_C \frac{(z-z_0)^n}{(z-z_1)^2}\mathrm{d}z \tag{19.14}$$

The numerator is a polynomial and therefore entire, so we use the Cauchy formula extension and rewrite the integrand as a derivative at z_1:

$$S'(z_1) = \frac{1}{2\pi i}\sum_{n=0}^{\infty} a_n 2\pi i \frac{\mathrm{d}}{\mathrm{d}z}\,(z-z_0)^n|_{z=z_1} \tag{19.15}$$

$$= \sum_{n=1}^{\infty} a_n n(z_1-z_0)^{n-1}, \tag{19.16}$$

where we reindex as the $n=0$ case just evaluates to 0. □

Remark 19.5. *The derivative has the same convergence radius as the function itself. If we say $S'(z)$ has a convergence radius R', then by the differentiation theorem we have $R' \geq R$ but by the integration theorem we have $R' \leq R$, so $R' = R$.*

Example 19.2. Consider $\frac{1}{z}$. Expand this using a geometric series:

$$\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n, |z-1| < 1. \tag{19.17}$$

Taking a derivative, we get

$$-\frac{1}{z^2} = \sum_{n=1}^{\infty}(-1)^n n(z-1)^{n-1}, |z-1| < 1. \tag{19.18}$$

We could also integrate, to get $\int_C \frac{1}{z}\mathrm{d}z = \operatorname{Log} z$. Let C be the line segment from 1 to z. $\frac{1}{z}$ has an antiderivative $\operatorname{Log} z$ in $D = \{|z-1| < 1\}$. By integrating by terms, we get

$$\int_C \frac{1}{z}\mathrm{d}z = \sum_{n=0}^{\infty}(-1)^n \int_C (z-1)^n \mathrm{d}z = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(z-1)^{n+1}, \tag{19.19}$$

and therefore we've derived the Taylor series for Log.

□

Remark 19.6. *Integration by terms also works for Laurent series: if we let* $\omega = \frac{1}{z-z_0}$*, the series is* $\sum_{n=1}^{\infty} b_n \omega^n$*, and the above results apply for* $|\omega| < r \implies |z - z_0| > \frac{1}{r}$.

19.2 Uniqueness of Taylor/Laurent Series

Theorem 19.7. *Let* $\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ *for* $|z-z_0| < R$*. Then* $a_n = b_n \ \forall n$.

Proof.

$$a_n = b_n = \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z \underbrace{=}_{\text{Cauchy integral formula}} \frac{f(z_0)}{n!} \tag{19.20}$$

□

Theorem 19.8. *Let* $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n$ *for* $R_1 < |z-z_0| < R_2$*. Then* $a_n = b_n \ \forall n$

Proof. Apply the Laurent theorem:

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z = \frac{1}{2\pi i}\sum_{k=-\infty}^{\infty} a_n \int_C \frac{(z-z_0)^k}{(z-z_0)^{n+1}}\mathrm{d}z \tag{19.21}$$

For convenience, let $I_k \triangleq \int_C (z-z_0)^{k-(n+1)}\mathrm{d}z$.

$$I_k = \begin{cases} 0 & k \geq n+1 \text{ by Cauchy-Goursat} \\ \frac{1}{2\pi i}\int_C \frac{1}{z-z_0}\mathrm{d}z = 1 & k = n \text{ by Cauchy formula extension at 0} \\ \frac{1}{2\pi i}\int_C \frac{(z-z_0)^k}{(z-z_0)^{n+1}}\mathrm{d}z = \frac{\mathrm{d}^n}{\mathrm{d}z^n}(z-z_0)^k = 0 & k \leq n-1 \end{cases} \tag{19.22}$$

Therefore, $I_k = \delta_{nk}$, so the right hand side is just a_n and the proof is complete. □

19.3 Multiplying and Dividing Power Series

If f, g are analytic on some domains $D_1 = \{|z - z_0| < R_1\}$ and $D_2 = \{|z - z_0| < R_2\}$ respectively, then fg is analytic on $\{|z - z_0| < \min\{R_1, R_2\}\}$ and fg has a Taylor series on this domain given by $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ where

$$c_n = \frac{[f(z_0)g(z_0)]^{(n)}}{n!} = \frac{1}{n!}\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z_0) g^{(n-k)}(z_0) = \sum_{k=0}^{n} a_k b_{n-k} \tag{19.23}$$

,

i.e. c_n is the convolution of a_n and b_n.

Example 19.3. Consider $f(z) = \frac{\sinh z}{1+z}$ around $z = 0$. The Taylor series of $\sinh z$ and $\frac{1}{1+z}$ are known:

$$\sinh z = z + \frac{1}{6}z^3 + \frac{120}{z}^5 + \dots \qquad |z| < \infty \tag{19.24}$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots \qquad |z| < 1 \tag{19.25}$$

Therefore, their product has a Taylor series that converges for $|z| < 1$, with coefficients resulting from the convolution operation. Doing this by hand for a few terms, we get

$$\frac{\sinh z}{1+z} = z - z^2 + \frac{7}{6}z^3 - \frac{7}{6}z^4 + \dots (|z| < 1) \tag{19.26}$$

□

We can use this in conjunction with the Cauchy integral formula to extract contour integral values, for example:

$$\oint_{|z|=\frac{1}{2}} \frac{\sinh z}{(1+z)z^4} \mathrm{d}z \underbrace{=}_{n=3} = 2\pi i a_3 = \frac{7\pi i}{3} \tag{19.27}$$

because more generally,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \tag{19.28}$$

and

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\mathrm{d}z \tag{19.29}$$

Example 19.4. Consider $\frac{1}{\sinh z} = \frac{1}{z+\frac{1}{6}z^3+\frac{1}{120}z^5}$. The zeros of $\sinh z$ are $z = n\pi i, n \in \mathbb{Z}$, so we can work with the expansion in the punctured disk $0 < |z| < \pi$. In this domain, we can get the Laurent series by polynomial long division.

