Analysis of Partial Differential Equations
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Chapter 1

Introduction

Contents

1.1 Overview of PDEs

A partial differential equation is an equation that defines some relationship between partial derivatives of a multivariable function. This is in contrast to ordinary differential equations, which have only one independent variable (and its derivatives).

We'll cluster these independent variables together and call them all x (except sometimes we'll separate out one variable called t), and we'll call the solution u.

Definition 1.1. *Suppose $U \subset \mathbb{R}^n$ is an open set. A partial differential equation (PDE) of order k is an expression of the form*

$$F(x, u(x), Du(x), \ldots, D^k u(x)) = 0. \tag{1.1}$$

where $u : U \to \mathbb{R}$ is unknown and $F : U \times \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}$ is a given function.

We say that u is a classical solution of the PDE if on substitution of u and its partial derivatives, the equation is satisfied identically in U.

We can also consider the case where $u(x) \in \mathbb{R}^p, F \in \mathbb{R}^q$ which we call a *system of PDEs*.

The main reason we care about PDEs is they model real-life phenomena in ways that ordinary differential equations can't. Here's a bunch of examples of how that works!

Example 1.1.

1. Systems of ordinary differential equations can be cast as PDEs, for instance $\frac{\mathrm{d}}{\mathrm{d}t}u = f(u, v)$, $\frac{\mathrm{d}}{\mathrm{d}t}v = g(u, v)$. Predator-prey models and SIR models are of this form.
2. The simplest possible PDE is the heat equation,

$$u_t = D\Delta u.$$

If $F : \mathbb{R} \to \mathbb{R}$, the average value of F on $[-h, h]$ for some small $h > 0$,

$$F_{Av} = \frac{1}{2h} \int_{-h}^{h} F(x)\mathrm{d}x.$$

If we do a Taylor expansion,

$$\Delta F|_{x=0} = F''(0) = \frac{12}{h^2}(F_{Av} - F(0)) + \mathcal{O}\big(h^2\big),$$

so the Laplacian measures a "local average deviation". This is relevant for physical intuition about the heat equation, as it describes a diffusion process: $\partial_t u > 0$ if the neighbourhood is hotter.

We can make this a stochastic PDE by adding a random term, $u_t = D\Delta u + \xi$. This is beyond our scope.

3. We can make a simple PDE system from the heat equation,

$$\begin{aligned} u_t &= D_1 \Delta u + f(u,v) \\ v_t &= D_2 \Delta v + g(u,v) \end{aligned}$$

Reaction-diffusion equations have this form. Turing instabilities arise from this.

4. The transport equation:

$$\partial_t u = D\Delta u + \vec{v} \cdot Du + f(u)$$

This tells us how the concentration of some chemical in a fluid of velocity $\vec{v}$ changes with production rate f.

5. Laplace's equation and Poisson's equation, respectively:

$$\begin{aligned} \Delta u = \sum_i \partial^2_{x_1 x_2} u = 0 \\ \Delta u = f, \vec{E} = \nabla u. \end{aligned}$$

Note that Δ is always a spatial Laplacian.

6. Euler-Poisson equation:

$$\begin{aligned} \rho(\partial_t u + (u \cdot \nabla)u) + \nabla(\rho^\gamma) = -\rho \nabla \phi \\ \Delta \phi = \rho \\ \partial_t \rho = \mathrm{d}r(\rho u). \end{aligned}$$

This models a star held together by gravity, in a time-dependent region $U(t)$.

7. The wave equation:

$$\Box u = -\frac{1}{c^2} \partial^2_{tt} u + \Delta u = 0$$

and, closely associated, the Klein-Gordon equation: $(\Box - m^2)u = 0$.

The wave equation is used for sound waves and seismic waves, for examples. The Klein-Gordon equation models some phenomena in QFT.

8. The KdV equation, $\partial_t u + \partial_x^3 u - bu\partial_t u = 0$, admits soliton-like solutions. This describes a kind of wave first seen in the Edinburgh canal (Russell 1834).

9. Maxwell's equations: $\nabla \cdot E = \rho, \nabla \cdot B = 0, \partial_t E = \nabla \times B - J, \partial_t B = -\nabla \times E$, the Schrodinger equation, $\partial_t u + \Delta u = V(u)$, and the Einstein equations in GR, $Ric(g)_{\mu\nu} = 0$. One of the postulates in SR is there's no canonical choice of time and space. What do the ∂_t, ∂_x operators mean in this context?

   Stuff to google: constraint equation, Jang's equation, Penrose inequalities.

10. The Black-Scholes equation (finance).

11. The Navier-Stokes equations (fluids).

$\square$

In all these examples, it is not sufficient to just know the PDE. We need additional information to solve them: the initial temperature in some domain, boundary values $u|_{\partial U}$, and so on. We refer to these as the *data*.

An important component of analysis of PDEs is understanding what data are appropriate. For example, the positive mass theorem in GR tells us that our data cannot be compactly-supported infinitely-differentiable functions.

Our guiding principle through this is Hadamard's notion of well-posedness (although this is informal). We say a PDE problem (the equation and the data) is well posed if

(a) a solution exists in some function space C^*

(b) for the given data, the solution is unique (in some function space)

(c) the solution depends continuously on the data (for the data and solution both respectively in some function spaces.)

The third condition can especially be open to interpretation.

We want to choose a function space that is large enough that a solution exists, but small enough that it is unique.

1.2 Multi-index notation

Earlier, we said we'd include a bunch of other variables under the single name x. This isn't quite what we're used to from multivariable calculus, where we would have an expression like $\frac{\partial^2 f}{\partial x \partial y}$. If we're combining x, y, z, etc into just x, how do we express derivatives of mixed orders between different components of x? The answer is a multi-index, which basically expresses the exponents like an array like you might do in programming. These are all useful notational conveniences.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ (starting at 0). α is called a multi-index.

We define $|\alpha| = \sum_i \alpha_i$ to be the order of α.

The derivative with respect to a multi-index is

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial_{x_i}^{n_i}} \tag{1.2}$$

If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then $x^\alpha = \prod_i x_i^{\alpha_i}$.

Finally, $\alpha! = \prod_i \alpha_i!$.

1.3 Classifying PDEs

A PDE is linear if F is a linear function of u and its derivatives.

$$\sum_{\alpha:|\alpha|=k} a_\alpha(x) \frac{\partial^\alpha u}{\partial x^\alpha} = 0, \tag{1.3}$$

For example, $\Delta u = 0$ is a linear PDE, as it can be expanded into

$$\sum_{i=1}^n \frac{\mathrm{d}^2 u}{\mathrm{d}x_i^2} = 0$$

which has the form required, where $a_\alpha(x) = 1$ if the multi-index α has the form $(0, 0, \dots, 2, \dots, 0)$ and 0 otherwise.

A PDE is semilinear if higher-order derivatives appear linearly with coefficients that depend only on x

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + a_0(x, u, \dots, D^{k-1}u) = 0. \tag{1.4}$$

For a few examples:

1. $\Delta u = u_x^2$
2. $\Box u = v$
3. $(\Box - 1)v = u\partial_t v$

Essentially, this is anything without mixed $D^\alpha u D^\beta u$ terms.

A PDE is quasilinear if the highest order derivative coefficients depend linearly on lower-order derivatives of u, i.e.

$$\sum_{|\alpha|=k} a_\alpha(x, u(x), \dots, D^{k-1}u(x)) \frac{\partial^\alpha u}{\partial x^\alpha} + a_0(x, u, \dots, D^k u) = 0, \tag{1.5}$$

For example, $uu_{xx} + u_{yy} = u_x^2$ is quasilinear. This relaxes another layer of constraints; we're allowing mixed terms under the constraint of linearity, but nothing like u_{xx}^2.

If a PDE is none of these, we say it is nonlinear.

1.4 Real analysis, topology, and functional analysis notes

A Banach space is a complete normed vector space. That is, it's the pairing of a vector space X and a function $\|\cdot\| : X \to \mathbb{R}_+$ (the norm) such that $\|x\| = 0 \iff x = 0$, the triangle inequality holds, and it obeys the scaling property up to absolute value ($\|cx\| = |c|\|x\|$), and such that Cauchy sequences in X converge to limits in X.

$U \subset \mathbb{R}^n$ is open if and only if it is the union of a countable collection of open balls. The forward direction follows from the union of open sets being open. The backward direction follows if we take $x \in U \cap Q^n$ (the n-dimensional rationals, countable and dense in $\mathbb{R}^n$) and take the union of open balls U_x (which exist by the definition of openness) around each one.

Bounded sequences in reflexive Banach spaces (L^p and Hilbert spaces) have weakly convergent subsequences.

Chapter 2

The Cauchy-Kovalevskaya theorem

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2.1 ODE theory refresher

Most tools don't generalize between types of PDEs, but there is one that does: the Cauchy-Kovalevskaya Theorem. This is a generalization of the Picard-Lindelöf theorem from ODE theory.

A quick reminder of what it means to be Lipschitz: f is Lipschitz above u_0 if for all $x, y \in B_r(u_0)$, $\|f(x) - f(y)\| \leq K\|x - y\|$. We call K the Lipschitz constant here. This bounds how fast a function can deviate from its value at an initial point, which is a useful property to have when trying to propagate an ODE solution through time.

Now, we can introduce our first function spaces. A function is in[1] C^0 (or $C^0(U)$ if its domain is U) if it's continuous, and it's C^k if it's differentiable and its derivative is C^{k-1}. Infinitely differentiable functions are said to be C^∞. In practice, we'll probably only directly deal with C^0, C^1, C^2, C^∞.

Fix $U \subset \mathbb{R}^n$ open and take some $f : U \to \mathbb{R}^n$. Consider the ODE

$$u_t = f(u(t)), u(0) = u_0 \in U \tag{2.1}$$

Theorem 2.1 (Picard-Lindelöf)**.** *Suppose there exist constants $r, k > 0$ such that $B_r(u_0) \subset U$, and f is locally Lipschitz above u_0 with Lipschitz constant k. Then there exists some $\epsilon(r, k) > 0$ and a unique C^1 function u such that $u : (-\epsilon, \epsilon) \to U$ solves 2.1.*

Proof sketch.
If u solves 2.1 then by the fundamental theorem of calculus,

$$u(t) = u_0 + \int_0^t f(u(s))\mathrm{d}s \tag{2.2}$$

If u is a C^0 solution to 2.2, then it solves 2.1. Then u, if it exists, is a fixed point of the following map:

$$G(w(t)) = u_0 + \int_0^t f(w(s))\mathrm{d}s \tag{2.3}$$

Here, $G : S \to S$, where $S = \{w : (-\epsilon, \epsilon) \to \overline{B_r(u_0)} : w \in C^0\}$.
Since we care about the uniqueness of a fixed point of this map, we make use of the contraction mapping theorem (also known as the Banach fixed-point theorem), which requires that the space (the domain and the range, which are the same) is a complete metric space with a sup norm. In this case, we show that's true of S.
From this, we show that $G : S \to S$ is a contraction for small ϵ, that u is a fixed point by the contraction mapping theorem, and then that $u \in C^1$ by FTC.

This cannot always be a global solution; for instance, check what happens if $u_t = u(t)^2, u(0) = 0$. Also, it may not be unique. For the example $u_t = u(t)^{1/2}, u(0) = u_0$, you can find two solutions.

[1] in shorthand we just say the function *is* X if it is part of the function space X

We've established an existence theorem for the solutions of ODEs, so let's start extending it. What happens if we introduce one extra independent variable, towards making it a PDE?

If we add in dependence on some other parameter, i.e. $\dot{u}(t,\lambda) = f(u(t,\lambda))$ and the initial condition $u(0,\lambda) = u_0(\lambda)$, then if f, u_0 are locally Lipschitz and continuous "as needed", then the solution $u(t,\lambda)$ is C^0 in λ.

If f is more regular, we expect u to be more regular as well. We can make this precise just by differentiating repeatedly. Let $f \in C^\infty(U)$ and $u \in C^1((-\epsilon,\epsilon))$. Then, by the chain rule,

$$\ddot{u} = Dfu \times \dot{u} := F_2(u(t), \dot{u}(t)) \tag{2.4}$$

Therefore $\ddot{u}$ exists and is C^0, implying that $u \in C^2((-\epsilon,\epsilon))$. We can repeat this process, showing that $u^{(3)}$ exists and is C^0, so $u \in C^3((-\epsilon,\epsilon))$, and so on. Therefore $u \in C^k$ for all k. So if $f \in C^\infty(U)$, then so is u.

This implies that given $u_0 = u(0)$, we can determine $u^{(k)}(0) = F_k(u, Du, \ldots, D^{k-1}u)\big|_{t=0}$, and so we have a *formal power series solution*,

$$u(t) = \sum_{k\geq 0} \frac{u^{(k)}(0)}{k!} t^k. \tag{2.5}$$

When does this power series actually solve the ODE? The Cauchy-Kovalevskaya theorem gives us an answer!

Theorem 2.2 (Cauchy-Kovalevskaya for simple ODEs)**.** *Suppose $U \subset \mathbb{R}^n$ is open, and $u_0 \in U$. If $f : U \to \mathbb{R}$ is real analytic near u_0, and $u(t)$ is the unique solution of $\dot{u}(t) = f(u(t)), u(0) = u_0$ given by Picard-Lindelöf, then $u(t)$ is also real analytic near $t = 0$.*

Another way of saying this is if $f(u)$ is real analytic in a neighbourhood of u_0, then the series $\sum \frac{u^{(k)}(0)}{k!} t^k$ converges in some neighbourhood of $t = 0$ to the unique solution of the ODE as established by Picard-Lindelöf.

We just introduced the term *real analytic*: what does that mean and why does it matter?

2.2 Real analyticity and majorants

Not all C^∞ functions are well described by their Taylor series, unfortunately.

Suppose $f : (-\epsilon,\epsilon) \to \mathbb{R}$ is infinitely differentiable. Then $f^{(n)}(0)$ exists for all $n \geq 0$. We would like it if for $0 < |x| < \delta$ for some δ, $\sum \frac{f^{(n)}(0)}{n!} x^n \to f(x)$. However, this is not always the case.

