Mathematics 1B, Calculus II at UC Berkeley

Part I: Techniques of Integration (Lectures 1-9)

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Contents

1 Lecture 1: Overview, Integration by Parts

1.1 Overview

This course is divided into three overall parts:

1.1.1 Techniques of Integration

- How to compute $\int f(x)dx$ for many different types of functions $f(x)$
- How to compute the definite integral $\int_b^a f(x)dx$ approximately
- $\int_a^\infty f(x)dx$

etc.

1.1.2 Sequences and Series

- Convergence, divergence, absolute and conditional convergence of series and sequences.
- Power series
- Taylor series

1.1.3 Differential Equations

- Separation of variables, integrating factor method
- Homogeneous second-order differential equations
- Non-homogeneous differential equations

1.2 Review of Math 1A

1.2.1 Continuity

A function $f(x)$ is continuous at $x = a$ if

- $f(a)$ is defined
- $\lim\limits_{x \to a} f(x) = f(a)$

$f(x)$ is continuous on a closed interval $[c, d]$ if it is continuous at every point on that interval.

1.2.2 Differentiation

$$f'(a) = \frac{df(a)}{dx} := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

When the above limit exists, $f(x)$ is said to be *differentiable* at $x = a$.

$f(x)$ is differentiable on a closed interval $[c, d]$ if it is differentiable at each point of $[c, d]$.

1.3 Rules of differentiation and integration

Recall three basic rules of differentiation:

$$(f + g)' = f' + g'$$

$$f(g(x)) = f'(g(x))g(x)$$

$$(fg)' = f'g + fg'$$

Integration has three corresponding rules,

$$\int (f + g)dx = \int f dx + \int g dx$$

$$\int f(g(x))'dx = \int f'(g(x))g'(x)dx = f(g(x)) + C$$

The third corresponding rule is referred to as "integration by parts" and is covered in a later section of this lecture.

1.4 Basics of Integration

The *indefinite integral* or *antiderivative* of a function $f(x)$ is defined as the function $F(x)$ given that

$$F'(x) = f(x)$$

Note that in general, we can add an arbitrary constant C to $F(x)$ without affecting this relation, as constants vanish during differentiation of the antiderivative to get the function back again.

For example, if $f(x) = x^n$,

$$F(x) = \int f(x)dx = \frac{x^{n+1}}{n+1} + C$$

In general, finding $F(x)$ is difficult, and is sometimes not possible analytically, e.g. for $f(x) = e^{-x^2}$.

1.5 The definite integral

Geometrically, the antiderivative of a function is the area under the curve defined by that function. We can find the area enclosed between two specific limits - call the lower limit a and the upper limit b - by taking the area of an infinitesimally small slice and adding up the areas of each slice between a and b.

Let there be n slices, where $x_0 = a$ and $x_n = b$. Then, the width of the i-th slice is

$$\Delta x_i = x_i - x_{i-1}$$

Then, to find the area of the slice, we evaluate the function $f(x)$ at some intermediate value x_i^* where $x_{i-1} \leq x_i^* \leq x_i$.

The value of the area under the curve is then

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

and we define the definite integral as this value as n goes to infinity:

$$\int_a^b f(x)dx := \lim_{n\to\infty,\ \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

1.6 Fundamental Theorem of Calculus

If $F(x)$ is an antiderivative of $f(x)$, then

$$F(x)\Big|_a^b := F(b) - F(a) = \int_a^b f(x)dx$$

The key point in this is that it does not matter which of *infinitely many* antiderivatives is chosen for this. In indefinite integrals, an arbitrary constant is added to represent the entire set of possible antiderivatives, but this is not a consideration in definite integration.

1.7 Integration by Parts

We start with the product rule for differentiation, stated above as

$$(fg)' = f'g + fg'$$

Then, as stated above, the antiderivative of the sum of two functions is the sum of the individual antiderivatives of each function; therefore, we can integrate this expression on both sides:

$$\int (fg)'dx = \int f'gdx + \int fg'dx$$

The left-hand side of this expression corresponds simply to fg, as we are taking the integral of a derivative of the original function. Therefore, we can rearrange this:

$$\int fg'dx = fg - \int f'gdx$$

This gives us a way to simplify the antiderivatives of the products of some functions. In general, we want to choose f and g so that f' is "simpler" than f.

1.8 Examples

1

$$I = \int \ln x dx = \int \ln x(x')dx$$

We choose $f = \ln x$, and $g' = dx = (x')dx$:

$$I = x\ln x - \int (\ln x)' x dx = x\ln x - \int \frac{1}{x} x dx$$

$$= x\ln x - x + C$$

2

$$I = \int x \sin x dx$$

Choose $f = x$, $g' = \sin x dx$:

$$I = -x\cos x - \int (-\cos x) dx = -x\cos x + \sin x + C$$

3

$$I = \int x e^x dx$$

We choose $f = x$, and $g' = e^x dx$:

$$I = xe^x - \int e^x dx = (x-1)e^x + C$$

Note: a more compact way of writing the general form of integration by parts is $\int udv = uv - \int vdu$, in which you choose $g' = dv$ and find v as the antiderivative of g' or dv. This makes no functional difference, but may be a little easier to understand.

2 Lecture 2: Trigonometric Integrals

2.1 A specific case of Integration by Parts

$$I = \int e^x \sin x dx$$

Choose $f = e^x$, $g' = \sin x dx$:

$$I = -e^x \cos x - \int -e^x \cos x dx$$

This requires that we use integration by parts again, with $f = e^x$ and $g' = \cos x dx$:

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

The integral in this is the original problem, which we represented by I. Therefore,

$$I = -e^x \cos x + e^x \sin x - I \implies 2I = e^x(\sin x - \cos x)$$

So the original integral is

$$I = \frac{e^x}{2}(\sin x - \cos x) + C$$

Note that the practice of referring to the value of the required antiderivative as I is not standard for this course, but as can be seen above, it can simplify problems.