□

Math 185: Complex Analysis **Spring 2021**

Lecture 20: Residues and poles

Lecturer: Di Fang *15 April* *Aditya Sengupta*

20.1 Singular points

The Cauchy-Goursat theorem tells us that if a function f is analytic at all points on C and interior to C, then $\int_C f(z)\mathrm{d}z = 0$. This is a restrictive assumption and not a very useful result, so let's try and extend it. In particular, let's first consider the case where we have some singular points.

With one singular point, we can sometimes write the function as $f(z) = \frac{g(z)}{(z-z_0)^n}$, where g is analytic in $\bar{D}$, and then we can apply the Cauchy integral extension to get that the contour integral is equal to $\frac{2\pi i}{(n-1)!} g^{(n-1)}(z_0)$. In the past, we've only looked at this case; for example, when integrating trigonometric or hyperbolic functions with zeros on the real axis, we've restricted ourselves to contours with only one singularity. What if we wanted to extend this?

Consider $\int_C \frac{1}{z(z-1)}\mathrm{d}z$ where $C = \{|z| = 2\}$ positively oriented. This has two singular points, so how do we do this integral? Another representative example is $\oint_{|z|=1} e^{\frac{1}{z}}\mathrm{d}z$, which has a singular point at $z = 0$. This is not expressible in terms of the form we used above. Let's try and come up with a way to fix this!

Definition 20.1. *A singular point z_0 is isolated if there exists some punctured neighborhood of z_0 such that f is analytic with in it.*

Example 20.1. Let $f(z) = \frac{z-1}{z^5(z^2+9)}$. $z = 0, \pm 3i$ are singular points. These are isolated, as 1-neighborhoods around each of these points satisfy analyticity. □

Example 20.2. $f(z) = \operatorname{Log} z$ does not have $z = 0$ as an isolated singular point, due to the branch cut (no neighborhood of $z = 0$ that can exclude it.) □

Example 20.3. Let $f(z) = \frac{1}{\sin \frac{\pi}{z}}$ has singularities at $\frac{\pi}{z} = n\pi, n \in \mathbb{Z}$, i.e. $z = \frac{1}{n}$. Therefore $z = 0$ is not isolated but the others are. □

20.2 Residues

Let z_0 be an isolated singular point, and let the corresponding punctured neighborhood be B'. Then we can expand f in B' in a Laurent series around the singular point:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{20.1}$$

where $b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} \mathrm{d}z$ and C is any positively oriented simple closed contour lying in B'.

Notice that if we take b_1, then we see that

$$2\pi i b_1 = \int_C f(z) \mathrm{d}z, \tag{20.2}$$

i.e. if we can expand the function into its Laurent series, we get the *residue* at a pole by taking the coefficient b_1.

Definition 20.2. *The residue of a function f at a point z_0 is defined as the coefficient of the $\frac{1}{z-z_0}$ term in the Laurent series of f around z_0: $\mathrm{Res}_{z=z_0} f(z) = b_1$.*

Remark 20.1. *If f is analytic at z_0 as well as in the neighborhood, the Laurnt series reduces to just a Taylor series, $b_1 = 0$, so the residue is 0 and the contour integral is also 0, consistent with the Cauchy-Goursat theorem.*

Remark 20.2. *If we can write f as $\frac{g(z)}{(z-z_0)^n}$, then the contour integral is $2\pi i$ times the $n-1$th coefficient of the Laurent series of g.*

Example 20.4. Consider the integral $I = \int_C \frac{e^z - 1}{z^4} \mathrm{d}z$, with $C : |z| = 1$ positively oriented. Using the Cauchy integral extension:

$$I = \left. \frac{\mathrm{d}^3 (e^z - 1)}{\mathrm{d}z^3} \right|_{z=0} = \frac{2\pi i}{3!} = \frac{\pi i}{3} \tag{20.3}$$

and using the residue:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \implies \frac{e^z - 1}{z^4} = \sum_{n=1}^{\infty} \frac{z^{n-4}}{n!}, \tag{20.4}$$

i.e the residue $\mathrm{Res}_{z=0} f(z) = \frac{1}{3!}$ and $I = \frac{2pii}{3!} = \frac{\pi i}{3}$

□

Example 20.5. We can now do the integral we weren't able to do before: $I = \int_C e^{1/z} \mathrm{d}z$ on $C : |z| = 1$ positively oriented. The integrand can be expanded to $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$, so $I = 2\pi i \cdot 1 = 2\pi i$.

□

Example 20.6. The integral $I = \int_C e^{\frac{1}{z^2}} \mathrm{d}z$ has an integrand that can be expanded to $\sum_{n=0}^{\infty} \frac{1}{n!} z^{-2n}$, which has $b_1 = 0$, therefore $I = 0$.

□

20.3 Residue theorem

What if we have more than one isolated singular points? We can generalize the above result:

Theorem 20.3 (Residue theorem)**.** *Let C be a simple closed contour that is positively oriented. Let f be analytic inside and on C except for a finite number of isolated singular points $z_k (k = 1, \dots, n)$ inside C. Then*

$$\int_C f(z) \mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z) \tag{20.5}$$

Proof. This is Cauchy-Goursat v3, considering the overall contour with the neighborhoods around each isolated singularity removed.

$$\int_C f(z) \mathrm{d}z - \sum_{k=1}^{n} \int_{C_k} f(z) \mathrm{d}z = 0 \tag{20.6}$$

$$\int_C f(z) \mathrm{d}z = \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z) \tag{20.7}$$