Example 2.2. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

This is a smooth function; to either side, it's smooth, and you can check that the derivatives at zero all match from the left and the right. However, $f^{(n)}(0) = 0$ for all n, so we have

$$\sum \frac{f^{(n)}(0)}{n!} x^n = 0 \neq f(x) \text{ for } x > 0.$$

□

Functions that do equal their Taylor expansions are called *real analytic*.

Definition 2.1. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. We say f is real analytic near $x_0 \in U$ if there exist constants $r > 0$ and $f_\alpha \in \mathbb{R}$ such that*

$$f(x) = \sum_{\alpha \in \mathbb{R}^n} f_\alpha (x - x_0)^\alpha \text{ for all } x \text{ such that } |x - x_0| < r.$$

Real analytic functions have important properties.

1. On a ball about x_0, we can write f as a convergent power series, $f = \sum_\alpha f_\alpha x^\alpha$, whose coefficients are
$$f_\alpha = \frac{D^\alpha f(x_0)}{\alpha!} \tag{2.6}$$
2. f is real analytic on an open set U if it is real analytic near any point $x_0 \in U$. We denote the set of real analytic functions on U by $C^\omega(U)$.
3. If $f \in C^\omega(U)$ then $f \in C^\infty(U)$. (My best guess: either it terminates somewhere and the high-order derivatives are all 0, or it doesn't and you can keep writing down the derivatives in terms of f_α indefinitely.)
4. f is real analytic near x_0 if and only if f is smooth near x_0 and there exist constants $s, c, r > 0$ such that
$$\sup_{|x - x_0| \leq s} |D^\alpha f(x)| \leq \frac{c|\alpha|!}{r^{|\alpha|}}.$$
Proving this is on Example Sheet 1.
5. If $f \in C^\omega(U)$ and U is a connected open subset of $\mathbb{R}^n$, then f is uniquely determined in U if we know $D^\alpha f(z)$ $\forall \alpha \in \mathbb{N}^n$ and some $z \in U$.
6. If a function is infinitely differentiable and has compact support, it cannot be real analytic: $C^w(U) \cap C_c^\infty(U) = \varnothing$.

Exercise 2.3. *Show that $f(x) = \frac{1}{x}$ and $f(x) = \sqrt{x}$ are real analytic for $x \in \mathbb{R}, x > 0$.*

In addition, we need a tool to analyze multidimensional power series. To do this, let's look at how we do it in a simple case.

Example 2.3. Recall that $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$. Let $r > 0$.

Consider

$$f(x) = \frac{r}{r - (x_1 + \cdots + x_n)} = \frac{1}{1 - \frac{x_1 + \cdots + x_n}{r}} = \sum_{k \geq 0} \left(\frac{x_1 + \cdots + x_n}{r} \right)^k$$

This is valid provided (by the Cauchy-Schwarz inequality)

$$|x_1 + \cdots + x_n| \leq \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} (\sum 1^2)^{1/2} = \|x\| \sqrt{n} < r.$$

By the multinomial theorem,

$$f(x) = \sum_{k \geq 0} \frac{1}{r^k} \left(\sum_{|\alpha| = k} \binom{|\alpha|}{\alpha} x^\alpha \right),$$

so by pattern-matching, $D^\alpha f(0) = \frac{|\alpha|!}{r^{|\alpha|}}$. We claim that the series $f(x) = \sum_\alpha f_\alpha x^\alpha$, where $f_\alpha = \frac{D^\alpha f(0)}{\alpha!}$ is absolutely convergent near 0.

$$\sum_{\alpha} \frac{|\alpha|!}{\alpha!} \frac{|x^\alpha|}{r^{|\alpha|}} = \sum_{k \geq 0} \left(\frac{|x_1| + \cdots + |x_n|}{r} \right)^k < \infty$$

□

This is useful to have as a standard reference point for multidimensional problems, similar to how we had the one-dimensional geometric series in $\mathbb{R}$.

For more complicated cases, we can use a similar idea to series tests in single-variable calculus.

Definition 2.2. *Let $f = \sum_\alpha f_\alpha x^\alpha$, $g = \sum_\alpha g_\alpha x^\alpha$ for $f_\alpha, g_\alpha \in \mathbb{R}$. We say g majorizes f or g is a majorant of f if $g_\alpha \geq |f_\alpha|$ for all α.*

This is denoted $g \gg f$. We can extend the term to vector-valued cases, in which case $g \gg f$ if $g^i \gg f^i$ for all i.

Lemma 2.4 (Properties of Majorants). *(i) If $g \gg f$ and g converges for $|x| < r$, then so does f.*

(ii) If f converges for $|x| < r$ and $s \in \left(0, \frac{r}{\sqrt{n}}\right)$, then there exists a majorant of f which converges for $\|x\| \leq \frac{s}{\sqrt{n}}$.

Proof .

(i) We note that

$$\sum_{|\alpha|\leq k} |f_\alpha x^\alpha| = \sum_{|\alpha|\leq k} |f_\alpha| \prod_{i=1}^{n} |x_i|^{\alpha_i} \tag{2.7}$$

$$\leq \sum_{\alpha} g_\alpha \prod_{i=1}^{n} |x_i|^{\alpha_i} \tag{2.8}$$

$$= g(\tilde{x}), \tag{2.9}$$

where $\tilde{x} = (|x_i|)_{i=1}^n$. Therefore, $\|\tilde{x}\| = \|x\|$, so if $\|x\| < r$ then $\|\tilde{x}\| < r$. We know that g converges for $|x| < r$, so $f(x) \leq g(\tilde{x}) < \infty$ means $f(x)$ also converges.

(ii) Let $s \in \left(0, \frac{r}{\sqrt{n}}\right)$ and set $y = (s, \dots, s)$. Then $\|y\| = s\sqrt{n}$. By assumption, $f(y)$ converges, and a convergent series has uniformly bounded terms, so there exists C such that $|f_\alpha y^\alpha| \leq C$ for all α.

$$|f_\alpha| \leq \frac{C}{|y^\alpha|} \frac{C}{|y_1|^{\alpha_1} \dots |y_n|^{\alpha_n}} = \frac{C}{s^{|\alpha|}} \leq C \frac{1}{s^{|\alpha|}} \cdot \frac{|\alpha|!}{\alpha!} := g_\alpha, \tag{2.10}$$

and this defines a majorant of f.

$$g(x) = \sum_\alpha g_\alpha x^\alpha \frac{Cs}{s - (x_1 + \dots + x_n)} = C \sum_\alpha \frac{|\alpha|!}{s^{|\alpha|}\alpha!} x^\alpha \tag{2.11}$$

From the example, we know this converges for $\|x\| < \frac{s}{\sqrt{n}}$, and $g \gg f$, so we're done!

We can use the method of majorants to prove the Cauchy-Kovalevskaya theorem for ODEs, 2.2.

Proof of CK-ODE.
We want to show that a formal power series will actually solve the ODE when f is real analytic. To do this, we'll first find what the coefficients are, and then we'll see where the power series with those coefficients solves the ODE.
Without loss of generality let $u_0 = 0$, and for simplicity let $n = 1$. (Notation comment: primes will be used for derivatives of the explicit argument, and dots for time derivatives.)
To find the indices, we'll basically just repeatedly apply the ODE. The zeroth order is set by the initial condition, so let's look at the first order.

$$\dot{u} = f(u) \implies \dot{u}(0) = f(0) \implies u_1 = f(0). \tag{2.12}$$

We can take a second derivative and look at the second order,

$$\ddot{u}(t) = f'(u(t))\dot{u}(t) \implies \ddot{u}(0) = f'(0)\dot{u} = f'(0)f(0) \implies u_2 = \frac{1}{2!}f'(0)f(0), \tag{2.13}$$

and continuing in the same way,

$$\dddot{u}(t) = f''(u(t))\dot{u}(t)f(u(t)) + f(u(t))^2\dot{u}(t) \tag{2.14}$$
$$\dddot{u}(0) = f'''(0)(f(0))^2 + (f'(0))^2 f(0) \tag{2.15}$$
$$u_3 = \frac{1}{3!}(f''(0)f(0)^2 + (f'(0))^2 f(0)). \tag{2.16}$$

By induction, we can show that $u_k = P_k\big(f(0), f'(0), \dots, f^{(k-1)}(0)\big)$ where P_k is a polynomial of k variables and nonnegative coefficients (as they come out of the Leibniz rule, which doesn't allow for negative coefficients), starting

$$P_1(x) = x$$
$$P_2(x,y) = \frac{1}{2!}xy$$
$$P_3(x,y,z) = \frac{1}{3!}(x^2 z + xy^2).$$

Since f is real analytic, we know that we can write it as a series $f(v) = \sum_{k\geq 0} f_k v^k$ with $f_k = \frac{f^{(k)}(0)}{k!}$. We can invert this to get $f^{(k)}(0) = k! f_k$ for all k. So we can use this to write u_k as a polynomial in the f_is: we get $u_k = Q_k(f_0, f_1, \dots, f_{k-1})$, where Q_k is also a polynomial with non-negative coefficients. The polynomial Q_k is universal, meaning that if we have another ODE system $\dot{v} = h(v), v(0)$ where $h(v) = \sum h_k v^k$ and $v(t) = \sum v_k h^k$, then $v_k = Q_k(h_0, h_1, \dots, h_{k-1})$. This polynomial doesn't depend on the problem: it comes strictly out of the Leibniz rule.
We've established a relationship between the coefficients in the solution power series u_k and f's (rescaled) known derivatives f_k. This will help us show the main statement, as it makes it easy to apply real analyticity. We want to show that $\sum_{k\geq 0} u_k t^k$ converges in a neighbourhood of $t = 0$ and solves $\dot{u} = f(u)$. Since f is real analytic, $\sum f_k u^k$ converges for $|u| < r$ for some $r > 0$.
Fix $s \in (0, r)$. Then $f(s)$ converges absolutely, and therefore $\left|f_k s^k\right| \leq C$ uniformly in k for all $k \geq 0$ and some $C > 0$ (the terms are bounded). Therefore

$$|f_k| \leq \frac{C}{s^k} \triangleq g_k \in \mathbb{R}. \tag{2.17}$$

This defines our majorant:

$$g(u) = \sum_{k\geq 0} g_k u^k = C \sum_{k\geq 0} \left(\frac{u}{s}\right)^k = \frac{Cs}{s-u} \quad \text{for} \quad |u| < s \tag{2.18}$$

Consider the auxiliary ("additional that will end up being quite useful") differential equation with trivial initial data,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = g(w), w(0) = 0. \tag{2.19}$$

By separation of variables, and by imposing the initial condition, we can show this is solved by $w(t) = s - \sqrt{s^2 - 2Cst}$ (where we take the negative square root in order to impose the initial condition). This is real analytic for $|t| < \frac{s}{2C}$. Therefore $w(t) = \sum_k w_k t^k$ converges for $|t| < \frac{s}{2C}$, and recalling that Q_k is universal, $w_k = Q_k(g_0, g_1, \dots, g_{k-1})$.
We claim that w majorizes u. By construction, $g_k \geq |f_k|$ for all k, and since Q_k has no non-negative coefficients,

$$w_k = Q_k(g_0, g_1, \dots, g_{k-1}) \geq Q_k(|f_0|, |f_1|, \dots, |f_{k-1}|) \geq |Q_k(f_0, f_1, \dots, f_{k-1})| = |u_k| \tag{2.20}$$

Example 2.4. A quick example of this kind of triangle-inequality argument is $\left|3x + 3y^2\right| \leq 3|x| + 3\left|y^2\right| \leq 3|a| + 3b^2 = 3a + 3b^2$. □

Since $\sum_k w_k t^k$ converges for $|t| < \frac{s}{2C}$, we know that $\sum_k u_k t^k$ converges for $|t| < \frac{s}{2C}$ by Lemma 2.4. We claim that $u(t) = \sum_{k\geq 0} u_k t^k$ solves the ODE. Both sides are analytic, so it suffices to substitute $u(t)$ into the ODE and show that the derivatives of each side agree at $t = 0$ to all orders, which we have by construction.

This argument can be extended to systems of dimension n. Instead of polynomials, we have systems of polynomials depending on multi-indices: substitute $u_k \to u_k^j = Q_k^j(D_u^\alpha f(0))_{|\alpha|<k-1}$ and $w \to w^j = w'$ for all j. The argument then follows from generalising majorants to vector-valued majorants that hold for each component individually.

In the non-autonomous case, $\dot{u} = f(u,t), u(0) = 0$, we just expand the vector x: we solve for $v(t) = (u(t), t)$, where $\dot{v}(t) = (u, 1) = (f(u,t), 1) = F(v) \in \mathbb{R}^{n+1}$.

2.3 Cauchy-Kovalevskaya for PDEs

Let our unknown function be $\vec{u} : \mathbb{R}^n \to \mathbb{R}^m$. Let $r > 0$. Our differential equation is

$$\begin{aligned} \text{solve } \partial_t \vec{u} &= \sum_{j=1}^{n-1} B_j(\vec{u}, x')\partial_{x_j} u + \vec{C}(\vec{u}, x), \\ \text{subject to } \vec{u}(x', t=0) &= 0 \quad \text{on} \quad x' \in B_r^{n-1}(0) \end{aligned} \tag{2.21}$$

where $x' \in \mathbb{R}^{n-1}$ and $t = x^n$. We want a solution to the differential equation 2.21 on the n-dimensional ball $B_r^n(0) = \{x \in \mathbb{R}^n \mid \|x\| = \sqrt{t^2 + |x'|^2} < r\}$.

Theorem 2.5 (Cauchy-Kovalevskaya for first-order systems)**.** *Assume $\{B_1, \dots, B_{n-1}, C\}$ are all real analytic in $\|u\|^2 + \|x'\|^2 < \rho^2$ for some $\rho > 0$. Then there exists $r \in (0, \rho)$ such that there exists a unique real analytic function $\vec{u} = \sum u_\alpha x^\alpha$ that solves Equation 2.21.*

The idea we'll use here is similar to the ODE case: we'll compute the components $u_\alpha = \frac{D^\alpha u(0)}{\alpha!}$ in terms of B_j, C. To construct the derivatives we need for this, we'll use the PDE directly.