2.2 Trigonometric Integrals

This name refers to any integrals involving the functions $\cos x$ or $\sin x$. They can frequently be simplified using trigonometric identities:

2.2.1 A first example

$$I = \int \sin^3 x dx = \int (1 - \cos^2 x) \sin x dx$$

$(-\cos x)' = \sin x dx$, therefore,

$$I = \int (1 - \cos^2 x)(-\cos x)' = \int (\cos^2 x - 1)(\cos x)'$$

We make the substitution $u = \cos x$:

$$I = \int (u^2 - 1) du = \frac{u^3}{3} - u + C$$

Now, reverse the substitution:

$$I = \frac{\cos^3 x}{3} - \cos x + C$$

2.3 Trigonometric identities

The above example used the identity $\sin^2 x + \cos^2 x = 1$. Other important identities include:

- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(x + y) = \sin x \cos y + \sin y \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$

Also noteworthy is Euler's formula, which is not strictly necessary but may still be useful:

$$e^{ix} = \cos x + i \sin x$$

This can be restated as:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Some other important formulas are:

$$(\tan x)' = \frac{1}{\cos^2 x}$$
$$\tan^2 x = \sec^2 x - 1, \sec x = \frac{1}{\cos x}$$
$$(\sec x)' = \sec x \tan x$$

2.3.1 Examples using identities

1

$$I = \int \sin^2 x dx = \int \frac{1}{2}(1 - \cos 2x) dx$$

$$I = \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x dx$$

$$I = \frac{x}{2} - \frac{1}{4} \int \cos(2x) d(2x)$$

$$\therefore I = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

2

$$\int \frac{dx}{\cos^2 x} = \int (\tan x)' dx = \tan x + C$$

2.4 General forms of trigonometric integrals

1

$$\int \sin^m x \cos^{2k+1} x dx = \int \sin^m x \cos^{2k} x \cos x dx$$

$$= \int \sin^m x (1 - \sin^2 x)^k \cos x dx$$

Substitute $u = \sin x,\ u' = \cos x dx$:

$$= \int u^m (1 - u^2)^k du$$

upon which it reduces to a polynomial, which can be easily solved and the substitution reversed using $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

2

$$\int (\sin x)^{2k+1} \cos^n x dx = \int \cos^n x(1 - \cos^2 x)^k \sin x dx$$

As above, use the substitution $u = cosx$, and the expression reduces to a polynomial.

3

$$I = \int (\tan x)^m (\sec x)^{2k} dx (k \geq 2)$$

$$I = (\tan x)^m (\sec x)^{2k-2} \sec^2 x dx$$

$$I = (\tan x)^m (\tan^2 x + 1)^{k-1} d(tanx)$$

Substitute $u = \tan x$ and solve the polynomial antiderivative.

2.5 A note on trigonometric integrals

It is possible for one integral to be solved in multiple ways, which may result in different-looking answers which are really the same. For example:

$$I = \int \frac{dx}{\cos x} = \int \frac{\cos x dx}{\cos^2 x} = \int \frac{d(\sin x)}{1 - \sin^2 x} = \int \frac{du}{1 - u^2} = \frac{1}{2}(\log(u+1) - \log(1-u)) + C$$

The last step above was done using partial fractions, which will be covered later. Re-substituting $u = \sin x$, we get

$$I = \frac{\log(\sin x + 1) - \log(1 - \sin x)}{2} + C$$

Alternatively, we can multiply and divide by $\sec x + \tan x$:

$$I = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

$$I = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

Let $u = \sec x + \tan x$, then $\frac{du}{dx} = \sec x \tan x + \sec^2 x$:

$$I = \int \frac{du}{u} = \log u + C = \log(\sec x + \tan x) + C$$

It is possible to show that these two expressions are really the same, even though the procedure for both is very different.

3 Lecture 3: Trigonometric Substitutions

3.1 Overall Goal

Suppose we have a function of the form

$$\int R(x)\sqrt{Ax^2 + Bx + C}dx$$

where $R(x)$ is any rational function. Then, our goal is to convert this into a trigonometric integral of the form

$$\int F(\cos x, \sin x)dx$$

since in general these functions are easier to compute.

3.2 Rational Functions

A *rational function* is any function of the form

$$R(x) = \frac{P(x)}{Q(x)}$$

where P, Q are polynomials, which means they can be represented as $f(x) = \sum_{k=1}^{n} a_k x^k$ with the condition that $a_n \neq 0$.

3.3 Procedure for trigonometric substitution

1: Complete the square

$$ax^2 + bx + c = |a|\Big(\pm \Big(x + \frac{b}{2a}\Big)^2 + \Big(\frac{c}{|a|}\Big) \mp \frac{b^2}{4a^2}\Big)$$

We define new variables X and A, as follows:

$$X = x + \frac{b}{2a}$$

$$\pm A^2 = \frac{c}{|a|} \mp \frac{b^2}{4a^2}$$

Then, the original expression ax^2+bx+c becomes one of three expressions, which we convert into a trigonometric expression according to the table in the next step.