□

Example 20.7. Consider $\oint_{|z|=2} \frac{4z-5}{z(z-1)} \mathrm{d}z$. There are two poles, at $z = 0$ and $z = 1$, so summing their residues gets us

$$I = 2\pi i (\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=1} f(z)) = 2\pi i (5 - 1) = 8\pi i, \tag{20.8}$$

where the coefficient s come from expanding the integrand using geometric series,

$$\frac{4z-5}{z(z-1)} = \left(4 - \frac{5}{z}\right)\frac{-1}{1-z} = \left(4 \underbrace{-\frac{5}{z}}\right)(\underbrace{-1} - z - z^2 - \dots), \tag{20.9}$$

and

$$\frac{4z-5}{z(z-1)}\frac{1}{z-1}\frac{4(z-1)-1}{z} = \left(4 - \frac{1}{z-1}\right)\frac{1}{1+(z-1)}\left(4 \underbrace{-\frac{1}{z-1}}\right)(\underbrace{1} + (z-1) + \dots) \tag{20.10}$$

□

20.4 Residues at infinity

Definition 20.3. *Let f be analytic for $R_1 < |z| < \infty$. Then we call "∞" an isolated singular point of f, and we define its residue according to*

$$\operatorname{Res}_{z=\infty} f(z) \triangleq -\frac{1}{2\pi i}\int_{C_0} f(z)\mathrm{d}z, \tag{20.11}$$

where C_0 is $|z| = R_0 > R_1$. We further claim this is equal to $-\operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$.

Theorem 20.4. *Let f be analytic in $\mathbb{C} \setminus \{z_1, \dots, z_n\}$ where the z_ks are isolated singular points interior to some contour C. Then*

$$\int_C f(z)\mathrm{d}z = 2\pi i \operatorname{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right) \tag{20.12}$$

This is an alternate formulation of the residue theorem that might be easier in some cases, where $f(1/z)$ is easier to deal with than $f(z)$, where it's not obvious how to deal with the general Laurent series, or just where we only want to deal with one residue instead of n residues for n singularities.

Proof. By the Laurent theorem,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \tag{20.13}$$

and so

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} c_n \frac{1}{z^{n+2}} \tag{20.14}$$

The residue of this function at $z = 0$ is c_{-1}, which by the residue theorem is also $\int_C f(z)\mathrm{d}z$. □

Example 20.8. Let $f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}$ and let $C : |z| = 3$ (pos). The integral can be found as follows:

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z} \frac{z-3}{(z+1)(z^4+2)} \tag{20.15}$$

The $\frac{1}{z}$ term is equal to $\frac{1}{z}$ times the constant term in the Taylor expansion of $\frac{z-3}{(z+1)(z^4+2)}$, so evaluating this at $z = 0$ yields a residue of $-\frac{3}{2}$. Therefore $I = -3\pi i$. □

20.5 Types of isolated singular points

Let z_0 be an isolated singular point. f can be expanded into a Laurent series around z_0 with coefficients a_n in the analytic part and b_n in the principal part.

The three types of isolated singular points can be classified based on how many of the principal coefficients b_n are nonzero.

1. A *removable* isolated singular point has no nonzero b_ns, i.e. the Laurent series is a Taylor series. For example, $z = 0$ is removable in both $f(z) = \begin{cases} e^z & z \neq 0 \\ 0 & z = 0 \end{cases}$ and in $f(z) = \frac{\sin z}{z}, z \neq 0$.
2. An *essential* isolated singular point has an infinite number of $b_n \neq 0$.
3. A *pole of order m* has a nonzero finite number of b_n nonzero, and m is the maximum index that is nonzero. Specifically, $b_m \neq 0$ and $b_k = 0$ for all $k > m$.

Example 20.9. $e^{1/z}$ can be expanded into $\sum_{n=0}^{i} nfty \frac{1}{n!} \frac{1}{z^n}, 0 < |z| < \infty$, so $z = 0$ is essential. □

Example 20.10. Let $f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2} + frac1z + \sum_{n=0}^{\infty} z^n$. $z = 0$ is a pole of order 2. □

Example 20.11. $f(z) = \frac{z^2+z-2}{z+1} = z - \frac{2}{z+1}$. $z = -1$ is a pole of order 1, or a simple pole. □

Math 185: Complex Analysis **Spring 2021**

Lecture 21: Residues at poles, analytic continuation

Lecturer: Di Fang *27 April* *Aditya Sengupta*

21.1 Residues at poles

Theorem 21.1. *Let z_0 be an isolated singular point of f. Then, the following two statements are equivalent:*

1. *z_0 is a pole of order m $(m = 1, 2, \dots)$ of f.*
2. *Let $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ $(m = 1, 2, \dots)$. Then $\phi(z)$ is analytic and nonzero at z_0.*

Moreover, if either of the above holds, then

$$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \tag{21.1}$$

This gives us a standard way to rewrite a function in terms of its poles, and we can then use the Cauchy integral formula (for $m = 1$) or extension (for $m > 1$) to evaluate its residue.

Proof. We show 1 implies 2.

Let z_0 be a pole of order m. We expand f into its Laurent series about m, and because z_0 is a pole of order m we only have principal parts up to the mth power:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z - z_0)^n}. \tag{21.2}$$

Pull out a factor of $\frac{1}{(z-z_0)^m}$:

$$f(z) = \frac{1}{(z - z_0)^m} \underbrace{(g(z)(z - z_0)^m + b_1(z - z_0)^{m-1} + \cdots + b_m)}_{\triangleq \phi(z)}. \tag{21.3}$$

We see that because $\phi(z)$ is a polynomial centered at z_0, it is analytic at z_0. Further, $\phi(z_0) \neq 0$ as $\phi(z_0) = b_m \neq 0$.

We show 2 implies 1.