Example 2.5. Consider the system

$$\begin{aligned} u_t &= v_x - f \\ v_t &= -u_x \\ u = v = 0 \quad \text{on} \quad &\{t = 0\}. \end{aligned} \tag{2.22}$$

The boundary conditions are $u(x, 0) = v(x, 0) = 0$, and so $u(0, 0) = v(0, 0) = 0$.

Our aim is to determine u_α for all α using the PDE.

By differentiating the boundary condition, we have

$$(\partial_x)^n u(x, 0) = (\partial_x)^n v(x, 0) = 0 \, \forall n \geq 0. \tag{2.23}$$

This gives us all derivatives of the form $\alpha = (n, 0)$.

Then, from the PDE itself, we get

$$u_t(x, 0) = 0 - f(x, 0) \tag{2.24}$$
$$v_t(x, 0) = 0 \tag{2.25}$$

Assuming $f \in C^\infty$, we can differentiate these as many times as we like.

$$(\partial_x)^n \partial_t u(x, 0) = -(\partial_x)^n f(x, 0) \tag{2.26}$$
$$(\partial_x)^n \partial_t v(x, 0) = 0. \tag{2.27}$$

This holds for all $n \geq 0$, so this defines the derivatives of the form $\alpha = (n, 1)$. Doing this one more time, we get

$$u_{tt} = v_{xt} - f_t \qquad \implies u_{tt}(x, 0) = -f_t(x, 0) v_{tt} = -u_{xt} \qquad \implies v_{tt}(x, 0) = f_x(x, 0) \tag{2.28}$$

and this gives us

$$(\partial_x)^n (\partial_t)^2 u(x, 0) = -(\partial_x)^n \partial_t f(x, 0) \tag{2.29}$$
$$(\partial_x)^n (\partial_t)^2 v(x0) = (\partial_x)^{n+1} f(x, 0). \tag{2.30}$$

Therefore we have all the derivatives of the form $\alpha = (n, 2)$. We can keep doing this to get any derivative we need. $\square$

The idea of Cauchy-Kovalevskaya is that we use the PDE to get all the coefficients needed to construct the series, then we use the fact that everything is real analytic to extend the idea from the ODE case to show convergence to a solution in the PDE case.

Note that C-K guarantees uniqueness only in the class of real analytic functions: we may have other solutions, but they won't be real analytic.

2.4 Reduction to first-order system

In ODE theory, we reduced nth order differential equations to coupled systems of first-order ones. We can do something similar for PDEs.

Consider the system

$$u_{tt} = f(u, u_t) \tag{2.31}$$
$$u(0) = u_0 \tag{2.32}$$
$$u_t(0) = u_1. \tag{2.33}$$

We enlarge the system to use a primary variable $w = (u, v) \in \mathbb{R}^{2n}$, where we set $v = u_t$. Then the system becomes

$$\begin{aligned} w_t = (u_t, v_t) = (v, f(u, v)) = g(w, t) \\ w(0) = (u_0, u_1). \end{aligned} \tag{2.34}$$

Example 2.6. Consider $u : \mathbb{R}^3 \to \mathbb{R}$,

$$\begin{aligned} u_{tt} &= uu_{xy} - u_{xx} + u_t \\ u\Big|_{t=0} &= u_0(x, y) \\ u_t\Big|_{t=0} &= u_1(x, y) \end{aligned} \tag{2.35}$$

where we assume that u_0, u_1 are real analytic in a neighbourhood of $0 \in \mathbb{R}^2$.

First, we zero out the boundary conditions by a change of variables. Note that $f(t, x, y) = u_0 + tu_1$ is analytic in a neighbourhood of $0 \in \mathbb{R}^3$, and $f\big|_{t=0} = u_0$, $f\big|_{t=0} = u_1$, so we can enlarge the system without losing real analyticity.

Set $w(t, x, y) = u - f$. This gives us the following differential equation in w:

$$
\begin{aligned}
w_{tt} &= ww_{xy} - w_{xx} + w_t + fw_{xy} + f_{xy}w + F \\
w\Big|_{t=0} &= w_t\Big|_{t=0} = 0 \in \mathbb{R},
\end{aligned} \tag{2.36}
$$

where $F = ff_{xy} - f_{xx} + f_t$ is also analytic and independent of w and its derivatives.

By making the vectors of parameters and derivatives (except for the highest order) explicit, we can reduce the system order. Let $\vec{x} = (x, y, t) = (x^1, x^2, x^3)$, and let $\vec{v} = (w, w_x, w_y, w_z)$. Then

$$
\begin{aligned}
v^1_t &= w_t = v^4 \\
v^2_t &= w_{xt} = v^4_{x_1} \\
v^3_t &= w_{yt} = v^4_{x_2} \\
v^4_t &= w_{tt} = v^1 v^2_{x_2} - v^2_{x_1} + v^4 + fv^2_{x_2} + f_{xy}v^1 + F.
\end{aligned} \tag{2.37}
$$

This can be put into the standard form

$$
\begin{aligned}
\vec{v}_{x_3} &= \sum_{j=1}^{2} B_j \vec{v}_{x_j} + C \\
\vec{v} = 0 &\quad \text{for} \quad B_r(0) \cap \{x_3 = 0\}
\end{aligned} \tag{2.38}
$$

with

$$
B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ v^1 + f & 0 & 0 & 0 \end{bmatrix} \tag{2.39}
$$

$$
C = \begin{bmatrix} v_4 \\ 0 \\ 0 \\ v^4 + f_{xy}v^1 + F \end{bmatrix}. \tag{2.40}
$$

We conclude that by Cauchy-Kovalevskaya there exists a unique real analytic solution to the PDE problem in a neighbourhood of $0 \in \mathbb{R}^3$. $\square$

This procedure relied on being able to solve for u_{tt} in terms of at most two spatial derivatives of u.

Also, note that we did this in two steps: zeroing out the initial data, and rewriting into a first-order system. If we did this in reverse, it would stil work, but we would get a different parameterisation.

What's the greatest generality in which we can do this? Consider the scalar quasilinear problem

$$\sum_{\alpha||\alpha|=k} a_\alpha(D^{k-1}u,\ldots,u,x)D^\alpha u + a_0(D^{k-1}u,\ldots,u,x) = 0, \tag{2.41}$$

where $u : B_r(0) \subset \mathbb{R}^n \to \mathbb{R}$, and $u = \partial_{x_n} u = \cdots = (\partial_{x_n})^{k-1}u = 0$ for $\|x'\| < r, x_n = 0$. The combination of this PDE and its data is called a *Cauchy problem*. We'll introduce a vector consisting of u and all of its derivatives up to order $k-1$,

$$\vec{v} = \left(u, \frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_1^2},\ldots,\frac{\partial^2 u}{\partial x_1 \partial x_n},\ldots,\frac{\partial^2 u}{\partial x_n^2},\ldots,\frac{\partial^{k-1}u}{\partial x_n^{k-1}}\right). \tag{2.42}$$

This is a vector in $\mathbb{R}^m$ with components $v^1,\ldots,v^m$.

Now, to convert the PDE to a first-order system, we need to express $\frac{\partial v}{\partial x_n}$ in terms of $\vec{v}$ and its spatial derivatives. To do this, we consider cases. Let $j \in \{1,\ldots,m-1\}$ run over the derivative components except the last one.

If $j = 1$, we have $v^1 = u$, so we get $\frac{\partial v^1}{\partial x_n} = \frac{\partial u}{\partial x_n} = v^l$, for some $l \in \{1,\ldots,m\}$.

If $2 \le j \le m-1$, then we can describe this derivative in terms of some multi-index: $v^j = D^\alpha u$ where $1 \le |\alpha| \le k-1$ and $\alpha_n < k-1$ (it's not the very last component). So

$$\frac{\partial v^j}{\partial x_n} = D^\alpha \frac{\partial u}{\partial x_n} = \frac{\partial^{|\alpha|+1}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n+1}}u. \tag{2.43}$$

Now, we consider some further cases. If $|\alpha| \le k-2$, then $|\alpha|+1 \le k-1$, so we once again have $\frac{\partial v_j}{\partial x_n} = v^l$ for some $l \in \{1,\ldots,m\}$. Otherwise, $|\alpha| = k-1$. We know that $\alpha_n < k-1$ so there is some other nonzero component, say p such that $\alpha_p \ge 1$. So

$$\frac{\partial v^j}{\partial x_n} = \frac{\partial}{\partial x_n}\frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} = \frac{\partial}{\partial x_p}\frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\ldots\partial x_p^{\alpha_p-1}\ldots\partial x_n^{\alpha_n+1}} = \frac{\partial v^l}{\partial x_p} \tag{2.44}$$

for some $l \in \{1,\ldots,n\}$. Now, only $j = m$ remains, for which we use the PDE. To do this, we need to derive a condition on a certain term in the PDE. Recall that the coefficients are $a_\alpha(\vec{v},x)$ for $v \in \mathbb{R}^m$. We are assuming through all of this that $a_\alpha : B_\rho(0) \to \mathbb{R}$ are analytic near zero. We'll further assume that a certain one of these is nonzero near the origin, $a_c := a_{(0,\ldots,0,k)}(0,\ldots,0) \neq 0$. a_α being real analytic implies they are continuous, so this implies $a_{(0,\ldots,0,k)}(z,x) \neq 0$ for all $\|z\|^2 + \|w\|^2 \le \delta^2$ where $\delta < \rho$.

We can now explicitly write out the PDE, slightly rearranged:

$$a_c \frac{\partial^k u}{\partial x_n^k} = -\left(\sum_{|\alpha|=k,\alpha_n<k} a_\alpha D^\alpha u + a_0\right) \tag{2.45}$$

$$\frac{\partial^k u}{\partial x_n^k} = -\frac{1}{a_c}\left(\sum_{|\alpha|=k,\alpha_n<k} a_\alpha D^\alpha u + a_0\right) \tag{2.46}$$

The RHS consists of terms in v, $\frac{\partial v^l}{\partial x_p}$ with $p < n$, which we've seen how to analyse. Therefore, as long as we can assume $a_c \neq 0$, we've turned the PDE into a first-order system.

Definition 2.3. *If $a_{(0,\ldots,0,k)}(0,\ldots,0) \neq 0$, then we say the plane $\{x_n = 0\}$ is non-characteristic (good). Otherwise it is characteristic (bad).*

2.5 Exotic boundary conditions

So far, we've just set our initial data on $\{x_n = 0\}$, which is relatively limiting; what if initial data are specified on a more general surface? Does C-K still hold?

We'll find that we can generalise to other surfaces, but they need some structure. This is analogous to the idea in ODE theory that we can solve ODEs along certain integral curves.

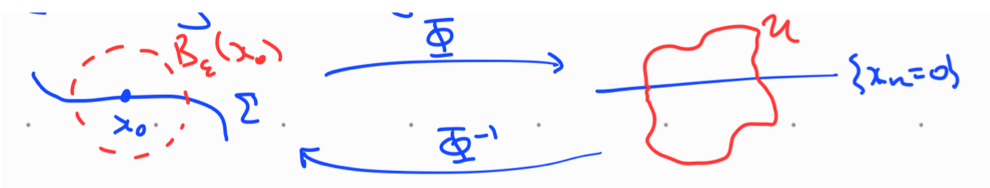
Definition 2.4. *We say that $\Sigma \subset \mathbb{R}^n$ is a real analytic hypersurface near $x \in \Sigma$ if there exists $\epsilon > 0$ and a real analytic map $\Phi : B_\epsilon(x_0) \to U \subset \mathbb{R}^n$, where $U = \Phi(B_\epsilon(x_0))$, such that*

(i) *Φ is a bijection*

(ii) *$\Phi^{-1} : U \to B_\epsilon(x_0)$ is real analytic*

(iii) *$\Phi(\Sigma \cap B_\epsilon(x_0)) = \{x_n = 0\} \cap U$.*

The idea here is we want to have a real analytic chart mapping the surface Σ to a hyperplane. For instance, in 2D, Σ is a curve, and Φ takes it to something that's locally a line: Φ "straightens out" Σ.



Spheres, planes, and tori are all real analytic surfaces.

With this idea, let's see how C-K extends to these surfaces. Let $\gamma = (\gamma^1, \ldots, \gamma^n)$ be the unit normal to Σ, and suppose u solves

$$\begin{aligned} \sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \ldots, u, x) D^\alpha u + a_0(D^{k-1}u, \ldots, u, x) = 0 \\ u = \gamma^i \partial_i u = \cdots = (\gamma^i \partial_i)^{k-1} u = 0 \quad \text{on} \quad \Sigma \end{aligned} \tag{2.47}$$

where Σ is a RA hypersurface.

Note that $\gamma^i \partial_i u = \gamma \cdot \nabla u$ is the directional derivative of u in the direction γ.

Let $v = u \circ \Phi^{-1}$ on u. Then $u(x) = v(\Phi(x))$ for $x \in B_\epsilon(x_0)$, and we can write the PDE on u as one on v using the chain rule:

$$\partial u x_i = \sum_{j=1}^{n} \partial u y_i \partial \Phi^{(j)} x_i \tag{2.48}$$

where for notational convenience we say $\Phi(x_i) = y_i$. Plugging this into the equation, we can see that v solves a quasilinear equation of the form

$$\begin{gathered} \sum_{|\alpha|=k} b_\alpha D^\alpha v + b_0 = 0 \\ v = \partial_i v = \cdots = (\partial_i)^{k-1} v = 0 \quad \text{on} \quad \{y_n = 0\} \end{gathered} \tag{2.49}$$

noting now that partial derivatives are with respect to y.