2: Substitute

Expression	**Substitution**	**Result**
$\sqrt{A^2 - X^2}$	$X = A\sin\theta$	$A\cos\theta$
$\sqrt{A^2 + X^2}$	$X = A\tan\theta$	$A\sec\theta$
$\sqrt{X^2 - A^2}$	$X = A\sec\theta$	$A\tan\theta$

3.4 Examples

1

$$I = \int \sqrt{1 - x^2} dx$$

We consider the substitution $x = \sin\theta$, then $dx = \cos\theta d\theta$:

$$I = \int \sqrt{1 - \sin^2\theta} \cos\theta = \int \cos^2\theta d\theta$$

$$I = \int \frac{1}{2}(1 + \cos 2\theta) d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

Then we reverse the substitution by setting $\theta = \arcsin x$:

$$I = \frac{\arcsin x}{2} + \frac{2\sin(\arcsin x)\cos(\arcsin x)}{4} + C$$

$$I = \frac{\arcsin x}{2} + \frac{x\sqrt{1 - x^2}}{2} + C$$

2

$$I = \int \frac{dx}{\sqrt{1+x^2}}$$

We substitute $x = \tan\theta$, then $dx = \sec^2\theta d\theta$:

$$I = \int \frac{\sec^2\theta d\theta}{\sqrt{1+\tan^2\theta}} = \int \sec\theta d\theta$$

$$I = \ln(\sec\theta + \tan\theta) + C$$

Then, we reverse the substitution:

$$I = \ln(\sec(\arctan x) + \tan(\arctan x)) + C = \ln(\sqrt{x^2+1} + x) + C$$

3

$$I = \int \frac{dx}{\sqrt{9x^2+6x-8}}$$

In this case, we have to start by completing the square:

$$9x^2 + 6x - 8 = (3x)^2 + 2(3x)(1) + 1^2 - 9 = (3x+1)^2 - 3^2$$

We can do one substitution to directly get a trigonometric integral, but it may be easier to substitute twice, as follows: $u = 3x + 1$, $3dx = du$:

$$I = \int \frac{du/3}{\sqrt{u^2-9}} = \frac{1}{3}\int \frac{du}{\sqrt{u^2-9}}$$

Then, the next substitution: let $u = 3\sec\theta$, then $du = 3\sec\theta\tan\theta d\theta$, and:

$$I = \frac{1}{3}\int \frac{3\sec\theta\tan\theta d\theta}{\sqrt{9\sec^2\theta - 9}} = \frac{1}{3}\int \frac{3\sec\theta\tan\theta d\theta}{3\tan\theta} = \frac{1}{3}\int \sec\theta d\theta$$

As explained in section 2.5, this is equivalent to

$$I = \frac{1}{3}\Big(\log(\sec\theta + \tan\theta)\Big) + C$$

We then reverse two substitutions: take $\theta = \sec^{-1} u$, then

$$I = \frac{1}{3}\Big(\log(u + \sqrt{u^2 - 1})\Big) + C$$

Now, take $u = 3x + 1$:

$$I = \frac{1}{3}\Big(\log(3x + 1 + \sqrt{9x^2 + 6x - 8})\Big) + C$$

Note that it is possible to do this with one substitution, $3x + 1 = 3\sec\theta$, but multiple substitutions (although it may be more time consuming) are less prone to error.

4

$$I = \int \frac{x-2}{x+2}\frac{dx}{\sqrt{1-x^2}}$$

Substitute $x = \sin\theta$, then $dx = \cos\theta d\theta$:

$$I = \int \frac{\sin\theta - 2}{\sin\theta + 2}\frac{\cos\theta d\theta}{\cos\theta}$$

$$I = \int \frac{\sin\theta - 2}{\sin\theta + 2}d\theta$$

This function can be split as follows:

$$I = \int \Big(1 - \frac{4}{\sin\theta + 2}\Big)d\theta$$

$$I = \theta - 4\int \frac{d\theta}{\sin\theta + 2}$$

We use the standard Weierstrass substitution (see next section), $t = \tan\frac{\theta}{2}$ and $d\theta = \frac{2dt}{1+t^2}$:

$$I = \theta - 4\int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2} + 2}$$

Then, multiplying throughout by $1 + t^2$:

$$I = \theta - 4\int \frac{dt}{1 + t + t^2}$$

Since θ is out of the integral, we do not need to change it so that it is in terms of t; once we reverse all the substitutions, it will convert back into terms of x either way.

Completing the square yields:

$$I = \theta - 4\int \frac{dt}{(t+\frac{1}{2})^2 + \frac{3}{4}}$$

This suggests the substitution $u = \frac{t+\frac{1}{2}}{\sqrt{3}/2} = \frac{2t+1}{\sqrt{3}}$, $dt = \frac{\sqrt{3}du}{2}$:

$$I = \theta - 4\int \frac{\sqrt{3}du/2}{\frac{3u^2}{4} + \frac{3}{4}}$$

$$I = \theta - \frac{4\sqrt{3}/2}{3/4}\int \frac{du}{u^2+1}$$

$$\therefore I = \theta - \frac{8}{\sqrt{3}}\arctan u + C$$

Reversing substitutions,

$$I = \theta - \frac{8}{\sqrt{3}}\arctan\left(\frac{2t+1}{\sqrt{3}}\right) + C$$

$$I = \theta - \frac{8}{\sqrt{3}}\arctan\left(\frac{2\tan(\frac{\theta}{2})+1}{\sqrt{3}}\right) + C$$

$$\therefore I = \arcsin x - \frac{8}{\sqrt{3}}\arctan\left(\frac{2\tan(\frac{\arcsin x}{2})+1}{\sqrt{3}}\right) + C$$

3.5 Weierstrass Substitution

When a trigonometric substitution is required, but no selection of new variable is apparently correct, we can use the Weierstrass substitution as a "silver bullet":

$$t = \tan\left(\frac{\theta}{2}\right) \implies d\theta = \frac{1}{2}\sec^2\frac{\theta}{2}dx \implies d\theta = \frac{2dt}{1+t^2}$$

The advantage of this substitution is that it always gives a relatively simple expression for $\sin\theta$ and $\cos\theta$, as follows:

$$\sin\theta = \frac{2t}{1+t^2},\ \cos\theta = \frac{1-t^2}{1+t^2}$$

The result, in general, is:

$$\int R(\sin\theta,\cos\theta)d\theta = \int R\Big(\frac{1-t^2}{1+t^2},\frac{2t}{1+t^2}\Big)\frac{2dt}{1+t^2}$$

which is a rational function in t. In the next lecture, we will cover a method to simplify rational functions.