Let $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ be analytic at z_0. Taylor expand ϕ in a neighborhood of z_0:

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + \dots \tag{21.4}$$

and so

$$\frac{\phi(z)}{(z - z_0)^m} = \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \cdots + a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + \dots \tag{21.5}$$

and therefore we get that z_0 is a pole of order m. □

Example 21.1. Let $f(z) = \frac{z+4}{z^2+1}$. We get that $z = \pm i$ are isolated singular points. At $z = i$, we rewrite the function in the form suggested by the theorem, $f(z) = \frac{\frac{z+4}{z+i}}{z-i}$. Similarly at $z = -i$, $f(z) = \frac{\frac{z+4}{z-i}}{z+i}$. This makes computing residues a lot simpler:

$$\operatorname{Res}_{z=i} f(z) = \left.\frac{z+4}{z+i}\right|_{z=i} = \frac{i+4}{2i} \tag{21.6}$$

$$\operatorname{Res}_{z=-i} f(z) = \left.\frac{z+4}{z-i}\right|_{z=-i} = \frac{-i+4}{-2i} \tag{21.7}$$

$$\tag{21.8}$$

□

Example 21.2. Let $f(z) = \frac{(\log z)^3}{z^2+1}$, where $\log z = \ln r + i\theta(r > 0, \theta \in (0, 2\pi))$ (i.e. a branch cut excluding the positive reals). The poles are $z = \pm i$, and so the residue at i is

$$\operatorname{Res}_{z=i} f(z) = \left.\frac{(\log z)^3}{z+i}\right|_{z=i} = \frac{\left(i\frac{\pi}{2}\right)^3}{2i} = -\frac{\pi^3}{16} \tag{21.9}$$

and similar for $z = -i$.

□

21.2 Zeros of analytic functions

Definition 21.1. *z_0 is a zero of order m if $0 = f(z_0) = f^{(1)}(z_0) = \cdots = f^{(m-1)}(z_0)$ and $f^{(m)}(z_0) \neq 0$.*

Remark 21.2. *Note that if $f \equiv 0$ (identically zero), we cannot say f has zeros of any finite order, because there are no derivatives that are nonzero.*

Theorem 21.3. *Let f be analytic at z_0. Then the following statements are equivalent:*

1. *f has a zero of order m at z_0.*
2. *$f(z) = (z - z_0)^m g(z)$, where g is analytic and nonzero at z_0.*

This resembles the statement we had for a pole of order m.

Proof. We show 1 implies 2.

Let f have a zero of order m. Since f is analytic at z_0, it can be Taylor expanded in a neighborhood of z_0:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{21.10}$$

$$= \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m + \frac{f^{m+1}(z_0)}{(m+1)!}(z - z_0)^{m+1} + \dots \tag{21.11}$$

$$= (z - z_0)^m \underbrace{\left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!} + \dots\right]}_{\triangleq g(z)} \tag{21.12}$$

and we know $g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0$ and g is analytic as it is a Taylor series in a neighborhood around z_0.

To show 2 implies 1, we Taylor expand $g(z)$ about z_0, and inside of that Taylor expansion the first term corresponds to the nonzero term $f^{(m)}(z_0)$. ☐

Theorem 21.4. *Let two functions p, q be analytic at z_0. Let $p(z_0) \neq 0$ and let q have a zero of order m at z_0. Then $\frac{p(z)}{q(z)}$ has a pole of order m at z_0.*

Proof. (sketch) We can represent $q(z) = (z - z_0)^m g(z)$, where g is analytic and nonzero at z_0. Therefore, the quotient can be rewritten as

$$\frac{p(z)}{q(z)} = \frac{\frac{p(z)}{g(z)}}{(z - z_0)^m}, \tag{21.13}$$

where we know the numerator is analytic at z_0 and nonzero as it is the quotient of nonzero functions that are analytic at z_0. ☐

Theorem 21.5. *Let p, q be analytic at z_0, let $p(z_0) \neq 0, q(z_0) = 0, q'(z_0) \neq 0$, i.e. z_0 is a zero of order 1 of q. Then z_0 is a simple pole of $\frac{p(z)}{q(z)}$, and $\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$.*

Proof. By the previous theorem, z_0 is a simple pole of $\frac{p}{q}$, so it suffices to show that the residue is what was claimed above.

$$\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \text{Res}_{z=z_0} \frac{p(z)}{(z-z_0)g(z)} = \frac{p(z_0)}{g(z_0)}, \tag{21.14}$$

where $q(z) = (z - z_0)g(z)$. Take a derivative at $z = z_0$:

$$(z - z_0)g'(z) + g(z) = q'(z) \implies g(z_0) = q'(z_0). \tag{21.15}$$

Therefore $\text{Res}_{z=z_0} \frac{p(z_0)}{q'(z_0)}$. □

Lemma 21.6. *Let f be analytic at z_0 and let $f(z_0) = 0$. Then either $f \equiv 0$ in some neighborhood of z_0, or there exists some deleted neighborhood $0 < |z - z_0| < \epsilon$ such that $f(z) \neq 0$ in it.*

A simpler way of saying this is that analytic functions that are not identically zero have only isolated zeros.

Proof. Suppose f is not identically zero in any neighborhood of z_0, and z_0 is a zero of order $m \geq 1$. Then we can write $f(z) = (z - z_0)^m g(z)$ where g is analytic and nonzero at z_0. This tells us that g is continuous at z_0 and $g(z_0) \neq 0$. Therefore, there exists a neighborhood $B_\epsilon(z_0)$ such that $g(z) \neq 0$ for any $z \in B_\epsilon(z_0)$. So, for all $z \in B'_\epsilon(z_0)$ (i.e. $0 < |z - z_0| < \epsilon$) we just showed that $g(z) \neq 0$, and we know that $(z - z_0)^m \neq 0$, and so $f(z) \neq 0$, which is what we wanted. □

21.3 Analytic Continuation

We also call this Miracle #4!

Theorem 21.7. *A function analytic in a domain D is uniquely determined over D by its values on a smaller domain contained in D, or along a line segment contained in D.*

A different way of stating this is: Suppose f, g are analytic in D. If $f = g$ on a smaller domain in D, or on a line segment continued in D, then $f \equiv g$ in D.

Let f be defined in some subset of D. Its analytic continuation is a new function which

1. agrees with the old function in the original subset
2. but also makes sense elsewhere

Example 21.3. Consider the geometric series $\sum_{n=0}^{\infty} z^n$. This only converges for $|z| < 1$. But we know that within $|z| < 1$, this is equal to $\frac{1}{1-z}$, which exists as long as $z \neq 1$. Therefore $\frac{1}{1-z}$ is the analytic continuation of the power series in the domain $D = C \setminus \{1\}$.