We've transformed our PDE with exotic boundary conditions into one on a hyperplane, so the only remaining aspect is seeing how the non-characteristic condition transforms. We do this by computing the top-order coefficient $b_{(0,\ldots,0,k)}$. If α is of order k, then we can directly do the transformation:

$$D^\alpha u = \frac{\partial^k v}{\partial y_n^k}^{\alpha} + \text{terms not involving } \frac{\partial^k v}{\partial y_n^k}. \tag{2.50}$$

Therefore the coefficient of $\frac{\partial^k v}{\partial y_n^k}$ is $b_{(0,\ldots,0,k)} = \sum_{|\alpha|=k} a_\alpha (\nabla \Phi^{(n)})^\alpha$.

Definition 2.5. *Σ is non-characteristic at $x_0 \in \Sigma$ for the above equation if*

$$b_{(0,\ldots,0,k)}(0,\ldots,0,x_0) = \sum_{|\alpha|=k} a_\alpha(0,\ldots,0,x_0)(\nabla \Phi^{(n)})^\alpha(x_0) \neq 0.$$

Note that $\Sigma = \{x \mid \Phi^{(n)}(x) = 0\}$: Σ are the xs whose nth component is zero under the map Φ. This means it's the graph of that function, so $D\Phi^{(n)}(x) = c(x)\gamma(x)$, where γ is the unit normal of Σ. This lets us rewrite the non-characteristic condition: the surface is non-characteristic if and only if $\sum_{|\alpha|=k} a_\alpha(0,\ldots,0,x_0), \gamma(x_0)^\alpha \neq 0$.

Σ is non-characteristic at $x_0 \in \Sigma$ if $\sum a_\alpha \big(\nabla \Phi^{(n)}(x_0)\big)^\alpha \neq 0$. By the condition that $x_0 \in \Sigma$, we see that $\nabla \Phi^{(n)}(x_0) = \underbrace{c(x_0)}_{\neq 0} \gamma(x_0)$, where γ is the unit normal of Σ. Being non-characteristic is therefore equivalent to

$$\sum_{|\alpha|=k} a_\alpha(0,\ldots,0,x_0)\gamma(x_0)^\alpha \neq 0. \tag{2.51}$$

All this gives us a more general form of Cauchy-Kovalevskaya for non-characteristic hypersurfaces:

Theorem 2.6 (C-K for data on non-characteristic hypersurfaces)**.** *Suppose $\Sigma \subset \mathbb{R}^n$ is a RA hypersurface, and consider the quasilinear PDE*

$$\begin{cases} \sum a_\alpha D^\alpha u(x) + a_0 = 0 \\ u = \gamma \nabla u = \cdots = (\gamma\nabla)^{k-1} u = 0 \quad \textit{on} \quad \Sigma \end{cases} \tag{2.52}$$

where γ is the unit normal to Σ. Suppose also a_0, a_α are RA at $x_0 \in \Sigma$, and Σ is non-characteristic at $x_0 \in \Sigma$. Then there exists a unique RA solution to the PDE in a neighbourhood of x_0.

2.6 Characteristic surfaces

We've seen what happens when the surface on which the boundary condition is defined is non-characteristic, but what if it is characteristic?

Suppose we have a Cauchy problem

$$\begin{cases} Lu = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = f \\ u = \gamma^i \partial_i u = 0 \quad \text{on} \quad \Pi_\gamma = \{x \mid \vec{x} \cdot \vec{\gamma} = 0\} \end{cases} \tag{2.53}$$

where $a_{ij} \in \mathbb{R}$ and without loss of generality $a_{ij} = a_{ji}$ (mixed partials commute) and $\|\gamma\| = 1$.

Π_γ is non-characteristic if and only if $\sum_{i,j=1}^n a^{ij}\gamma^i\gamma^j \neq 0$ (the zero gamma case is covered by the normalisation condition.)

Let's try to find non-characteristic surfaces. The condition on Π_γ being characteristic can be written in terms of the $\mathbb{R}^n$ inner product as $\langle A\gamma, \gamma\rangle$ where $A = (a_{ij})$. Because A is symmetric, it is diagonalisable, so we can write it in the form $A = P^\intercal D P$. The condition then becomes

$$\langle A\gamma, \gamma\rangle = \langle P^\intercal D P\gamma, \gamma\rangle = \langle DP\gamma, \gamma\rangle = \langle Dv, v\rangle \tag{2.54}$$

where $v = P\gamma$. We see v is a unit vector, and so this has the form $v^\intercal D v = 0$, but D is diagonal so we can write it out in terms of the eigenvalues $\{\lambda_i\}_i$ of A:

$$\Pi_\gamma \text{ non-characteristic } \iff \sum_{i=1}^n \lambda_i (v_i)^2 \neq 0. \tag{2.55}$$

Therefore, we can look at this in terms of the eigenvalues. We'll look at two interesting cases:

1. If all $\lambda_i > 0$ or $\lambda_i < 0$, there are no characteristic hyperplanes because the condition $\lambda_i v_i^2 = 0$ can't be met. In this case, L is called an *elliptic operator*. The simplest example is the Laplacian $\triangle = \sum \partial_{ii}^2$.

2. If one eigenvalue is negative and the rest are positive (or vice versa), we call L a *hyperbolic operator*. The simplest example is $-\partial_t^2 + \triangle$. These *do* admit characteristic hyperplanes, defined by the intersections of the cone $|x|^2 - t^2 = 0$ and the unit ball $\|x\|^2 + t^2 = 1$.

Our aim from here will be to focus on interesting properties of elliptic and hyperbolic equations. For the moment, let's ignore boundary conditions, and look for wave-like solutions of the form $u(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}$ to the PDE form we've been considering:

$$L\left(e^{i\vec{k}\cdot\vec{x}}\right) \sum_{j,l=1}^{n} a_{jl} k_j k_l \underbrace{=}_{\text{we'd like}} 0 \tag{2.56}$$

This happens if $\sum a_{jl} k_j k_l = 0$. If k is parallel to the normal of a characteristic surface γ, then we can write $k = c\gamma$ and this reduces to $\sum a_{jl} \gamma_j \gamma_l = 0$.

If L is elliptic, however, there is no such γ, so we must have that $c = 0$. This implies $\vec{k} = \vec{0}$, and so $u(x) = 1$. There are no wave-like (periodic in x) solutions to an elliptic PDE.

On the other hand, if L is hyperbolic, it does admit wave-like solutions $u(\vec{x}) = e^{i(\lambda\vec{\gamma})\cdot\vec{x}}$ because it does have characteristic surfaces. This gives a family of solutions per characteristic surface, parameterised by $\lambda \in \mathbb{R}$.

λ controls the size of oscillations; for instance, by taking λ large, we can give the solution large oscillations in the γ direction. Also, $u'(x) \sim \lambda$, so for large λ, the solutions become "rougher" (integrability gets harder because of the large variations in the derivative).

Can we take an infinite sum of such solutions?

Exercise 2.7. *Consider* $u(x, y) = \sum_{n=1}^{\infty} n^{-\frac{7}{2}} \cos(n(x + y))$. *This solves the hyperbolic equation* $u_{xx} - u_{yy} = 0$ *and corresponds to the characteristic line whose normal is* $\gamma = (1, 1)$. *Show* $u \in C^2$ *but* $u \notin C^3$.

The idea here is that hyperbolic equations allow for rough solutions, whereas solutions to elliptic equations tend to be smooth.

2.7 Limitations of Cauchy-Kovalevskaya

There are two main reasons why C-K alone is an insufficient theory.

1. We have no control over how long a solution exists; we only solve a PDE in a neighbourhood of an initial point.
2. We have no guarantees that the solution has continuous dependence on the data.

Example 2.7. Consider $u_{xx} + u_{yy} = 0$ subject to $u(x, 0) = \varphi(x)$ and $\partial_y u(x, 0) = 0$. We're parameterising the boundary condition using φ.

If $\varphi(x) = 0$, then $u(x, y) \equiv 0$ solves the PDE, and by Cauchy-Kovalevskaya this must be the only solution. If we varied φ a little bit so that it's close to, but not exactly zero, then will $u(x, y)$ also be close to zero? According to Hadamard's notion of well-posedness, we'd like this to be the case, but we can show this isn't always the case.

Let $\varphi_k(x) = e^{-\sqrt{k}} \cos(kx)$. A solution to this PDE is given by

$$u_k(x,y) = e^{-\sqrt{k}} \cos(kx) \cosh(ky). \tag{2.57}$$

Since φ_k and u_k are real analytic for any k, this is the solution guaranteed by Cauchy-Kovalevskaya. However, it does not satisfy well-posedness. As we take $k \to \infty$, the initial data function goes to 0:

$$\sup_{x \in \mathbb{R}^2} |\partial_x^n \varphi_k(x)| = k^n e^{-\sqrt{k}} \to 0. \tag{2.58}$$

But this isn't the case for the solution: fix $y = \epsilon > 0$ small, then

$$\sup_{x \in \mathbb{R}} |u_k(x,\epsilon)| = e^{-\sqrt{k}} \cosh(k\epsilon) \geq e^{-\sqrt{k}} \frac{1}{2} e^{\epsilon k} \to \infty. \tag{2.59}$$

□

The problem here is the restriction to real analyticity: the Cauchy-Kovalevskaya theorem gives us a solution under these constraints, but not always a particularly physically useful one. If we want to satisfy Hadamard's notion of well-posedness, we need to weaken this assumption and try to reduce regularity.

Chapter 3

Function spaces

Contents

3.1 Motivation

At the end of the last chapter, we saw that the Cauchy-Kovalevskaya theorem doesn't always help us generate well-posed solutions. However, it would be an incomplete theory even without this, because of its restrictions on boundary conditions. We know that elliptic equations admit real analytic solutions, so surely C-K is a complete theory there?

It turns out it isn't. Solving for the electrostatic potential of a conductor living in a larger space requires solving a PDE of the form

$$\begin{aligned} \triangle u &= f \quad \text{in} \quad U \\ u\Big|_{\partial U} &= 0 \quad \text{on} \quad \partial U \end{aligned} \tag{3.1}$$

which is the *Dirichlet problem*, where we can identify u as the electrostatic potential φ and f as the charge distribution ρ.

C-K can't guarantee a solution here, because we would also need data about what happens to u in the normal direction to U. That is, we would also need $\frac{\partial u}{\partial \gamma} = 0$ on ∂U. Electromagnetic theory tells us we don't actually need this data, so C-K would have us overspecify conditions we don't really need. It prioritises regularity over being physically realistic. We need a more general tool.

3.2 Hölder spaces

Another way of saying real analyticity is too strict a condition is: *not every solution we may care about necessarily lives in the space of real analytic functions*. So to start finding other solutions, we need to build some spaces in which they might be able to live. We'll impose less regular conditions in order to build larger spaces and see what existence results we can place on those.

The first of these is the Hölder spaces, which generalise continuity. Recall that C^k is defined as follows:

Definition 3.1. *Let $U \subset \mathbb{R}^n$ be open and let $k \in \mathbb{N}$.*

$$C^k(U) = \{u : U \to \mathbb{R} \mid u \textit{ is } k \textit{ times differentiable}, D^\alpha u \textit{ continuous } \forall |\alpha| \leq k\}. \tag{3.2}$$

This is a good start, but it is not a Banach space with the sup norm $\|\cdot\|_\infty$ because it doesn't say anything about what might happen on the boundary. For instance, $u(x) = \frac{1}{x}$ is in $U = (0,1)$, but the sup norm does not exist because $\sup_{x \in U} u(x) = \infty$.

We can rule this out with the next definition:

Definition 3.2. *Let $U \subset \mathbb{R}^n$ be open and let $k \in \mathbb{N}$. The space of bounded k-continuous functions is given by*

$$C^k(\overline{U}) = \{u \in C^k(U) \mid D^\alpha u \textit{ bounded and uniformly continuous on } U \forall |\alpha| \leq k\}. \tag{3.3}$$

together with the norm

$$\|u\|_{C^k(\overline{U})} = \sum_{|\alpha|\leq k} \sup_{x\in U} |D^\alpha u(x)|. \tag{3.4}$$

This requires that we can uniformly extend u to the closure ∂U.

Boundedness and uniform continuity are quite strict conditions if U is unbounded: this is a more restrictive definition than "the set of k-differentiable functions such that its derivatives of all orders are continuous".

We can show that $(C^k(\overline{U}), \|\cdot\|_{C^k(\overline{U})})$ is a Banach space. (Note that Evans' book doesn't use quite the same definition.)

Uniform continuity being a strict condition prompts us to try and relax it a bit while keeping the space Banach. This leads us to Hölder continuity.

Definition 3.3. *We say that $u : U \to \mathbb{R}$ is Hölder continuous of index $\gamma \in (0, 1]$ if there exists a constant $c \geq 0$ such that*

$$|u(x) - u(y)| \leq c|x - y|^\gamma \ \forall x, y \in U. \tag{3.5}$$

If $\gamma = 1$, we say the function is *Lipschitz continuous*. If $\gamma > 1$, it is possible to show that u must be constant: between $x = x_0$ and $y = x_n$, define $x_i = x + \frac{i}{n}(y - x)$. Then

$$|u(x) - u(y)| \leq C\|x - y\|^\gamma \underbrace{\leq}_{\text{Minkowski's inequality}} C\sum_{i=0}^{n-1} \|x_{i+1} - x_i\|^\gamma = nC\frac{\|x - y\|}{n^\gamma} \xrightarrow{n\to\infty} 0. \tag{3.6}$$

We can define the space of functions that are Hölder continuous:

Definition 3.4. *Let $\gamma \in (0, 1]$. We say*

$$C^{0,\gamma}(\overline{U}) = \{u \in C^0(\overline{U}) \mid u \textit{ is } \gamma - \textit{Hölder continuous}\} \tag{3.7}$$

is the 0-Hölder space.

We'd like to make this a Banach space, so we try to make a choice for the norm. We choose the smallest value of C in the Hölder inequality:

$$[u]_{C^{0,\gamma}(\overline{U})} = \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\gamma} \tag{3.8}$$

However, this is not a norm, because constant functions are zero under it. We therefore call it the *γ-Hölder semi-norm*, and we make it into a norm by adding the C^0 norm to it:

$$\|u\|_{C^{0,\gamma}(\overline{U})} = [u]_{C^{0,\gamma}(\overline{U})} + \|u\|_{C^0(\overline{U})}. \tag{3.9}$$

With this choice, $C^{0,\gamma}(\overline{U})$ is a Banach space.