4 Lecture 4: Partial Fraction Decomposition

4.1 Step 1: Long Division

Suppose we have a function $f = \frac{P}{Q}$ in which P and Q are polynomials of the form

$$P(x) = \sum_{k=0}^{n} a_k x^k,\ a_n \neq 0$$

$$Q(x) = \sum_{k=0}^{m} a_k x^k,\ a_m \neq 0$$

We can then carry out long division to obtain the form

$$f = S + \frac{R}{Q}$$

where the degree of R is strictly less than that of Q, yielding a *proper rational function.*

Example 1

$$\frac{x^2+x+1}{x-1} = \frac{x(x-1)+2x+1}{x-1} = \frac{x(x-1)+2(x-1)+3}{x-1} = x+2+\frac{3}{x-1}$$

Example 2

$$\frac{x^3+1}{x^2+x+1} = \frac{x(x^2+x+1)-x^2-x+1}{x^2+x+1} = \frac{x(x^2+x+1)-1(x^2+x+1)+2}{x^2+x+1}$$

$$= x-1+\frac{2}{x^2+x+1}$$

4.2 Step 2: Fundamental Theorem of Algebra

Any polynomial with real coefficients can be expressed as a product, as follows:

$$Q(x) = (a_1x+b_1)^{r_1}(a_2x+b_2)^{r_2}...(c_1x^2+d_1x+e_1)^{s_1}(c_2x^2+d_2x+e_2)^{s_2}...$$

The polynomials of the form $cx^2 + dx + e$ are not decomposable any further.

In general, for quadratic polynomials, they can be decomposed as follows:

$$ax^2+bx+c = a(x-x_1)(x-x_2)$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

When $b^2 - 4ac > 0$, the solutions are both real. When $b^2 - 4ac = 0$, the solutions are both real and equal. When $b^2 - 4ac < 0$, the solutions are complex, i.e. for our purposes it cannot be decomposed.

4.3 Step 3: Decomposition

Assuming that $\frac{P}{Q}$ is proper, then it can be decomposed:

$$\frac{P}{Q} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_2x+b_2)^2}+...+\frac{A_{r_1}}{(a_1x+b_1)^{r_1}}+\frac{B_1x+C_1}{c_1x^2+d_1x+e_1}+...+\frac{B_{s_1}x+C_{s_1}}{(c_1x^2+d_1x+e_1)^{s_1}}+...$$

Example 1

$$\frac{1}{2x^2+3x-2} = \frac{1}{(2x-1)(x+2)} = \frac{A}{2x-1} + \frac{B}{x+2}$$

To find the coefficients, we multiply by $(2x-1)(x+2)$ throughout:

$$1 = A(x+2) + B(2x-1) = (A+2B)x + (2A-B)$$

$$A+2B = 0,\ 2A-B = 1 \implies A = \frac{2}{5},\ B = \frac{-1}{5}$$

This means the decomposition into partial fractions is

$$\frac{2/5}{2x-1} - \frac{1/5}{x+2}$$

Example 2

$$\frac{x}{x^3-1} = \frac{x}{(x-1)(x^2+x+1)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}$$

$$x = (Ax+B)(x-1) + C(x^2+x+1) = (A+C)x^2 + (B-A+C)x + (-B+C)$$

$$\therefore A+C = 0,\ B-A+C = 1,\ -B+C = 0$$

$$A = -\frac{1}{3},\ B = C = \frac{1}{3}$$

$$\frac{x}{x^3-1} = \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+x+1} + \frac{\frac{1}{3}}{x-1}$$

5 Lecture 5: Using PFD to Solve Integrals

Once we use partial fraction decomposition to simplify a high-degree polynomial, it is relatively easy to integrate each of the resultant fractions.

Example 1

$$I = \int \frac{dx}{1-x^2}$$

We decompose this into partial fractions as follows:

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \implies 1 = A(1+x) + B(1-x)$$

$$A - B = 0,\ A + B = 1 \implies A = B = \frac{1}{2}$$

Therefore,

$$I = \int \frac{dx}{1-x^2} = \int \frac{dx}{2(1-x)} + \int \frac{dx}{2(1+x)}$$

Evaluating both of these and adding the result, we get

$$I = \frac{\log(1+x) + \log(1-x)}{2} + C$$

Example 2

$$I = \int \frac{dx}{x^4-1}$$

$$\frac{1}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$1 = A(x^2+1)(x+1) + B(x^2+1)(x-1) + (Cx+D)(x^2-1)$$

$$1 = A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + Cx^3 + Dx^2 - Cx - D$$

$$1 = (A+B+C)x^3 + (A-B+D)x^2 + (A+B-C)x + (A-B-D)$$

$$\therefore A = \frac{1}{4},\ B = \frac{-1}{4},\ C = 0,\ D = \frac{-1}{2}$$

$$I = \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

Integrating term by term,

$$I = \frac{1}{4}\log|x-1| - \frac{1}{4}\log|x+1| - \frac{1}{2}\arctan x + C$$

Example 3

$$I = \int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$$

We split this using partial fraction decomposition,

$$I = \int \Big(\frac{1}{2x} + \frac{1}{5(2x+1)} - \frac{1}{10(x+2)}\Big)dx$$

$$I = \frac{1}{2}\ln|x| + \frac{1}{5}\frac{1}{2}\ln|2x+1| - \frac{1}{10}\ln|x+2| + C$$

$$\therefore I = \frac{1}{2}\ln|x| + \frac{1}{10}\ln\left|\frac{2x-1}{x+2}\right| + C$$