□

Lemma 21.8. *Let f be analytic in a domain D. If $f = 0$ at each point of a domain/line segment contained in D, then $f \equiv 0$ in D.*

Proof. We will first prove this for a disk centered at z_0, where $f(z_0) = 0$. Let the disk be B. f is analytic inside of B, so we Taylor expand:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{21.16}$$

If z_0 is not an isolated zero (which it is not by the statement of the lemma), then there exists some neighborhood $B_\epsilon(z_0) \subseteq B$ such that $f \equiv 0$ in $B_\epsilon(z_0)$. Further, because f is exactly its Taylor series within B and therefore within $B_\epsilon(z_0)$, all of the Taylor series coefficients must be zero. Therefore $f = 0$ in B.

Next, we generalize to an arbitrary domain D. We use the connectedness of the domain: we can draw a multilinear path between z_0 and any $\tilde{z} \in D$, and around each point on this path we draw a ball in which the function is 0, so it is 0 throughout. □

21.4 Reflection Principle

Let f be analytic in some domain D that contains a segment of the x axis and whose lower half is symmetric to the upper half with respect to the x axis. If $f(x)$ is real for each x on this segmnet, then $\overline{f(z)} = f(\bar{z})$

Remark 21.9. *The above condition is generally not true; for instance, let $f(z) = z + i$. Then $f(\overline{z}) = \overline{z} + i$ but $\overline{f(z)} = \overline{z} - i$.*

Proof. Let $F(z) \triangleq \overline{f(\overline{z})}$. We want to show that F is analytic in D, and we further want to check that $f(x) = F(x)$ (analytic continuation using the segment of the $x-$axis).

Start with the fact that f is analytic at $z_0, \overline{z_0} inD$. Then $f(\overline{z}) = \sum_{n=0}^{\infty} a_n(\bar{z} - \bar{z_0})^n$, and $F(z) = \overline{f(\overline{z})} = \sum_{n=0}^{\infty} \overline{a_n}(z - z_0)^n$ in a neighborhood of z_0. Therefore $F(z)$ is analytic at z_0, and therefore analytic in D.

Further, since f is real-valued on the segment, we know that $f(x) = \overline{f(x)} = F(x)$. □

Math 185: Complex Analysis **Spring 2021**

Lecture 22: Behaviour of functions near isolated singularities

Lecturer: Di Fang *29 April* *Aditya Sengupta*

22.1 Isolated Singularities

Let's first look at how functions behave near removable singularities.

Theorem 22.1. *Let z_0 be a removable singularity of a function f. Then f is bounded and analytic in some deleted neighborhood $B'_\epsilon(z_0)$.*

We almost have the converse:

Theorem 22.2 (Riemann's theorem of removable singularities)**.** *Suppose f is bounded and analytic in some $B'_\epsilon(z_0)$ and f is not analytic at z_0. Then z_0 is a removable singularity of f.*

Proof. (of the first theorem) Let g be a function that matches f except at z_0, where we set $g(z_0)$ such that it is analytic in $B_\epsilon(z_0)$. Then g is continuous in $\overline{B_{\epsilon_0}}(z_0)$ where $0 < \epsilon_0 < \epsilon$, which implies g is bounded in $\overline{B_{\epsilon_0}}(z_0)$ and so f is bounded in the punctured neighborhood $\overline{B_{\epsilon_0}(z_0)} \setminus \{z_0\}$, and therefore f is bounded in $B'_\epsilon(z_0)$. □

Proof. Since f is analytic in the punctured neighborhood, there exists a Laurent series such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, z \in B'_\epsilon(z_0). \tag{22.1}$$

In order for z_0 to be a removable singularity, we would like to show that the b_ns are all 0, i.e.

$$\frac{1}{2\pi i} \int_C \frac{f(z)\mathrm{d}z}{(z - z_0)^{-n+1}} = 0. \tag{22.2}$$

To do this, we use the M-L estimate on the contour $|z - z_0 = \epsilon_0$ for some arbitrary ϵ_0 such that $0 < \epsilon_0 < \epsilon$. First, we know that because f is bounded, there exists some M such that $|f(z)| \leq M$ for all $z \in C$, so using that,

$$\begin{aligned} |b_n| &= \left| \frac{1}{2\pi} \int_C \frac{f(z)\mathrm{d}z}{(z - z_0)^{-n+1}} \right| \leq \frac{1}{2\pi} \underbrace{\frac{M}{\epsilon_0}^{-n+1}}_{M} \underbrace{2\pi\epsilon_0}_{L} && (22.3) \\ &= M\epsilon_0^n, && (22.4) \end{aligned}$$

i.e. by choosing $\epsilon_0 \to 0$, we get that $|b_n|$ is arbitrarily small and so $b_n = 0$, which completes the proof. □

22.2 Pole Singularities

Theorem 22.3. *If z_0 is a pole of f, then* $\lim_{z \to z_0} f(z) = \infty$.

This takes away the boundedness we relied on before.

Proof. It suffices to show $\lim_{z \to z_0} \frac{1}{f(z)} = 0$. Suppose f has z_0 as a pole of order m. Then, the function can be written as $f(z) = \frac{\phi(z)}{(z-z_0)^m}$, and so

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z-z_0)^m}{\phi(z)} = \frac{0}{\phi(z_0)} = 0. \tag{22.5}$$

□

22.3 Essential Singularities

Theorem 22.4 (Casorati-Weierstrass Theorem)**.** *Let z_0 be an essential singularity of f (this tells us that f is analytic in $B'_{\delta_0}(z)$ because essential singularities are isolated) and let ω_0 be any complex number. Then, for all $\epsilon > 0$ and for all $0 < \delta < \delta_0$, there exists $z \in B'_\delta(z_0)$ such that $|f(z) - \omega_0| < \epsilon$*

That seems powerful and cool! What does it mean?