We can extend this to differentiation at higher orders:

Definition 3.5. *The $k-$Hölder space is given by*

$$C^{k,\gamma}(\overline{U}) = \{u \in C^k(\overline{U}), D^\alpha u \in C^{0,\gamma}(\overline{U})\ \forall |\alpha| = k\} \tag{3.10}$$

with the norm

$$\|u\|_{C^{k,\gamma}(\overline{U})} = \|u\|_{C^k(\overline{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})}. \tag{3.11}$$

$C^{k,\gamma}(\overline{U})$ is a Banach space.

3.3 Lebesgue spaces

We've built spaces that guarantee differentiability, but we need our solutions to also be integrable, so we review the L^p spaces.

Definition 3.6. *Let $U \subset \mathbb{R}^n$ be open and let $1 \leq p \leq \infty$. We define the space $L^p(U)$ by*

$$L^p(U) = \{u : U \to \mathbb{R} \textit{ measurable} \mid \|u\|_{L^p(U)} < \infty\} \backslash \sim \tag{3.12}$$

where the norm is given by

$$\|u\|_{L^p(U)} = \begin{cases} \left(\int_U |u(x)|^p \mathrm{d}x\right)^{1/p} & 1 \leq p < \infty \\ \operatorname{ess\,sup}_{x \in U} |u(x)| = \inf\{c \geq 0 \mid |u(x)| \leq c \textit{ pointwise a.e.}\} \end{cases} \tag{3.13}$$

and where $u_1 \sim u_2$ if $u_1 = u_2$ a.e.

$L^p(U)$ with this norm is a Banach space: scaling and superposition are clear, and we can show completeness using the monotone and dominated convergence theorems.

Here, too, we have potential issues at the boundary, but in the other direction: we're now being potentially too restrictive about what happens at ∂U. What we really care about is *local* integrability, on parts of U that avoid the boundary, so we make this idea concrete.

We say $u \in L^p_{\text{loc}}(U)$ if $u \in L^p(V)$ for every $V \Subset U$ ("V compactly contained in U"). This means there exists a compact set K such that $V \subset K \subset U$. Another way of writing this is $L^p_{\text{loc}}(U) = \cap_{V \Subset U} L^p(V)$.

We might imagine that "cutting out" the boundary would lead to problems with completeness, and in fact it does: $L^p_{\text{loc}}(U)$ is not a Banach space. The problem actually starts even before completeness, because there's no really useful way to put a norm on it. The norm on the parent space came from integrating over U, but we can't do that here if we want to rule out boundary dependence.

The tradeoff here is avoiding the boundary is often useful anyway. Let $u(x) = \frac{1}{x^2}$ on $U = (0, 1)$. We can't integrate this globally, but we can integrate it anywhere we'd like locally. Another example is $u(x) = 1$ on $U = \mathbb{R}$. This is not in $L^1(\mathbb{R})$ because we can't integrate it, but we can integrate over any subset of a compact set, so $u \in L^1_{\text{loc}}(\mathbb{R})$.

Observe that if K is compact, $K \subset U$, and U is open, then

$$d(K, \partial U) = \inf\{|x - y| \mid x \in K, y \in U^{\complement}\} > 0 \tag{3.14}$$

That is, every compact set K has a nontrivial "buffer zone" between it and the boundary.

3.4 Weak derivatives

Now we've got a good theory of integrable functions, and we'd like to also make them differentiable. One way to do that is to just require that functions are both differentiable and locally integrable to whatever order we'd like. But these two ideas are slightly incompatible. We only understand local integrability through its behaviour over a whole set at once, by requiring that a certain integral is finite. On the other hand, when we look at nondifferentiable functions in single-variable calculus, it's always things like $f(x) = |x|$; it's continuous except for some "problem points". Issues at individual points (or in a more general setting, at any subset of measure zero) don't matter to local integrability. There's no reason to impose the high standard of differentiability everywhere on a condition that only ever requires that something be true *almost* everywhere. So we'd like an idea of what it means to be differentiable that works with the integral formulation, so that eventually we have a more expansive view of potential PDE solutions.

So what are the fundamental properties of a derivative, especially as it relates to integration? A major one is the Leibniz rule or product rule,

$$\frac{\mathrm{d}f(x)g(x)}{\mathrm{d}x} = f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x} + \frac{\mathrm{d}f(x)}{\mathrm{d}x}g(x) \tag{3.15}$$

which if we rearrange and integrate becomes *integration by parts*:

$$\int f(x)\mathrm{d}g(x) = f(x)g(x) - \int g(x)\mathrm{d}f(x) \tag{3.16}$$

(with some extra boundary terms).

The idea behind a weak derivative is we drop the limit/infinitesimal definition and say *a weak derivative is anything that satisfies integration by parts*. We'll see that this is unique and agrees with the usual derivative if it exists, but also gives us the ability to handle discontinuities.

Definition 3.7. *Let $u, v \in L^1_{\text{loc}}(U)$ and let α be a multi-index. We say v is the αth weak derivative of u if*

$$\int_U uD^\alpha\phi = (-1)^{|\alpha|}\int_U v\phi \mathrm{d}x\ \forall\phi \in C_c^\infty(U). \tag{3.17}$$

We write $v = D^\alpha u$.

We see that v takes the place that $D^\alpha u$ would have if it existed in the usual sense. Also, we've been able to get rid of the boundary term by requiring our test functions ϕ have compact support. Further, by requiring that they're C^∞, we've also imposed the requirement that the derivatives $D^\alpha\phi$ have compact support, so the left hand side of the definition makes sense.

We wanted to include $u(x) = |x|$ with this extension; does it? Yes! We can show its weak derivative is $v = \text{sgn}(x)$. Not everything is weakly differentiable, though: the Heaviside step function $H(x) = 1_{x>0}$ is not. We may know from distribution theory that the derivative of H is the Dirac delta δ_0, but δ_0 is not in L^1_{loc} and cannot be represented in the form $\int g\phi$ for some $g \in L^1_{\text{loc}}$.

Earlier we claimed weak derivatives are unique, and we can show this using the definition.

Lemma 3.1 (Uniqueness of weak derivatives)**.** *Let $v, \overline{v} \in L^1_{\text{loc}}$ both be αth weak derivatives of $u \in L^1_{\text{loc}}(U)$. Then $v = \overline{v}$ almost everywhere.*

Proof .
For all $\phi \in C_c^\infty(U)$, we have

$$\int_U v\phi \mathrm{d}x = \int_U \overline{v}\phi \mathrm{d}x = (-1)^{|\alpha|}\int_U uD^\alpha\phi \mathrm{d}x. \tag{3.18}$$

The right hand side doesn't matter beyond just being something that's independent of both v and $\overline{v}$). This gives us

$$\int_U (v - \overline{v})\phi \mathrm{d}x = 0 \tag{3.19}$$

We can use this to show that $v = \overline{v}$ almost everywhere. Suppose not. Then there exists a subset $E \subset U$ with nonzero measure, such that $v - \overline{v}$ is nonzero on E. Then, let ϕ be a smooth and arbitrarily accurate approximation of the indicator function 1_E. This would yield $\int_U (v - \overline{v})\phi \neq 0$, which would be a contradiction. Therefore $v = \overline{v}$ almost everywhere.

This also shows us that the weak derivative agrees with the usual derivative up to equality almost everywhere, because the usual derivative obeys the Leibniz rule and so the above uniqueness argument shows it must match the weak derivative.

3.5 Sobolev spaces

With an idea of derivatives that are more naturally compatible with integration, we can now build the idea of integrable functions that are differentiable up to a desired order.

Definition 3.8. *The Sobolev space $W^{k,p}(U)$ is defined by*

$$W^{k,p}(U) = \{u \in L^1_{\text{loc}}(U) \mid u \in L^p(U),\ \textit{weak derivatives } D^\alpha u \textit{ exist } \forall |\alpha| \leq k \textit{ with } D^\alpha u \in L^p(U)\} \tag{3.20}$$

and has norm

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p \mathrm{d}x\right)^{1/p} & 1 \leq p < \infty \\ \sum_{|\alpha|\leq k} \operatorname{ess\,sup}_U |D^\alpha u| & p = \infty. \end{cases} \tag{3.21}$$

There is an alternate definition of Sobolev spaces in terms of the Fourier transform, but the Fourier transform depends on a flat space, and we want a definition that works in curved spaces as well.

When $p = 2$, we write $H^k(U) = W^{k,2}(U)$. We choose H because this happens to be a Hilbert space. In practice, we'll later mostly use $H^1(U)$: the space of once-weakly-differentiable, locally-integrable and globally square-integrable functions on U.

Further, we denote by $W^{k,p}_0(U)$ the completion of $C^\infty_c(U)$ in the Sobolev(k,p) norm. The idea here is that functions in C^∞_c exist interior to U and vanish close to the boundary. If we take some sequence of them, their limit with respect to the Sobolev norm should also vanish near the boundary, and this completion enforces that. (I think).

Example 3.8. Let $U = B_1(0) \subset \mathbb{R}^n$, and consider

$$u(x) = \begin{cases} |x|^{-\lambda} & x \in B_1(0) \setminus \{0\} \\ \text{anything} & x = 0. \end{cases} \tag{3.22}$$

What Sobolev spaces does this live in? For $x \neq 0$, we can write a classical derivative in the ith direction,

$$D_i u = -\frac{\lambda x_i}{|x|^{\lambda+2}}. \tag{3.23}$$

By considering $\phi \in C^\infty_c(B_1(0) \setminus \{0\})$, we can see that if u has a weak derivative v then it matches this definition, i.e. $v_i = D_i u$. With the weak derivative, we get to ignore behaviour at the origin because it vanishes under the integral.

Note by moving to spherical coordinates that $u \in L^1(U) \iff \lambda < n$ and $v_i \in L^1(U) \iff \lambda < n - 1$. So we assume that $\lambda + 1 < n$.

We claim that the weak derivative of u in U is given by

$$v_i = \begin{cases} \frac{-\lambda x_i}{|x|^{\lambda+2}} & x \neq 0 \setminus \{0\} \\ \text{anything} & x = 0. \end{cases} \tag{3.24}$$

This follows from Stokes' theorem. Fix $\epsilon > 0$ to contain the singularity; then, we integrate the test function over $U \setminus B_\epsilon(0)$ so that we can use the classical derivative:

$$(-1)\int_{U\setminus B_\epsilon(0)} u\phi_{x_i}\mathrm{d}x = \int_{U\setminus B_\epsilon(0)} D_i u\phi \mathrm{d}x - \int_{\partial B_\epsilon} u\phi\vec{n}\cdot\mathrm{d}\vec{S}, \tag{3.25}$$

where $\vec{n}$ is the unit inward-pointing normal, and we removed the ∂U boundary term by letting ϕ have compact support.

Now, we estimate the difference between the two integrals and show it goes to 0, thereby showing that integration by parts holds meaning that v_i is the correct choice of weak derivative.

$$\left|\int_{\partial B_\epsilon(0)} u\phi\vec{n}\cdot\mathrm{d}\vec{S}\right| \leq \|\phi\|_{L^\infty}\int_{\partial B_\epsilon(0)} \epsilon^{-\lambda}\vec{n}\cdot\mathrm{d}\vec{S} \tag{3.26}$$

$$\tilde{C}\epsilon^{n-1-\lambda} \xrightarrow{\epsilon\to 0} 0 \tag{3.27}$$

where the last step uses the assumption that $n-1-\lambda > 0$.

Therefore, by the dominated convergence theorem, we get that

$$-\int_U u\phi_{x_i}\mathrm{d}x = \int_U v_i\phi\mathrm{d}x, \tag{3.28}$$

which means u has a weak derivative as above. □

We showed that a weak derivative exists for a nondifferentiable function with $|x|$, and further, the above example shows that a weakly differentiable function does not even have to be continuous.

We may also ask what function spaces $v_i = D_i u$ lives in. We can show by considering the integral that $D_i u \in L^p(U) \iff p(\lambda+1) < n$. Therefore, if $\lambda < \frac{n}{p} - 1$, then $u \in W^{1,p}(U)$. If we further impose $p > n$, then $\lambda < 0$ and $u \in C^0(U)$ so we have continuity. On the other hand, if $\lambda > \frac{n}{p} - 1$, then $u \notin W^{1,p}(U)$. In general, spaces with larger values of p are "better behaved".

Theorem 3.2. *For each $k \in \{0,1,2,\dots\}$ and $1 \leq p \leq \infty$, the Sobolev space $(W^{k,p}(U), \|\cdot\|_{W^{k,p}})$ is a Banach space.*

Proof .
To show it's a normed linear space, we use Minkowski's inequality,

$$\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{1/p} \leq \left(\sum |a_i|^p\right)^{1/p} + \left(\sum |b_i|^p\right)^{1/p}. \tag{3.29}$$

To show it is complete, let (u_j) be a Cauchy sequence in $W^{k,p}(U)$. Note that $\|D^\alpha u\|_{L^p(U)} \leq \|u\|_{W^{k,p}(U)}$ for $|\alpha| \leq k$. Set $v = u_j$. This tells us that $(D^\alpha u_j)_j$ is Cauchy in $L^p(U)$.
By completeness of $L^p(U)$ there exists some $u^\alpha \in L^p(U)$ such that $D^\alpha u_j \to u^\alpha$ in L^p for each $\|\alpha\| \leq k$. Call $u = \lim_j u_j^{(0,\ldots,0)}$. We claim that u^α is the weak derivative of $D^\alpha u$ of the limit u. Now, it remains to prove that $u^\alpha = D^\alpha u$.
Let $\phi \in C_c^\infty(U)$ be a test function. Since $u_j \in W^{k,p}$, we can see that $D^\alpha u_j$ exist and satisfy the definition of the weak derivative:

$$(-1)^{|\alpha|}\int_U u_j D^\alpha \phi \mathrm{d}x = \int_U D^\alpha u_j \phi \mathrm{d}x \; \forall j. \tag{3.30}$$

Take $j \to \infty$, use $u_j \xrightarrow{L^p} u$, and show $D^\alpha u_j \xrightarrow{L^p} u^\alpha$ by Hölder's inequality. We can conclude the statement in equation 3.30. By the definition of the weak derivative, $D^\alpha u = u^\alpha \in L^p(U)$, and therefore $u \in W^{k,p}(U)$, which was what we wanted.