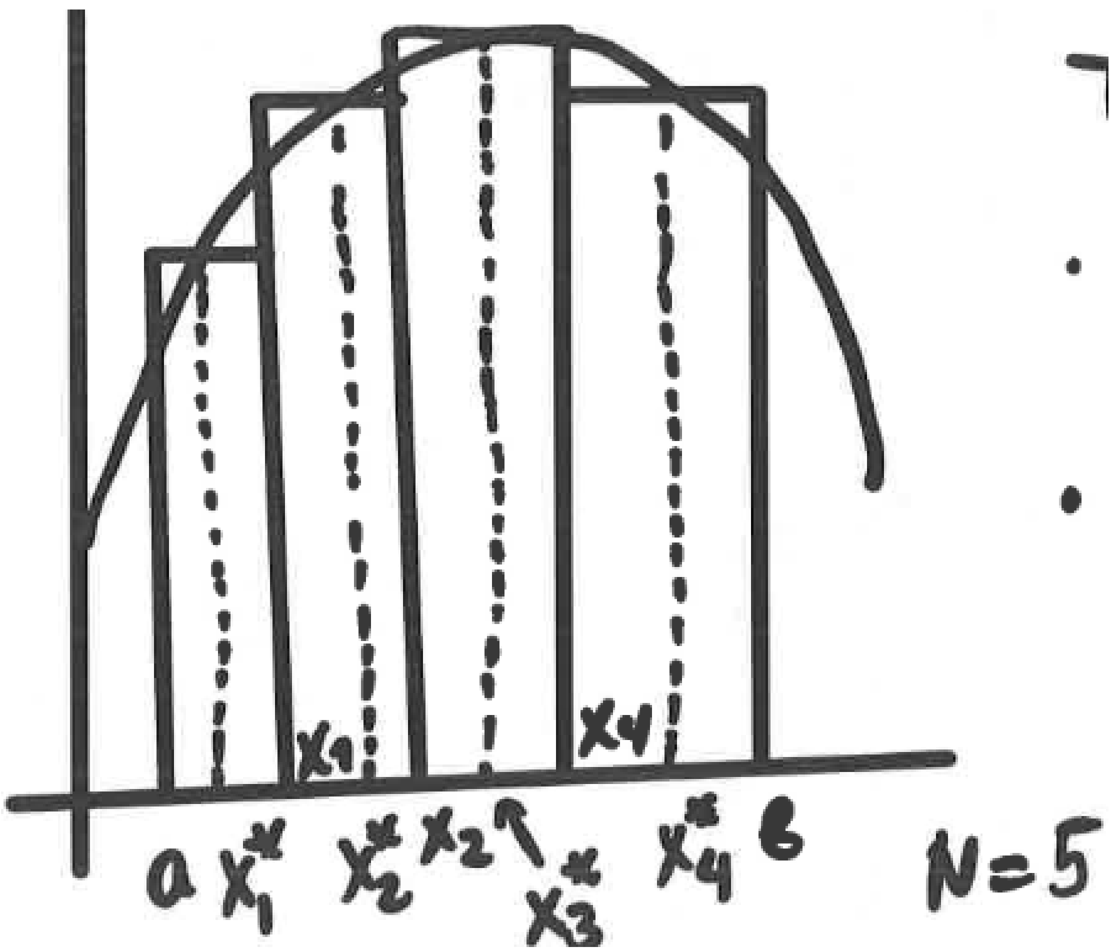
Note: the rest of this lecture was not exactly relevant to the course; the residue theorem (in Math 185) was briefly introduced, but it is not at all required for Math 1B.

6 Lecture 6: Approximate Integration

Some integrals are impossible to compute analytically, such as $\int e^{-x^2}dx$. However, a corresponding definite integral can be computed approximately, as follows:

$$\int_a^b f(x)dx = \lim_{n\to\infty,\ \Delta x_i\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

This definition of the definite integral was covered in Lecture 1. The difference here is that instead of taking limits, e.g. $n \to \infty$, we take particular values and see what the value of the definite integral approaches.