Suppose there is a function mapping the $x - y$ plane to the u_v plane. Consider a $\delta-$neighborhood of some essential singularity z_0. Then, we can get $\epsilon-$close to *whatever point we want* in the $u - v$ plane by choosing some point within the $\delta-$neighborhood of the singularity. Essentially, this maps the small neighborhood around z_0 to the *entire* $u - v$ plane!

Another way of saying this is the image of $B'_\delta(z_0)$ under f is dense in $\mathbb{C}$.

Proof. Towards a contradiction, say there exists some ω_0 that is *not* attainable for all ϵ, δ in this way, i.e. there exists an $\epsilon > 0$ and some $0 < \delta < \delta_0$ such that for all $z \in B'_\delta(z)$, $|f(z) - \omega_0| \geq \epsilon$.

Let $g(z) \triangleq \frac{1}{f(z)-\omega_0}$. Since $f(z)-\omega_0 \neq 0$ (otherwise we'd be done), this function is analytic as it's the quotient of analytic, nonzero functions. Further, it is bounded: since $|f(z) - \omega_0| \geq \epsilon$, we have that $\left|\frac{1}{f(z)-\omega_0}\right| \leq \frac{1}{\epsilon}$. Therefore, by Riemann's theorem, z_0 is a removable singularity of g.

Redefine $g(z_0)$ to make it analytic in $B_\delta(z_0)$. Now we have two cases.

If $g(z_0) \neq 0$, then we write $f(z) = \omega_0 + \frac{1}{g(z)}$, which is analytic in $B_\delta(z_0)$ if we say this also holds at z_0. In other words, z_0 is a removable singularity of f, which is a contradiction!

If $g(z_0) = 0$, then either $g(z)$ is identically zero (contradiction, as there are no essential singularities) or z_0 is a zero of some finite order m, and so $g(z) = (z - z_0)^m \phi(z)$. z_0 is therefore a pole of order m of the function f, which is a contradiction! □

Remark 22.5. *(Great Picard theorem, intuitive statement) in each neighborhood of an essential singularity, the function takes values of every complex number infinitely many times, with at most one complex number that is not attainable.*

Math 185: Complex Analysis **Spring 2021**

Lecture 23: Applications of Residues

Lecturer: Di Fang *4 May* *Aditya Sengupta*

We can use residues to compute difficult improper integrals in $\mathbb{R}$, and to prove even more powerful theorems. Today, we'll look at the first of these.

There are four common forms for integrals that can be done using residues; only the first three will be on our final.

Recall that an improper integral has a form something like $\int_0^\infty f(x)\mathrm{d}x = \lim\limits_{R\to\infty} \int_0^R f(x)\mathrm{d}x$, or $\int_{-\infty}^\infty f(x)\mathrm{d}x = \lim_{R_1\to\infty} \int_0^\infty f(x)\mathrm{d}x + \lim_{R_2\to\infty} \int_{R_2}^0 f(x)\mathrm{d}x$. When $R_1 = -R_2$ in the second of these, we call the integral the *Cauchy principal value* (P.V.) of the integral.

$$P.V. \int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\mathrm{d}x. \tag{23.1}$$

Remark 23.1. *1. The P.V. of an integral over all of R is not necessarily equal to the integral itself. If $\int_{-\infty}^\infty f(x)\mathrm{d}x$ converges, it is equal to its principal value, but the converse does not hold.*

> ***Example 23.1.*** *Look at the integral $\int_{-\infty}^\infty x\mathrm{d}x = \lim\limits_{R_1\to\infty} \int_0^{R_1} x\mathrm{d}x + \lim\limits_{R_2\to-\infty} \int_{R_2}^0 x\mathrm{d}x$. Both parts of this evaluate to $\frac{x^2}{2}$ evaluated between 0 and infinity, so the integral does not converge. However, the principal value exists, as that goes to $\lim\limits_{R\to\infty} \frac{x^2}{2}\Big|_{-R}^{R} = 0$.*
>
> □

2. Suppose f is continuous on $(-\infty,\infty)$ and odd. Then the principal value of its integral is 0. If it is continuous and even, then its principal value is double the integral over just the positive reals, i.e. P.V. $\int_{-\infty}^\infty f(x)\mathrm{d}x = 2\int_0^\infty f(x)\mathrm{d}x$.

Now, we consider the four forms. We motivate each one with examples.

> **Example 23.2.** We know that $\int_{-\infty}^\infty \frac{1}{1+x^2}\mathrm{d}x = \pi$. This integrand is even, and we know it converges so it should be equal to its principal value.
>
> $$P.V. \int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{d}x}{1+x^2}. \tag{23.2}$$
>
> Consider $f(z) = \frac{1}{1+z^2}$ and $\tilde{C_R}$ to be the positively-oriented semicircle of radius R in the upper half-plane ($0 \le \theta \le \pi$). Then the contour integral $\oint_{\tilde{C_R}} f(z)\mathrm{d}z$ is closed and positively oriented. Split the curve into the arc of the circle C_R, and the real

axis (from $-R \to R$) L:

$$\oint_{\tilde{C_R}} f(z)\mathrm{d}z = \int_{C_R} f(z)\mathrm{d}z + \int_L f(z)\mathrm{d}z \underbrace{=}_{residuetheorem} 2\pi i \operatorname{Res}_{z=i} f(z), \tag{23.3}$$

where $z = i$ is a simple pole of $\frac{1}{1+z^2} = \frac{1/(z+i)}{z-i}$, so we can evaluate the residue by plugging in $z = i$ to $\frac{1}{z+i}$, so we get that the residue is $2\pi i \left.\frac{1}{z+i}\right|_{z+i} = 2\pi i \frac{1}{2i} = \pi$.