3.6 Approximations of functions in Sobolev spaces

An obvious caveat when we're working with Sobolev spaces is the functions have limited regularity. We made these spaces with finite differentiability and integrability built in, which gives us a large set of functions to work with when solving PDEs, but often makes it difficult to make actual statements about them. One way of resolving this is smoothing functions out by convolving them with smooth mollifiers.

Definition 3.9. *Let*

$$\eta(x) = \begin{cases} C\exp\left(\frac{-1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}, \tag{3.31}$$

where we choose C such that $\int_{\mathbb{R}^n} \eta(x)\mathrm{d}x = 1$.

For each $\epsilon > 0$, let $\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)$.

We call η_ϵ the standard mollifier.

Exercise 3.3. *Show that $\eta_\epsilon \in C_c^\infty(\mathbb{R}^n)$, that $\operatorname{supp}(\eta_\epsilon) \subseteq B_\epsilon(0)$, and that $\int_{\mathbb{R}^n} \eta_\epsilon(x)\mathrm{d}x = 1$.*

Suppose $U \subset \mathbb{R}^n$ is open. Let $U_\epsilon = \{x \in U \mid \operatorname{dist}(x, \partial U) > \epsilon\}$.

Definition 3.10. *If $f \in L^1_{loc}(U)$, then the mollification of f is $f_\epsilon : U_\epsilon \to \mathbb{R}$, given by*

$$f_\epsilon(x) = \int_U \eta_\epsilon(x-y)f(y)\mathrm{d}y = \int_{B_\epsilon(x)} \eta_\epsilon(y)f(x-y)\mathrm{d}y. \tag{3.32}$$

Theorem 3.4 (Properties of mollifiers). *Let f, f_ϵ be defined as above.*

(i) $f_\epsilon \in C^\infty(U_\epsilon)$ *(massive improvement!)*

(ii) $f_\epsilon \xrightarrow{\epsilon\to 0} f$ *a.e. in* U

(iii) *if* $f \in C^0(U)$ *then* $f_\epsilon \to f$ *uniformly on compact subsets of* U

(iv) *if* $1 \leq p < \infty$ *and* $f \in L^p_{loc}(U)$*, then* $f_\epsilon \to f$ *in* $L^p_{loc}(U)$*, i.e.* $\|f_\epsilon - f\|_{L^p(V)} \to 0$ *for all* $V \subset\subset U$.

Lemma 3.5 (Local approximations of Sobolev functions away from ∂U). *Let* $u \in W^{k,p}(U)$ *for some* $1 \leq p < \infty$*, and set* $u_\epsilon = \eta_\epsilon * u$ *on* U_ϵ*. Then*

(i) $u_\epsilon \in C^\infty(U_\epsilon)$ *for each* $\epsilon > 0$

(ii) *if* $V \subset\subset U$ *then* $u_\epsilon \to u$ *in* $W^{k,p}(V)$.

Proof of (ii).
Since $u_\epsilon \in C^\infty$ we can compute its classical derivative,

$$\begin{aligned} D^\alpha_x u_\epsilon(x) &= D^\alpha_x \int_U \eta_\epsilon(x-y)u(y)\mathrm{d}y = \int_U D^\alpha_x \eta_\epsilon(x-y)u(y)\mathrm{d}y \\ &= (-1)^{|\alpha|}\int_U (D^\alpha_y \eta_\epsilon(x-y))u(y)\mathrm{d}y \\ &= (-1)^{|\alpha}(-1)^{|\alpha|}\int_U \eta_\epsilon(x-y)D^\alpha_y u(y)\mathrm{d}y \\ &= (\eta_\epsilon * D^\alpha u)(x). \end{aligned} \tag{3.33}$$

Fix $V \subset\subset U$. By the properties of mollifiers (iv), since $D^\alpha u \in L^p(U)$,

$$\begin{aligned} &D^\alpha u_\epsilon = \eta_\epsilon * D^\alpha u \to D^\alpha u \text{ in } L^pV \text{ on } \epsilon \to 0 \\ \implies &\forall V \subset\subset U \forall \delta > 0, \exists \epsilon_0 = \epsilon_0(\delta, V) \text{ s.t. } \|u_\epsilon - u\|^p_{W^{k,p}(U)} = \sum_{|\alpha|\leq k} \|D^\alpha u_\epsilon - D^\alpha u\|_{L^pV} \leq \delta \\ &\forall 0 < \epsilon < \epsilon_0. \end{aligned} \tag{3.34}$$

Theorem 3.6 (Global approximation by smooth functions). *Suppose* $U \subset \mathbb{R}^n$ *is open and bounded, and suppose* $u \in W^{k,p}(U)$ *for some* $1 \leq p < \infty$*. Then there exists* $(u_j) \in C^\infty(U) \cap W^{k,p}(U)$ *such that* $u_j \to u$ *in* $W^{k,p}(U)$.

Note that we don't yet assert that $u_j \in C^\infty(\overline{U})$.

Proof .
The idea is to exhaust U by compact sets.
We have $U = \cup_{j=1}^\infty U_j$ where $U_j = \{x \in U \mid \text{dist}(x, \partial U) > \frac{1}{j}\}$: each U_j gets closer to the boundary.
Let $V_j \triangleq U_{j+3} \setminus \overline{U_{j+1}}$, so that they include the regions between the Us and also overlap with one another if they're consecutive. U is a bounded set, so the U_js are precompact (their closure is compact), so the V_js are compactly contained in U (or symbolically $V_j \subset\subset U$). Choose $V_0 \subset\subset U$ to contain the interior, such that $U = \cup_{j=0}^\infty V_j$.

Now, we want to approximate u on each set V_j. Let $(\xi_j)_{j=0}^{\infty}$ be a partition of unity subordinate to V_j. That means they're functions of x such that $0 \le \xi_j \le 1$, they are smooth and compactly supported on V_j ($\xi_j \in C_c^{\infty}(V_j)$), and they add to 1: $\sum_{j=1}^{\infty} \xi_j(x) = 1$ for $x \in U$. [a]
Given $u \in W^{k,p}(U)$, consider the product $\xi_j u$: it's Sobolev and contained in V_j, or $\xi_j u \in W^{k,p}(U)$ and $\operatorname{supp}(\xi_j u) \subset V_j$. So we've localized u to V_j. Note that these don't agree on overlaps in general, as the partition of unity specifies nothing about that.
Next, we smooth out our split-up function. Let $W_j \triangleq U_{j+4} \setminus \overline{U_j}$ and note $\operatorname{supp}(\xi_j u) \subset U_j \Subset W_j$. By Lemma 3.5 , we know that we can mollify this function:

$$u_j := \eta_{\epsilon_j} * (\xi_j u) \to \xi_j u \text{ on } V_j. \tag{3.35}$$

Fix $\delta > 0$. For each j, we can choose ϵ_j sufficiently small such that $\operatorname{supp}(u_j) \subset W_j$ (convolution in general may wobble the support of u_j outside of W_j, but we can restrict that with an appropriate ϵ) and also such that

$$\|u_j - \xi_j u\|_{W^{k,p}(U)} = \|u_j - \xi_j u\|_{W^{k,p}(W_j)} \le \frac{\delta}{2^{j+1}}. \tag{3.36}$$

The last step, to close the argument, puts the u_js together to create the desired approximation. This uses the facts that $\sum \xi_j = 1$ and that $u_j \neq 0$ for only finitely many W_js. Let $v = \sum_{j=1}^{\infty} u_j$. Then $v \in C^{\infty}(U)$, since for each open set $V \Subset U$ this is a finite sum of smooth functions. Since the sum of this partition of unity is 1, we have that

$$u = u \cdot 1 = \sum_{j \ge 1} \xi_j u \text{ on } U. \tag{3.37}$$

So for any $V \Subset U$, we have

$$\|v - u\|_{W^{k,p}(V)} \underbrace{\le}_{\text{triangle inequality}} \sum_{j=0}^{\infty} \|u_j - \xi_j u\|_{W^{k,p}(V)} \le \delta \sum_{j \ge 0} 2^{-j+1} = \delta. \tag{3.38}$$

Note that δ is independent of V, so if we take a supremum over all such $V \Subset U$, we have some δ such that $\|v - u\|_{W^{k,p}(U)} \le \delta$. Therefore $v \in W^{k,p}(U)$, which was what we wanted.

[a] The existence of a partition of unity should be somewhere in Lee ISM, I think.

3.7 Approximations

By global approximation, we made a sequence of functions $(u_j) \in C^{\infty}(U) \cap W^{k,p}(U)$ converging to any $u \in W^{k,p}(U)$ for $1 \le p \le \infty$. Our next natural question is: can we approximate $W^{k,p}(U)$ by $C^{\infty}(\overline{U})$? There's pathological counterexamples that make this difficult, like the Cantor set in $\mathbb{R}^2$: let $U = \mathbb{R}^2 \setminus C$, then $\partial U = C$.

Definition 3.11. *Suppose $U \subset \mathbb{R}^n$ is open and bounded. We say that ∂U is a $C^{k,\infty}$ domain if $\forall p \in \partial U$ there exists $r > 0$ and a function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}, \gamma \in C^{k,\alpha}(\mathbb{R}^{n-1})$ and such that $U \cap B_r(p) = \{(x', x_n) \in B_r(p), x_n > \gamma(x')\}$ where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$.*

In words, locally ∂U is the graph of a $C^{k,\alpha}$ function: $\partial U \cap B_r(p) = \{x_n = \gamma(x')\}$.

Theorem 3.7 (Global approximation by smooth Sobolev functions)**.** *Let $U \in \mathbb{R}^n$ be open and bounded with Lipschitz boundary, i.e. ∂U is a $C^{0,1}$ domain. Let $u \in W^{k,p}(U), 1 \leq p < \infty$. Then there exists $(u_j)_j \subset C^\infty(\overline{U})$ such that $u_j \to u$ in $W^{k,p}(U)$.*

The idea of the proof is that for all $p \in \partial U$ there exists a local approximation. Since ∂U is compact, for every open cover there exists a finite subcover, and the interior of U is already covered.

Proof .
Fix $p \in \partial U$. Since $\partial U \in C^{0,1}$, there exists $r = r(p) > 0$ and a Lipschitz function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $U \cap B_r(p) = \{x \in B_r(p), x_n > \gamma(x_1, \dots, x_{n-1})\}$.
Set $V = U \cap B_{r/2}(p)$. Observe

$$|\gamma(x) - \gamma(p)| \leq L|x - p|. \tag{3.39}$$

In words, this tells us that the function can't grow faster away from p than a line with gradient L. In particular, we can always fit a cone that emanates from the point p (this idea is called the *method of cones*) inside the graph, such that its gradient is $2L$.
Let C_x be a cone of height $\frac{r}{4}$, with an opening angle 2θ such that $\cot\theta = 2L$, and with its vertex at x. Then $C_p \setminus \{p\} \subset V$.
From here, to deal with the boundary, we'll move points up and then mollify them. For instance, for $x \in V$, $C_{x+\delta\hat{e}_n} \subset U \cap B_r(p)$ for $0 \leq \delta \leq \frac{r}{8}$.
For the shifted point $x^\delta := x + \delta\hat{e}_n$ where $x \in V, 0 < \delta < \frac{r}{8}$, we have

$$V^\delta = \{x^\delta \mid x \in U\} \subset U \cap B_r(p). \tag{3.40}$$

Let $d(\delta) = \operatorname{dist}\left(\overline{V}^\delta, \partial(U \cap B_r(p))\right)$. For every $0 < \epsilon < \frac{1}{2}d(\delta)$ we have $B_\epsilon(x) \subset U \cap B_r(p)$ for $x \in V$, and in fact $\overline{V}^\delta \subset\subset U$.
Define $u_\delta(x) = u(x + \delta\hat{e}_n)$ for $x \in V$, $v_{\epsilon,\delta} = \eta_\epsilon * u_\delta$ for $0 < \epsilon < \frac{1}{2}d(\delta)$. Then $u_{\epsilon,\delta} \in C^\infty(\overline{V})$. FIx $\mu > 0$ small. Then note

$$\|v_{\epsilon,\delta} - u\|_{W^{k,p}(V)} \leq \|v_{\epsilon,\delta} - u_\delta\|_{W^{k,p}(V)} + \|u_\delta - u\|_{W^{k,p}(V)}. \tag{3.41}$$

The translator operator is constant in the L^p-norms for $p < \infty$ (handout) so we pick $\delta > 0$ such that $\|u_\delta - u\|_{W^{k,p}(V)} \leq \mu$. Fix such a δ, and choose $0 < \epsilon < \frac{1}{2}d(\delta)$ small enough such that $\|v_{\epsilon,\delta} - u_\delta\|_{W^{k,p}(V)} < \mu$.
Now, let p vary. We see that the sets V_p cover ∂U. But ∂U is compact, so we can find finitely many points p_i and radii $r_i > 0$ such that $V_i = U \cap B_{r_i/2}(p_i), i = 1, \dots, n$ is a cover of ∂U. Choose $V_0 \subset\subset U$ such that $U = \cup_{i=0}^n V_i$. By the above, we have that $v_i \in C^\infty(\bar(U_i))$ such that $\|v_i - u\|_{W^{k,p}(U_i)} \leq \mu$ for each i. By Lemma there exists $v_0 \in C^\infty(\overline{V_0})$ such that $\|v_0 - u\|_{W^{k,p}(V_0)} \leq \mu$.
Let $(\xi_i)_{i=0}^n$ be a smooth partition of unity subordinate to $\{U_0, \dots, U_n\}$. Define $v = \sum_{i=0}^n v_i\xi_i$, $v \in C^\infty(\overline{U})$. Further, for $|\alpha| \leq k$,

$$\|D^\alpha v - D^\alpha u\|_{L^p(U)} = \left\| D^\alpha(\sum_i v_i\xi_i) - D^\alpha(\sum_i u)\right\|_{L^p(U)} \tag{3.42}$$
$$\leq \sum_{i=0}^{n} \|D^\alpha(\xi_i(v_i - u))\|_{L^p(V_i)} \tag{3.43}$$
$$\leq C_k \sum_{i=0}^{n} \|v_i - u\|_{W^{k,p}(V_i)} \tag{3.44}$$
$$\leq C_k(1+N)\mu, \tag{3.45}$$

and by choosing μ as small as we want, we can show $\|v-u\|_{W^{k,p}(U)} \leq C\mu \to 0$, which was what we wanted.