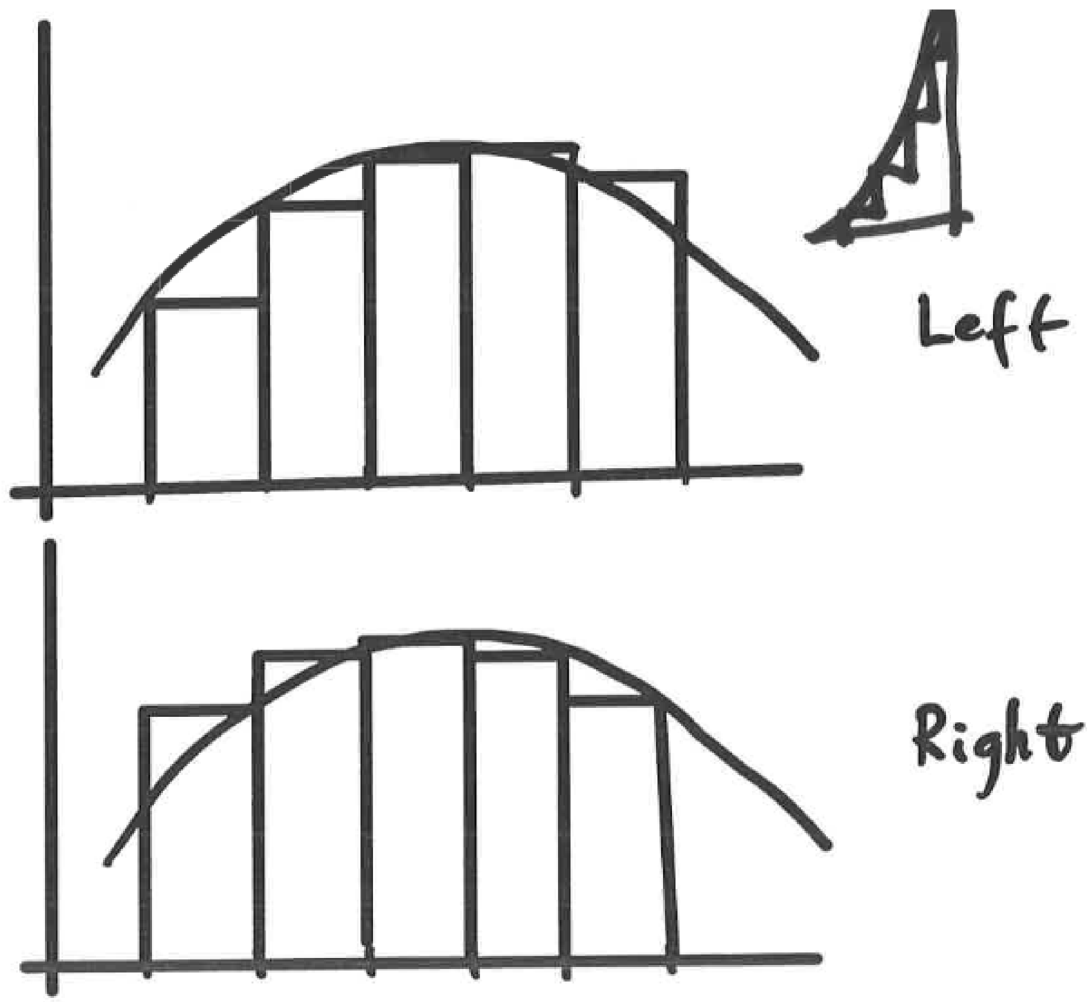
6.1 Left/Right End Point Approximation

If we choose a value of N, the rest of the approximation follows:

$$\Delta x_i = \frac{b-a}{N},\ x_i = a + \frac{i}{N}(b-a),\ i = 0, 1, ...N$$

Left Approximation: $x_i^* = x_i$, **Right Approximation**: $x_i^* = x_{i+1}$
Then, the approximations become

$$L_n = \sum_{i=0}^{n-1} f(x_i)\frac{b-a}{n},\ R_n = \sum_{i=0}^{n-1} f(x_{i+1})\frac{b-a}{n}$$



I don't really have the time or patience right now to figure out LaTeXgraphics, so I'm just doing this.

6.2 Midpoint Approximation

We can get a more accurate approximation, or rather lower the n required to get an equally accurate approximation and therefore save on computational effort, by taking the midpoint of each thin slice under the curve, as follows:

$$x_i^* := \frac{x_{i-1} + x_i}{2},\ i = 1, ..., n,\ x_i = a + \frac{i}{n}(b - a)$$

Then the approximation is

$$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)$$

6.3 Error

Since the above are all *approximations*, there will be some difference between the value to which the integral tends and the approximated value. This is characterised by a function E_n:

$$E_n^{(M)} = \left| \int_a^b f(x)dx - M_n \right| \leq \frac{K(b-a)^3}{24n^2}$$

The last expression is the *error bound*, which defines a maximum value that the error cannot exceed. K is defined as any integer that is always greater than the absolute value of the second derivative of f on the entire interval $[a, b]$:

$$K \geq |f''(x)| \forall x \in [a, b]$$

A greater value of K implies a greater value for the error bound, therefore we want to find the minimum possible value of K to get the most precise estimate we can.

6.4 Trapezoidal and Simpson's Rule

Rather than estimating the area under the curve of a function as a series of rectangles of infinitesimal width, it is possible to model it with a series of trapezoids or parabolas. The error bounds for both of these are

$$E_n^{(T)} = \left| \int_a^b f(x)dx - T_n \right| \leq \frac{K(b-a)^3}{12n^2}$$

$$E_n^{(S)} = \left| \int_a^b f(x)dx - M_n \right| \leq \frac{K(b-a)^5}{180n^4}$$

where K is defined as above.

For the error bound questions, it's not really that necessary to know the geometric interpretation of these approximations. You get the idea from the midpoint approximation.

6.5 Applications

In general, we are interested in the question: *How accurate an answer can we get for a given value of* n, or the converse: *Given an acceptable error bound, how many computations must we carry out (how large must we take n) to get an answer within that bound*?

Consider the problem $\int_1^2 \frac{dx}{x}$. Suppose we want to use the trapezoidal approximation to obtain an answer accurate to within 10^{-4}. Then, we look at the error bound:

$$E_n^{(T)} \leq \frac{K(b-a)^3}{12n^2}$$

We first need to find a value for K before we can use this. Therefore, we take the second derivative:

$$f'' = \frac{2}{x^3},\ x \in [1,2]$$

This function is monotonically decreasing, therefore its maximum value is at $x = 1$. $f''(1) \geq f''(x) \forall x \in [1,2] \implies K = 2$.

Note that this is just the minimum possible choice of K, not the only possibility. In general, though, we want to minimise K within the constraints. Therefore, we substitute $K = 2$, $a = 1$, $b = 2$ into the error bound and set it less than 10^{-4}:

$$E_n^{(T)} \leq \frac{2}{12n^2} = \frac{1}{6n^2} \leq 10^{-4}$$

Solving for n and rounding up to the nearest natural number yields

$$n \geq \frac{100}{\sqrt{6}} \approx 40.8 \implies n \geq 41$$

A choice of $n \geq 50$ is also acceptable (approximating $\sqrt{6} > 2$) if the required computation is difficult without a calculator.

6.6 Examples

1

Find the minimum n to obtain an approximate value for the integral from 0 to 1 of $f(x) = x^7 + x^5 - x^4$, accurate to within 10^{-4}.

$$f''(x) = 42x^5 + 20x^3 - 12x^2$$

We use the triangle inequality $|a+b| \leq |a| + |b|$ and consider the absolute value of the second derivative, which is maximum at x = 1. Therefore we take $K = 42 + 20 + 12 = 74$.

$$\therefore E_n^{(M)} \leq \frac{74}{24n^2} = \frac{37}{12n^2} \leq 10^{-4}$$

Solving for n yields

$$n \geq 176$$

2

How large should we take n in order to guarantee that the trapezoidal approximation to $\int_0^2 e^{x^2} dx$ is accurate to within 0.0001?