Having shown that the whole contour integral is π, we now want to show that the integral over C_R comes out to 0, so that we know that the integral over L (which is the real-valued integral we're interested in) comes out to π. We use the M-L estimate:

$$\left|\frac{1}{1+z^2}\right| \le \frac{1}{R^2-1}, |z| = R \left|\int_{C_R} f(z)\mathrm{d}z\right| \le \underbrace{\frac{1}{R^2-1}}_{M} \underbrace{\pi R}_{L} \xrightarrow{R\to\infty} 0. \tag{23.4}$$

and so we get the desired result.

□

This is form 1! In general, if we have $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\mathrm{d}x$ where the integrand is even with no singularities on the real axis, we first find all the poles in the upper half-plane and then compute the residues and apply the residue theorem.

$$\int_{-R}^{R} \cdots + \int_{C_R} \cdots = 2\pi i \left(\sum_j \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)}\right), \tag{23.5}$$

for poles at $z = z_j$. Finally, we let $R \to \infty$ and apply the M-L estimate to show the C_R integral is 0. For the M-L estimate, we need $\deg Q \ge \deg P + 2$. The reason for this is that $M = O\left(\frac{1}{R^{\deg Q - \deg P}}\right) = O\left(\frac{1}{R^a}\right), a \ge 2$ and $L = 2\pi R$, so their product is in $O\left(\frac{1}{R^{a-1}}\right)$, which implies $a - 1 \ge 1 \implies a \ge 2$.

For form 2, we look at functions of the form $\frac{P(x)}{Q(x)} \sin ax$ or $\frac{P(x)}{Q(x)} \cos ax$. This is a useful form for integrating Fourier series. Consider the function

$$f(z) = \frac{P(z)}{Q(z)} e^{iaz}, \tag{23.6}$$

which combines both at once - at the end we can take a real or imaginary part.

Along the upper-half plane, $|e^{iaz}| = |e^{ia(x+iy)}| = e^{-ay} \le 1$ and so this is bounded.

Example 23.3. Consider $\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2} \mathrm{d}x$. The answer is πe^{-a}, but how do we get there?

Consider the complex function $f(z) = \frac{e^{iaz}}{1+z^2}$ This has a UHP singularity at $z = i$. Split this as we did before:

$$\int_{C_R} f(z)\mathrm{d}z + \int_{-R}^{R} \frac{e^{iax}}{1+x^2}\mathrm{d}x = 2\pi i \operatorname{Res}_{z=i} \frac{e^{iaz}}{1+z^2} = 2\pi i \left.\frac{e^{iaz}}{z+i}\right|_{z+i} \tag{23.7}$$

$$= 2\pi i \frac{e^{-a}}{2i} = \pi e^{-a}. \tag{23.8}$$

Now it remains to show the C_R integral is 0. This is the case, similar to the above example:

$$\left|\int_{C_R} f(z)\mathrm{d}z\right| \le \underbrace{\frac{1}{R^2-1}}_{M} \underbrace{\pi R}_{L} \xrightarrow{R\to\infty} 0. \tag{23.9}$$

Therefore, we get that $\int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\mathrm{d}x = \pi e^{-a}$. Taking the real part, we get the desired result. Note that taking the imaginary part shows us that $\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\mathrm{d}x = 0$; this should be expected as sin is odd.

□

Generally, for M-L to work in these cases, we need $\deg Q \ge \deg P + 2$. Can we generalize this a bit? Yes; if $\deg Q = \deg P + 1$, we can still show that $\int_{C_R} f(z)\mathrm{d}z \to 0$, but not via the M-L estimate; the best bound that can be furnished by the M-L estimate is $O(1)$ whereas we would need at least $O(1/R)$. However, we can use the Jordan lemma to show this instead.

We first state the Jordan inequality: $\int_0^{\pi} e^{-R\sin\theta}\mathrm{d}\theta < \frac{\pi}{R}$.

Proof. Let $I \triangleq \int_0^{\pi} e^{-R\sin\theta}\mathrm{d}\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\mathrm{d}\theta$. Between 0 and $\frac{\pi}{2}$, we get that $\sin\theta > \frac{2\theta}{\pi}$, so $e^{-R\sin\theta} < e^{-\frac{2\theta}{\pi}R}$. Therefore

$$I < 2\int_0^{\frac{\pi}{2}} e^{-\frac{2\theta}{\pi}R}\mathrm{d}\theta = -2\frac{\pi}{2}\left. e^{-\frac{2\theta}{\pi}R}\right|_{\theta=0}^{\theta=\pi/2} \tag{23.10}$$

$$= \frac{\pi}{R}(1-e^{-R}) < \frac{\pi}{R}. \tag{23.11}$$

□

Lemma 23.2 (Jordan lemma)**.** *Let $C_R : z = Re^{i\theta}, R > 0, \theta \in [0,\pi]$. Then the following bound holds:*

$$I = \int_{C_R} g(z)e^{iaz}\mathrm{d}z \tag{23.12}$$

$$|I| \le M_R\frac{\pi}{a}, \tag{23.13}$$

where $M_R = \max_{z\in C_R} |g(z)|$.

Proof.

$$I = \int_{C_R} g(z)e^{iaz}\mathrm{d}z = \int_0^\pi g(Re^{i\theta})e^{iaRe^{i\theta}}Rie^{i\theta}\mathrm{d}\theta \tag{23.14}$$

We do something similar to the M-L estimate, except for a regular parameterized integral rather than a contour integral: bound $g(Re^{i\theta})$ by M_R, bound $|e^{iaRe^{i\theta}}| \leq e^{-aR\sin\theta}$ and bound $|Rie^{i\theta}| = R$.

$$|I| \leq \int_0^\pi M_R R e^{-aR\sin\theta} \underbrace{<}_{Jordanineq} M_R R\frac{\pi}{aR} = \frac{M_R\pi}{a}, \tag{23.15}$$

as desired. □

This inequality works as long as $M_R = O(1/R)$, i.e. $\deg Q = \deg P + 1$ at least.