The key idea here was the geometric one, where we took a smooth approximation of the function on the boundary and extended it with the cone to all of U.

3.8 Extensions and traces

Let u be Sobolev ($u \in W^{k,p}(U)$) where U is open and bounded in $\mathbb{R}^n$. We'd like to make a function $\bar{u}$ such that $\bar{u}\big|_U = u$ and it's defined on all of $\mathbb{R}^n$. To start with, how do we extend $\bar{u}$ to the boundary? We can't just set the function to $\bar{u} = 0$ on $\mathbb{R}^n \setminus U$ because we would get discontinuities.

Theorem 3.8 (Extension theorem)**.** *Suppose U is open and bounded and ∂U is $C^{1,0}$. Choose V such that $V \Subset U$ and let $1 \leq p < \infty$. Then there exists a bounded linear operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that for all $u \in W^{1,p}(U)$,*

1. *$E(u) = u$ a.e. in U*
2. $\operatorname{supp}(E(u)) \subset V$
3. *$\|E(u)\|_{W^{1,p}(\mathbb{R}^n)} \leq c\|u\|_{W^{1,p}(U)}$ where $C = C(U, u, p)$.*

We call Eu an extension of u to $\mathbb{R}^n$.

Proof sketch.

1. We're going to establish the result for C^1 functions first. Fix $x_0 \in \partial U$ and suppose ∂U flat, i.e. it's of the form $\{x_n = 0\}$. We can assume there exists $r > 0$ such that

$$B^+ = B_r(x_0) \cap \{x_n \geq 0\} \subset \overline{U}$$
$$B^- = B_r(x_0) \cap \{x_n < 0\} \subset \mathbb{R}^n \setminus \overline{U}.$$

Suppose also that $u \in C^1(\overline{U})$.

2. Define

$$\overline{u}(x) = \begin{cases} u(x) & x \in B^+ \\ -3u(x', -x_n) + 4u\big(x', -\frac{x_n}{2}\big) & x = (x', x^n) \in B^- \end{cases} \tag{3.46}$$

We call this a higher-order reflection of u from $B^+ \to B^-$. There's nothing special about the numbers $-3, 4$ - they're just the easiest choices.

We claim $\overline{u} \in C^1(B_r(x^0))$. Clearly $\overline{u} \in C^0(B_r(x_0))$ Then we compute the x_n derivative,

$$\partial_{x_n}\overline{u}(x) = \begin{cases} \partial_{x_n} u_x & x \in B^+ \\ 3\partial_{x_n} u(x', -x_n) - 2\partial_{x_n} u\big(x', -\frac{x_n}{2}\big) & x \in B^- \end{cases} \tag{3.47}$$

For any other coordinate $i = 1, \ldots, n-1$,

$$\partial_{x_i}\overline{u} = \begin{cases} \partial_{x_i} u & x \in B^+ \\ 3\partial_i u(x', -x_n) - 2\partial_i u\big(x', -\frac{x_n}{2}\big) & x \in B^- \end{cases} \tag{3.48}$$

and so we have that $D^\alpha \overline{u}\big|_{x_n = 0^+} = D^\alpha \overline{u}\big|_{x_n = 0^-}$ for $|\alpha| \leq 1$.

We can also check that $\|\overline{u}\|_{W^{1,p}(B_r(x_0))} \leq C\|u\|_{W^{1,p}(B^+)}$ for some constant C independent of u. Therefore, we define $E(u) = \overline{u}$.

The cool part here is the third condition in the theorem, which shows us that we can bound the function just by its values on the subset U.

3. Now, suppose ∂U is not flat near x_0. Using the fact it is C^1, there exist $r > 0, \gamma \in C^1$ such that $U \cap B_r(p) = \{\vec{x} \in B_r(p) \mid x_n > \gamma(x')\}$. By a change of coordinates Φ, we can shift this such that the image of this set $\Phi(U \cap B_r(p))$ is centered around p. This is achieved with the definitions

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \Phi(x) = y \tag{3.49}$$
$$y_i = x_i = \Phi_i(x), i = 1, \ldots, n-1 \tag{3.50}$$
$$y_n = x_n - \gamma(x') = \Phi_n(x) \tag{3.51}$$

and note that Φ is invertible, with inverse

$$\Psi : \mathbb{R}^n \to \mathbb{R}^n, \Psi(y) = x \tag{3.52}$$
$$x_i = y_i = \Psi_i(y), i = 1, \ldots, n-1 \tag{3.53}$$
$$x_n = y_n + \gamma(y_1, \ldots, y_{n-1}) = \Psi_n(x). \tag{3.54}$$

We can check that $\Phi \circ \Psi = \Psi \circ \Phi = id$, that the determinants of the derivatives are the identity, that $\Phi(U \cap B_r(x_0)) \subseteq \{y_n > 0\}$, and that Ψ, Φ are both C^1. We find that Φ is a diffeomorphism.

The image of $U \cap B_r(x_0)$ under Φ is the open ball $B_s(\tilde{x_0})$ for some $s > 0$ and $\Phi(p) = \tilde{p}$. Further,

$$\Phi(U \cap W) = B_s(\tilde{p}) \cap \{y_n > 0\} = B_+. \tag{3.55}$$

Now, define $v(y) = u(\Psi(y))$ for $y \in B_+$. Then $v \in C^1(B_+)$. There exists an extension $\tilde{v}(y) \in C^1(B_s(\tilde{p}))$ such that $\bar{v}\big|_{B^+} = v$ and $\|\bar{v}\|_{W^{1,p}(B_s(\tilde{p}))} \leq C\|v\|_{W^{1,p}(B_+)}$.

Mapping this backwards to the x coordinates (by something we'll do on Example Sheet 2) we can show we have a bound on u as well;

$$\|\bar{u}\|_{W^{1,p}(W)} \leq c\|u\|_{W^{1,p}(U)}. \tag{3.56}$$

4. Now, we have a local extension for all $p \in \partial U$ to $W = W_p$. Let $\{W_0, \dots, W_N\}$ form a finite subcover of U:

$$U \subset \cup_{i=0}^N W_i, \tag{3.57}$$

with extension $\bar{u}_i \in C^1(W_i)$ for $i = 1, \dots, N$. Let $\bar{u}_0 = u$.

Let $(\xi_i)_{i=0}^N$ be a partition of unity subordinate to $\{W_i\}$. That means $\operatorname{supp}(\xi_i) \Subset W_i$, $\sum \xi_i = 1$ on U. Then $\bar{u} = \sum_{i=0}^N \xi_i \bar{u}_i$. Then we have that $\bar{u}\big|_U = u$ and $\bar{u} \in C_c^1(\mathbb{R}^n)$. We also have

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq c\|u\|_{W^{1,p}(U)}, \tag{3.58}$$

by the inequality above that we'll do on example sheet 2.

5. Now we may assume that $\operatorname{supp}(\bar{u}) \subset V$ for some $U \Subset V$ by multiplying some cutoff function χ. Let $U \Subset S \Subset V$. Then $\chi\big|_U = 1$, $\chi\big|_{S^\complement} = 0$ by the mollifiers handout.

Next, there exists a sequence of functions in $C^\infty(\bar{U})$ such that $u_j \to u$ in $W^{1,p}(U)$, by the global approximation of smooth functions. We claim that $(E(u_j))_j$ is Cauchy in $W^{1,p}(\mathbb{R}^n)$.

Since $u_j \in C^\infty(\overline{U}) \subset C^1(\overline{U})$, by the above, $E(u_j) \in W^{1,p}(\mathbb{R}^n)$. Since $\overline{u}$ is defined by reflections and adding, we have linearity, so

$$\|E(u_j) - E(u_k)\|_{W^{1,p}(\mathbb{R}^n)} = \|E(u_j - u_k)\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u_j - u_k\|_{W^{1,p}(U)}. \tag{3.59}$$

$(u_j)_j$ is convergent in $W^{1,p}(U)$, so it is Cauchy. Therefore, $(E(u_j))_j$ is also Cauchy in the complete space $W^{1,p}(\mathbb{R}^n)$, so there exists a limit in $W^{(1,p)}(\mathbb{R}^n)$. Therefore, we can set $E(u) = \lim_{j\to\infty} E(u_j)$.

As an exercise, we can check that the limit is independent of the sequence approximating it.

In general, for $W^{k,p}(U)$, if we consider $\partial U \in C^k$ and $u \in C^k(\overline{U})$, then we can write the extension in the form

$$\overline{u}(x) = \begin{cases} u(x) & \text{in } B_+ \\ \sum_{j=1}^k c_j u\Big(x', -\frac{x_n}{j}\Big) & \text{in } B_- \end{cases} \tag{3.60}$$

where for matching at the boundary we end up needing to impose $\sum_{j=1}^k c_j \left(\frac{-1}{j}\right)^m = 1$ for all $m = 0, 1, \ldots, k-1$.

3.9 Traces

We'd like to talk about boundary values for elliptic PDEs. Previously, this was no problem, because we get the boundary values by just carrying out the restriction $u\big|_{\partial U}$ on functions in $C^0(\overline{U})$. However, this does not necessarilycarry over to Sobolev functions, because ∂U is a set of measure zero and u is only defined almost everywhere in U. So we replace the restriction with a trace operator T.

Theorem 3.9 (Trace theorem)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. Then there exists a bounded linear operator $T : W^{1,p}(U) \to L^p(\partial U), 1 \leq p < \infty$, called the trace of u on ∂U, such that*

(i) $T(u) = u\big|_{\partial U}$ *if* $u \in W^{1,p}(U) \cap C(\overline{U})$

(ii) $\|T(u)\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$ *for each* $u \in W^{1,p}(U)$*, where* C *depends only on* U *and* p*.*

We assume $u \in L^p$ and also $Du \in L^p$. We can't change u on the measure zero set ∂U.

Proof sketch.
Suppose $u \in C^1(\overline{U})$ and ∂U is flat near $x_0 \in \partial U$. Define the upper and lower half balls

$$\begin{aligned} B_+ &= B_r(x_0) \cap \{x_n \geq 0\} \subset \overline{U} \\ B_- &= B_r(x_0) \cap \{x_n \leq 0\} \subset \mathbb{R}^n \setminus U. \end{aligned} \tag{3.61}$$

Let Γ be the portion of ∂U within $B_{r/2}(p)$. Pick $\xi \in C_c^\infty(B_r(p))$ such that $0 \leq \xi \leq 1$ on $B_r(p)$ and $\xi = 1$ on $B_{r/2}(p)$. Then

$$\int_\Gamma |u(x', 0)|^p \mathrm{d}x' \leq \int_{B_r(p)\cap\{x_n=0\}} \xi |u(x', 0)|^p \mathrm{d}x' \underbrace{=}_{\text{FTC}} - \int_{B_+} \partial_{x_n}(\xi |u|^p) \mathrm{d}x_n \mathrm{d}x' \tag{3.62}$$

where the FTC step is possible because $u \in C^1(\overline{U})$, so $|u|$ is Lipschitz. We further simplify this (by a step on Sheet 2) to

$$- \int_{B_+} \partial_{x_n}(\xi |u|^p) \mathrm{d}x_n \mathrm{d}x' = - \int_{B_+} |u|^p \partial_{x_n} \xi + p|u|^{p-1} \operatorname{sgn}(u) \partial_{x_n} u \cdot \xi \mathrm{d}x \tag{3.63}$$

$$\leq c_p \int_{B_+} |u|^p + |Du|^p \mathrm{d}x \leq C_p \|u\|^p_{W^{1,p}(U)} \tag{3.64}$$

where the $\leq$ is because of Young's inequality $|ab| \leq \frac{|a|^m}{m} + \frac{|b|^n}{n}$, where $\frac{1}{m} + \frac{1}{n} = 1$ with $m = \frac{p}{p-1}, n = p$.
We identify this integral as T. To complete the proof, we extend this to general ∂U using the fact that ∂U is compact and that $C^\infty(\overline{U})$ is dense in $W^{1,p}(U)$.

Recall that $W_0^{k,p}(U)$ is the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$. So if $u \in W_0^{k,p}(U)$ then there exists $u_j \in C_c^\infty(U)$ converging to u in $W^{1,p}(U)$. Therefore, by the continuity (boundedness + linearity) of T we have

$$T(u) = T(\lim u_j) = \lim_j T(u_j) = \lim_j u_j \bigg|_{\partial U} = 0. \tag{3.65}$$

The converse also holds, so $T(u) = 0$ iff $u \in W_0^{k,p}(U)$. Finally, if $u \in W^{k,p}(U)$, we can define trace operators for derivatives of all orders $Du, \dots, D^{k-1}u$.

3.10 Sobolev inequalities

The general idea is we can trade differentiability (k) for integrability (p). Note that we can't go the other way.

Example 3.9. Let $f' \in L^1(\mathbb{R})$. Then $f \in L^\infty(\mathbb{R})$. However, $f \in L^\infty(\mathbb{R}) \not\Rightarrow f' \in L^1$. □

The idea is we'll show $\|u\|_{L^q(\mathbb{R}^n)} \leq \|Du\|_{L^p(\mathbb{R}^n)}$. We'll split this up into three cases: $1 \leq p < n$, $p = n$, and $n < p \leq \infty$.