We take the second derivative,

$$f''(x) = 2e^{x^2}(2x^2 + 1)$$

This is at a maximum on the given interval at $x = 2$, i.e.

$$K = 2e^4(9) = 18e^4 \approx 1000$$

Then,

$$E_n^{(T)} \leq \frac{1000 \cdot 2^3}{12n^2} = \frac{2000}{3n^2} \leq 10^{-4}$$

Solving for n, we get $n = 2582$.

7 Lecture 7: Improper Integrals

7.1 Improper Integrals

When one of the limits of an integral is ill-defined, or both are, we can take a limit to more accurately define the value of the definite integral, as follows:

$$\int_a^\infty f(x)dx := \lim_{t\to\infty} \int_a^t f(x)dx$$

Similarly for the lower limit,

$$\int_{-\infty}^{b} f(x)dx := \lim_{s\to\infty} \int_{-s}^{b} f(x)dx$$

We can put these together to define a definite integral across the entire real line,

$$\int_{-\infty}^{\infty} f(x)dx := \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

If both integrals exist, this is independent of the value of a.

If an improper integral exists, it is called *convergent*, else it is called *divergent*.

In general, we wish to find out whether an improper integral is convergent. If it is, we can compute it analytically or approximately.

7.2 A Standard Integral for Convergence Comparisons

$$I = \int_{1}^{\infty} \frac{dx}{x^p}$$

Depending on the value of p, we can evaluate this in two different ways, using the definition of a definite integral with one limit at $+\infty$ as above:

$$p \neq 1 \implies I = \frac{1}{1-p}(t^{1-p} - 1)$$

$$p = 1 \implies I = \ln t$$

Then, we let $t \to \infty$:

$$p > 1,\ \lim_{t\to\infty} t^{1-p} = 0$$

$$p = 1,\ \lim_{t\to\infty} \ln t = \infty$$

$$p < 1,\ \lim_{t\to\infty} t^{1-p} = \infty$$

Therefore, the limit does not exist and the integral diverges for $p \leq 1$, and the integral converges for $p > 1$.

7.3 Comparison Test for Integrals

Suppose we have two functions f and g such that $f \geq g \geq 0 \forall x \geq a$, then

$$\text{if} \int_a^\infty f dx \text{ converges, then} \int_a^\infty g dx \text{ converges}$$

$$\text{if} \int_a^\infty g dx \text{ diverges, then} \int_a^\infty f dx \text{ diverges}$$

Example 1

$$\int_1^\infty \frac{dx}{\sqrt{x}(1+x)}$$

We can approximately see whether the comparison test will work by comparing high values of x.

When $x >> 1$, $x + 1 \approx x$, therefore the integral becomes

$$\int_1^\infty \frac{dx}{x\sqrt{x}}$$

which we know to be convergent as $p > 1$.

We formalise this intuition by using the comparison test with $\int_1^\infty \frac{dx}{x^{\frac{3}{2}}}$:

$$\frac{1}{\sqrt{x}(1+x)} \leq \frac{1}{x^{\frac{3}{2}}}$$

$$\therefore \text{choose} f(x) = \frac{1}{x^{\frac{3}{2}}},\ g(x) = \int_1^\infty \frac{dx}{\sqrt{x}(1+x)}$$

Since $\int_1^\infty f(x)dx$ converges, and $f \geq g$, we can conclude that

$$\int_1^\infty \frac{dx}{\sqrt{x}(1+x)} \text{ converges.}$$

Example 2

$$\int_1^\infty \frac{x^2 + 100x + 51}{5x^3 + 50x^2 + 20x + 1} dx$$

As before, we take $x >> 1$, i.e. effectively only comparing the highest-degree terms, which yields

$$\frac{x^2}{5x^3} = \frac{1}{5x}$$

which is the case $p = 1$. We therefore expect divergence.

coming back to this later

Example 3

$$\int_1^\infty \frac{|\sin^3 x|}{x^2+1}dx \leq \int_1^\infty \frac{1}{x^2+1}dx \leq \int_1^\infty \frac{1}{x^2}dx \text{ which is convergent as } p > 1$$

8 Lecture 8: Discontinuous Integrals

8.1 Recap of last time: example of comparison test

Consider the integral

$$\int_1^\infty \frac{|\sin x^3| + x}{x^{\frac{5}{2}}}dx$$

Assuming $x >> 1$, take the highest order terms:

$$\int_1^\infty \frac{x}{x^{\frac{5}{2}}}$$

This has $p > 1$, therefore we expect convergence. We formalise this through a series of comparisons, as follows:

$$\int_1^\infty \frac{|\sin x^3| + x}{x^{\frac{5}{2}}}dx \leq \int_1^\infty \frac{x + x}{x^{\frac{5}{2}}}dx = 2\int_1^\infty \frac{dx}{x^{\frac{3}{2}}}$$

which has $p > 1$, therefore it converges, and so does the original integral.

8.2 Discontinuous Integrals

Let $f(x)$ be discontinuous at $x = b$. Then, we can still define a definite integral as follows:

$$\int_a^b f(x)dx := \lim_{t \to b^-} \int_a^t f(x)dx \text{ if the limit exists}$$

Similarly for the lower limit $x = a$,

$$\int_a^b f(x)dx := \lim_{t \to a^+} \int_t^b f(x)dx \text{ if the limit exists}$$

And for an intermediate point $c \in [a, b]$:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

In each of these cases, the integral is convergent if the limit exists, and divergent if it does not.

8.3 *p*-integrals on different intervals

We previously covered the p-integral on an interval $[1, \infty]$. Now we can use the definition of the discontinuous integral to define it over the interval $[0, 1]$, as follows:

$$I = \int_0^1 \frac{dx}{x^p} := \lim_{t \to 0^+} \int_t^1 \frac{dx}{x^p}$$

Evaluating this yields

$$p \neq 1 \implies I = \frac{1}{1-p}(1 - t^{1-p})$$

$$p = 1 \implies I = -\ln t$$

Therefore, taking the limit implies that the p-integral is divergent for $p \geq 1$ and convergent for $p < 1$.