Example 23.4. Consider $\int_0^\infty \frac{x\sin(2x)}{x^2+3}\mathrm{d}x$. This is not over all $\mathbb{R}$ but it is even, so we can proceed. The singularity is at $z = \sqrt{3}i$, and so

$$\int_{C_R} f(z)\mathrm{d}z + \int_{-R}^{R} f(x)\mathrm{d}x = 2\pi i \operatorname{Res}_{z=\sqrt{3}i} f(z) = 2\pi i \left.\frac{e^{i2z}z}{z+\sqrt{3}i}\right|_{z=\sqrt{3}i} = \pi i e^{-2\sqrt{3}}. \tag{23.16}$$

It remains to show that the C_R integral is 0, which is the case here by the Jordan inequality:

$$\left|\int_{C_R} f(z)\mathrm{d}z\right| \leq \frac{R}{R^2-3}\frac{\pi}{2} \xrightarrow{R\to\infty} 0. \tag{23.17}$$

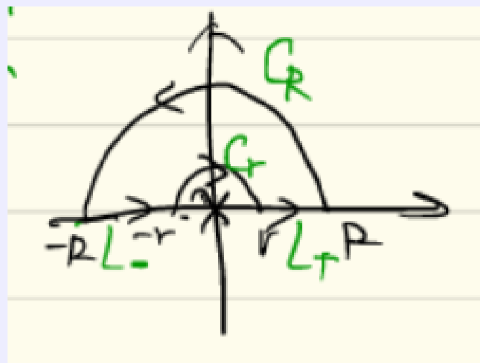
For the final integral value, we take the imaginary part of the result from the residue divided by 2:

$$I = \frac{\pi}{2}ie^{-2\sqrt{3}}. \tag{23.18}$$

□

For form 3, we look at integrals where the integrand f has a pole in $\mathbb{R}$. The residue theorem no longer applies in this case. To fix this, we take an indented path with a little semicircular detour around the pole.

Example 23.5. Consider the Dirichlet integral $\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$. We take $f(z) = e^{iz}z$



This does not actually include any poles, so the contour integral actually just reduces to the Cauchy-Goursat theorem:

$$0 = \int_{C_R} f(z)\mathrm{d}z + \int_{C_r} f(z)\mathrm{d}z + \int_{-R}^{-r} f(x)\mathrm{d}x + \int_r^R f(x)\mathrm{d}x. \tag{23.19}$$

Send $R \to \infty, r \to 0$. Using the Jordan lemma, the C_R integral has no contribution:

$$\left|\int_{C_R} \frac{e^{iz}}{z}\mathrm{d}z\right| \underbrace{\leq}_{\text{Jordan lemma}} \frac{1}{R}\pi \xrightarrow{R\to\infty} 0. \tag{23.20}$$

Now, we look at the second term. We claim the following:

$$\lim_{r\to 0}\int_{C_r} \frac{e^{iz}}{z}\mathrm{d}z = -\pi i. \tag{23.21}$$

We prove the claim by expanding the integrand in a power series:

$$\frac{e^{iz}}{z} = \frac{1}{z}(1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \dots) \tag{23.22}$$

$$= \frac{1}{z} + \underbrace{i + \frac{i^2}{2!}z + \frac{i^3}{3!}z^2 + \dots}_{\text{analytic part,}\triangleq h(z)} \tag{23.23}$$

and we can integrate by terms to get

$$\int_{C_r} \frac{e^{iz}}{z} \mathrm{d}z = \int_{C_r} \frac{1}{[} z] \mathrm{d}z + \int_{C_r} h(z) \mathrm{d}z. \tag{23.24}$$

We do this by parameterization, noting that C_r is negatively oriented:

$$\int_{C_r} \frac{1}{z} \mathrm{d}z = -\int_{-C_r} \frac{1}{z} \mathrm{d}z = -\int_0^{\pi} \frac{1}{re^{i\theta}} rie^{i\theta} \mathrm{d}\theta = -\pi i, \tag{23.25}$$

where $-C_r : z = re^{i\theta}, \theta \in [0, \pi]$.

Next, we look at the analytic term. Since $h(z)$ is analytic in $|z| < \infty$, it is continuous on C_r, so $|h(z)| \leq M_0$ on C_r.

$$\left| \int_{C_r} h(z) \mathrm{d}z \right| \leq M_0 \pi r \xrightarrow{r \to 0} 0. \tag{23.26}$$

Therefore, $\lim\limits_{r \to 0} \int_{C_r} \frac{e^{iz}}{z} \mathrm{d}z = -\pi i$.

Now, we're left with the two real parts, which sum to the real integral we are interested in. We work with the more general complex form, which we can split apart later:

$$\int_{-\infty}^{0} \frac{e^{ix}}{x} \mathrm{d}x + \int_0^{\infty} \frac{e^{ix}}{x} \mathrm{d}x = \int_{-\infty}^{0} \frac{\cos x}{x} \mathrm{d}x + \int_0^{\infty} \frac{\cos x}{x} \mathrm{d}x + i \left(\int_{-\infty}^{0} \frac{\sin x}{x} \mathrm{d}x + \int_0^{\infty} \frac{\sin x}{x} \mathrm{d}x \right), \tag{23.27}$$

where the last term is even and therefore sums to $2 \int_0^{\infty} \frac{\sin x}{x} \mathrm{d}x$. Therefore,

$$0 = 0 - \pi i + \int_{-\infty}^{\infty} \frac{\cos x}{x} \mathrm{d}x + i2 \int_0^{\infty} \frac{\sin x}{x} \mathrm{d}x. \tag{23.28}$$

Take the imaginary part:

$$0 = -\pi + 2 \int_0^{\infty} \frac{\sin x}{x} \mathrm{d}x, \tag{23.29}$$

which gives us

$$\int_0^\infty \frac{\sin x}{x} \mathrm{d}x = \frac{\pi}{2} \tag{23.30}$$

as desired.

□

Skipping form 4 for now for time (out of scope.)