8.4 Example using new *p*-integral

$$\int_0^1 \frac{dx}{\sqrt{x}(x+1)}$$

Instead of very large x, we now consider very small values of x, so that all but the lowest order terms vanish:

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

This is the case $p < 1$, which implies that the integral will converge.

We confirm this by taking $f(x) = \frac{1}{\sqrt{x}(x+1)}$ and selecting $g(x) = \frac{1}{\sqrt{x}}$. As g is convergent, and $\int g(x)dx \geq \int f(x)dx \geq 0$, the integral converges.

8.5 sin **function comparisons**

For high values of x, when we can neglect all but the highest order terms, we can use the inequality $\sin x < 1$ to justify ignoring the sine term. We see this in section 8.1's example of a comparison test question. However, with low values of x, we can use the inequality $\sin x < x$.

This is justified by the limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

which can be verified using L'Hopital's rule:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{(\sin x)'}{x'} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

Below is an example where this inequality is required:

Example

$$I = \int_0^1 \frac{\sin x^2}{x^3} dx$$

In this case, we could use the inequality $\sin x < 1$ as we did before, which yields

$$\int_0^1 \frac{\sin x^2}{x^3} dx < \int_0^1 \frac{1}{x^3} dx$$

However, this is the case $p > 1$ which is divergent on the interval $[0, 1]$. Therefore the comparison test is invalid for this case.

Instead, we consider the inequality $\sin x < x$:

$$I = \int_0^1 \frac{\sin x^2}{x^3} dx < \int_0^1 \frac{x^2}{x^3} dx = \int_0^1 \frac{dx}{x}$$

This indicates we should expect divergence.

8.6 Note

If you have an integral of the form

$$I = \int_a^\infty f(x)dx$$

then it can be expressed as

$$I = \int_a^T f(x)dx + \int_T^\infty f(x)dx$$

The original integral converges if and only if the integral from T to ∞ converges as well. Therefore, if one function is greater than another for small x, but in the long run the inequality reverses, it is possible to split it into two integrals, and only consider the region from some T to ∞ for the comparison test (more compactly, if $g(x) > f(x)$ up to a constant).

9 Lecture 9: Review

9.1 Integral Convergence

Example 1

$$I = \int_1^\infty \frac{x^2+5}{\sqrt{x}(x^3+1)(x+2)}dx$$

We first take the highest order terms,

$$I \approx \int_1^\infty \frac{dx}{x^{\frac{5}{2}}}$$

This suggests we should expect convergence.

A series of inequalities confirms this:

$$I = \int_1^\infty \frac{x^2+5}{\sqrt{x}(x^3+1)(x+2)}dx \leq \int_1^\infty \frac{x^2+5}{\sqrt{x}(x^3)(x)} \leq \int_1^\infty \frac{x^2+5x^2}{x^{\frac{9}{2}}}dx$$

The last of these inequalities is justified when $x \geq 1$, which it is, given the limits of integration. Simplifying this yields

$$I \leq \int_1^\infty \frac{6}{x^{\frac{5}{2}}}$$

which is the case $p > 1$, i.e. it converges.

Example 2

$$I = \int_1^\infty \frac{x^4 - 100x^2 - 200x - 300}{x^6 + 1} dx$$

At very high x, this function behaves as $\frac{1}{x^2}$ and is therefore convergent, but at smaller x, it is negative and so the comparison test does not apply. Therefore, we choose $T >> 1$ and redefine the integral:

$$I = \int_1^\infty f(x)dx = \int_1^T f(x)dx + \int_T^\infty f(x)dx$$

We can use the comparison test on the integral from T to ∞ (compare with $\frac{1}{x^2}$ and find it is convergent), and as there are no discontinuous points or infinite limits on the integral from 1 to T, that is also convergent. Therefore, the entire integral converges.

9.2 True-False Questions

1

If $f \geq g$ for $x \geq 1$ and $\int_1^\infty f^2 dx$ converges then $\int_1^\infty g^2 dx$ converges.

False: take $f = \frac{-1}{x}$ and $g = \frac{-1}{\sqrt{x}}$. (This would be true for g $\geq$ 0).

2

If $0 \leq f \leq M$ and $\int_1^\infty f dx$ converges, then $\int_1^\infty f^2 dx$ converges.

True: as $f \leq M$, $Mf \geq f^2$. We know that $\int f dx$ converges, therefore so does $\int Mf dx$, and by comparison test this is greater than $\int f^2 dx$, so $\int f^2 dx$ converges.

3

If $\int_1^\infty f dx$ converges, $f \geq 0$ then $\int_1^\infty f^2 dx$ converges.

False: take $f = \frac{1}{x\sqrt{x-1}}$. $\int f dx$ converges (comparison with $\frac{1}{x^{\frac{3}{2}}}$) but its square has a discontinuous point with a nonexistent limit near $x = 1$.

9.3 Approximate Integration

1

Find K for the function $f(x) = x\cos x^2$ **over the interval** $x \in [0, 1]$.

We take the second derivative using the product rule,

$$|f''(x)| = 6x|\sin x^2| + 4x^3|\cos x^2|$$

Then we use $|\sin x| < 1$, $|\cos x| < 1$:

$$|f''(x)| \leq 6x + 4x^3$$

which, over the given interval, has a maximum value of 10. Therefore we can take $K = 10$.

2

Find K for the function $\sin(\sqrt{x}) + 211x + 103$**,** $1 \leq x \leq 2$.

The second derivative is

$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}\cos(\sqrt{x}) - \frac{1}{4x}\sin\sqrt{x}$$

We take the absolute value and use $|\sin x| < 1$, $|\cos x| < 1$ as above:

$$|f''(x)| \leq \frac{1}{4x^{\frac{3}{2}}} + \frac{1}{4x}$$

This has a maximum value over the given interval at $x = 1$:

$$K = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$