Lemma 3.10. *Let $n \geq 2$ and let $f_1, \dots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. For any $1 \leq i \leq n$, we denote $\tilde{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$. Set*

$$f(x) = \prod_{i=1}^n f_i(\tilde{x}_i). \tag{3.66}$$

Then $f \in L^1(\mathbb{R}^n)$[1], and further,

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}. \tag{3.67}$$

Proof .
We proceed by induction starting with the case $n = 2$. Then

$$f(x) = f_1(x_2) f_2(x_1). \tag{3.68}$$

[1]each component is a function of $n-1$ variables, but each one gets used $n-1$ times so it's a function of n variables overall

and the norm is

$$\|f\|_{L^1(R^2)} = \int_{\mathbb{R}^2} |f(x_1, x_2)| \mathrm{d}x_1 \mathrm{d}x_2 \tag{3.69}$$

$$\underbrace{=}_{\text{Fubini}} \int |f_1(x_2)| \left(\int_{\mathbb{R}} |f_2(x_1)| \mathrm{d}x_1 \right) \mathrm{d}x_2 \tag{3.70}$$

$$= \|f_2\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} |f_1(x_2)| \mathrm{d}x_2 \tag{3.71}$$

$$= \|f_1\|_{L^1(\mathbb{R})} \|f_2\|_{L^1(\mathbb{R})}. \tag{3.72}$$

Suppose case n is true. Write $f(x) = \underbrace{f_1(\tilde{x_1}) \dots f_n(\tilde{x_n})}_{F(x)} f_{n+1}(\tilde{x_{n+1}})$. Then, fix x_{n+1} and integrate with respect to everything else:

$$\int_{\mathbb{R}^n} |f(\xi_1, \dots, \xi_n, x_{n+1})| \mathrm{d}\xi_1 \dots \mathrm{d}\xi_n = \int_{\mathbb{R}^n} F(\xi, x_{n+1}) f_{n+1}(\xi) \mathrm{d}^n \xi. \tag{3.73}$$

Let $p = n, q = \frac{n}{n-1}$. Applying Hölder's inequality,

$$\int_{\mathbb{R}^n} F(\xi, x_{n+1}) f_{n+1}(\xi) \mathrm{d}^n \xi \leq \|F(\cdot, x_{n+1})\|_{L^q(\mathbb{R}^n)} \|f_{n+1}\|_{L^p(\mathbb{R}^n)}. \tag{3.74}$$

Now, we apply the inductive hypothesis:

$$|F(\xi, x_{n+1})|^q = |f_1(\xi, x_{n+1})|^q \cdot |f_n(\xi, x_{n+1})|^q, \tag{3.75}$$

which implies

$$\|F(\cdot, x_{n+1})\|_{L^q(\mathbb{R}^n)} \leq \prod_{i=1}^n \|f_i(\cdot, x_{n+1})^{\frac{n}{n-1}}\|_{L^{n-1}(\mathbb{R}^{n-1})}^{\frac{n}{n-1}} \tag{3.76}$$

$$= \prod_{i=1}^n \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})}. \tag{3.77}$$

Now, we integrate over x_{n+1}.

$$\|f\|_{L^1(\mathbb{R}^{n+1})} \leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \int_{\mathbb{R}} \prod_{i=1}^n \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n+1})} \mathrm{d}x_n. \tag{3.78}$$

By the generalized Hölder inequality, the following holds:

$$\|f\|_{L^1(\mathbb{R}^{n+1})} \leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \left(\int_{\mathbb{R}} \prod_{i=1}^{n} \|f_i(\cdot, x_{n+1})\|^n_{L^n(\mathbb{R}^{n+1})} \mathrm{d}x_{n+1} \right)^{1/n} \tag{3.79}$$

$$= \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \prod_{i=1}^{n} \|f_i\|_{L^n(\mathbb{R}^n)} \tag{3.80}$$

$$= \prod_{i=1}^{n+1} \|f_i\|_{L^n(\mathbb{R}^n)}, \tag{3.81}$$

which was what we wanted.

This lemma will help us show the following important theorem:

Theorem 3.11 (Gagliardo-Nirenberg-Sobolev (GNS) Inequality)**.** *Assume $1 \leq p < n$. Then $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ where $p^* = \frac{np}{n-p}$ is the Sobolev conjugate to p. Moreover, this embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ is continuous, i.e. there exists $c = c(n,p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^n)$, $\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c\|Du\|_{L^p(\mathbb{R}^n)}$.*

p^* is more commonly written in the form $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. $p^* > p$, so differentiation implies more integrability. Note that nothing is said about the integrability of Du.

Example 3.10. One might ask: who cares that this embedding is continuous? It's actually very easy to break: $(C^0([0,1],\mathbb{R}), L^1)$ embeds discontinuously into $(C^0([0,1],\mathbb{R}), L^\infty)$. □

Let's gain some intuition for this theorem. Consider $f : \mathbb{R} \to \mathbb{R}$. We can use $L^p(\mathbb{R})$ to quantify the width and height of a function.

Example 3.11. If $f_1 = A1_w(x)$ for some interval w, then $\|f\|_p = \underbrace{|A|}_{\text{height}} \underbrace{(vol)(W)^{1/p}}_{\text{width}}$. □

Example 3.12. Let $\phi \in C_c^\infty(\mathbb{R})$ be a bump function and let ω be some large number. Then define $f_2(x) = \phi_x \sin(\omega x)$. The height of f_2 is bounded by 1, and the width of f_2 is bounded by C uniformly in ω.

However, we don't just want this bound: we also want to quantify the regularity and frequency scales. In the above example, the frequency is on the order ω, which we can see by taking a derivative:

$$f_2'(x) = \phi' \sin(\omega x) + \omega\phi \cos(\omega x) \tag{3.82}$$

and since this grows with ω, we can't say $|f_2'| \leq C$ in ω. □

Example 3.13. Consider the similar function

$$f_3(x) = \omega^{-k}\phi(x)\sin(\omega x), \tag{3.83}$$

for some $k \geq 0$. Then the frequency of f_3 is of the order ω, and we can have a constant bound on $\left|\partial_x^l f_3\right|$ if $l \leq k$. □

Example 3.14. Let $f_4(x) = A\phi\big(\frac{x}{R}\big)\sin(\omega x)$. Then

$$\|f_4\|_{W^{k,p}} \sim \left(\int_{|x|\leq R} |A\phi\sin(\omega x)|^p \mathrm{d}x + \dots\right)^{1/p} \sim |A|^p R^p |\omega|^k \gtrsim |A| R^{\frac{1}{p}-1} = \|f\|_{L^{p^*}(\mathbb{R})}, \tag{3.84}$$

for the case $n = 1$. □

The uncertainty principle tells us that the width times the frequency has to be greater than some constant, which is what we used in the $\gtrsim$ step.

Proof .
Assume $u \in C_c^\infty(\mathbb{R}^n) \underbrace{\subset}_{\text{dense}} W^{1,p}(\mathbb{R}^n)$ and consider $p = 1$. By the fundamental theorem of calculus, we can write

$$u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\mathrm{d}y_i \qquad i = 1, \dots, n. \tag{3.85}$$

This implies

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|\mathrm{d}y_i = f_i(\tilde{x}_i) \qquad i = 1, \dots, n. \tag{3.86}$$

This gives us

$$|u(x)|^n = |u| \dots |u| \leq f_1(\tilde{x}_1) \dots f_n(\tilde{x}_n) = \prod_{i=1}^n f_i(\tilde{x}_i). \tag{3.87}$$

Integrate over $x \in \mathbb{R}^n$:

$$\left\| |u|^{\frac{n}{n-1}} \right\|_{L^1(\mathbb{R}^n)} \underbrace{\leq}_{\text{Lemma}} \prod_{i=1}^n \left\| f_i^{\frac{1}{n-1}} \right\|_{L^{n-1}(\mathbb{R}^{n-1})} \underbrace{=}_{\text{definition of } f_i} \|Du\|_{L^1(\mathbb{R}^n)}^{\frac{n}{n-1}}. \tag{3.88}$$

This implies

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le |Du|_{L^1(R^n)}, \tag{3.89}$$

which was what we wanted.
Now, suppose $p > 1$. Consider $v(x) = |u(x)|^\gamma$ for some $\gamma > 1$ we'll choose later. We have (by a result from Example Sheet 2) $Dv = \gamma \quad \operatorname{sgn} \quad (u)|u|^{\gamma-1}Du$. Therefore

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} \mathrm{d}x\right)^{\frac{n-1}{n}} = \||u|^\gamma\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \tag{3.90}$$

$$\underbrace{\le}_{p=1 \quad \text{case}} \quad \|D(|u|^\gamma)\|_{L^1(\mathbb{R}^n)} \tag{3.91}$$

$$\le \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1}|Du|\mathrm{d}x \tag{3.92}$$

$$\le \gamma \left(\int_{\mathbb{R}}^n |u|^{(\gamma-1)\frac{p}{p-1}} \mathrm{d}x\right)^{1-\frac{1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p \mathrm{d}x\right)^{1/p}. \tag{3.93}$$

Now we choose γ such that $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} \implies \gamma = \frac{p(n-1)}{n-p} > 1$, and $\frac{\gamma n}{n-1} = \frac{np}{n-p} = p^*$. This gives us

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} \mathrm{d}x\right)^{\frac{n-1}{n}} \le \frac{p(n-1)}{n-p} \left(\int_{\mathbb{R}^n} |u|^{p^*} \mathrm{d}x\right)^{\frac{p-1}{p}} \|Du\|_{L^p(\mathbb{R}^n)} \tag{3.94}$$

which implies

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C(p,n)\|Du\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}, \tag{3.95}$$

which was what we wanted.

Corollary 3.12 (GNS for $U \subset \mathbb{R}^n$). *Suppose $U \subset \mathbb{R}^n$ is open and bounded with a C^1 boundary. Let $1 \le p < n$. If $p^* = \frac{np}{n-p}$ then we have $W^{1,p}(U) \hookrightarrow L^{p^*}(U)$, i.e. there exists $C = C(U,p,n)$ such that*

$$\|u\|_{L^{p^*}(U)} \le C\|u\|_{W^{1,p}(U)} \tag{3.96}$$

for all $u \in W^{1,p}(U)$.

Proof .
Let $u \in W^{1,p}(U)$. By the extension theorem, we have a bounded linear extension map $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ that matches u almost everywhere on U.

$$E(u) \in W^{1,p}(\mathbb{R}^n) \implies E(u) \in L^{p^*}(\mathbb{R}^n) \implies u \in L^{p^*}(U). \tag{3.97}$$

Also

$$\|u\|_{L^{p^*}(U)} = \|E(u)\|_{L^{p^*}(U)} \leq \|E(u)\|_{L^{p^*}(\mathbb{R}^n)} \tag{3.98}$$
$$\underbrace{\leq}_{\text{GNS}} C\|E(u)\|_{W^{1,p}(\mathbb{R}^n)} \tag{3.99}$$
$$\underbrace{\leq}_{\text{extension thm}} C\|u\|_{W^{1,p}(U)}. \tag{3.100}$$

When solving PDEs, the Poincaré inequality, which bounds u based on Du is helpful:

Corollary 3.13 (Poincaré inequality)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we have the estimate*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}\ \forall 1 \leq q \leq p^* \tag{3.101}$$

where the constant C depends on p, q, n, U.

In particular, as $1 \leq p \leq p^*$ (we take $q = p$),

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}. \tag{3.102}$$

Note that this only holds for functions in $W_0^{1,p}$ and not generally $W^{1,p}$, because this lets us kill off constants.

Proof .
Recall that $W_0^{1,p}(U)$ is the closure of $C_c^\infty(U)$ under the $W^{1,p}(U)$ norm. So there exists a sequence $(u_m)_m \subset C_c^\infty(U)$ such that $\|u_m - u\|_{W^{1,p}(U)} \to 0$. Since u_m vanish near ∂U, we can extend u_m to zero on $U^\complement$ to get $\overline{u}_m \in C_c^\infty(\mathbb{R}^n)$.
Apply GSN to find

$$\|u_m\|_{L^{p^*}(U)} \leq C\|Du_m\|_{L^p(U)}. \tag{3.103}$$

Send $m \to \infty$ to get

$$\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)}. \tag{3.104}$$

Since $\|u\| < \infty$, by Hölder, we get

$$\|u\|_{L^q(U)} \leq C\|u\|_{L^{p^*}} \leq C\|Du\|_{L^p(U)} \tag{3.105}$$

for $1 \leq q \leq p^*$.

We may wonder what happens if $p \geq n$. If $p = n$, we get $p^* \to \infty$. It turns out that the statement is false if $n > 1$. What about $n < p \leq \infty$?

Theorem 3.14 (Morrey's inequality)**.** *Let $n < p \leq \infty$. Then there exists a constant $C = C(p, n)$ such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{3.106}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$ where $\gamma = 1 - \frac{n}{p} < 1$.

This says these functions are Hölder continuous, meaning $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\gamma}(\mathbb{R}^n)$.

Proof .
First, we control the Hölder semi-norm.
Let Q be an open cube of side length $r > 0$, containing the origin, and set $\overline{u} = \frac{1}{|Q|}\int_Q u(x)\mathrm{d}x$. Then

$$|\overline{u} - u(0)| \leq \frac{1}{|Q|}\int_Q |u(x) - u(0)|\mathrm{d}x. \tag{3.107}$$

Since $u \in C_c^\infty$, we have the FTC, so

$$u(x) - u(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(u(tx))\mathrm{d}t = \sum_{i=1}^n \int_0^1 x^i \frac{\partial u}{\partial x^i}\mathrm{d}t. \tag{3.108}$$

So

$$|u(x) - u(0)| \underbrace{\leq}_{|x_i|<r \text{ in } Q} r\sum_{i=1}^n \int_0^1 |\partial_{x_i} u(tx)|\mathrm{d}t \tag{3.109}$$

$$|\overline{u} - u(0)| \leq \frac{r}{|Q|}\int_Q \int_0^1 \sum_{i=1}^n |\partial_{x_i} u(tx)|\mathrm{d}x\mathrm{d}t \tag{3.110}$$

$$= \frac{r}{|Q|}\int_0^1 t^{-n}\left(\sum_{i=1}^n \int_{tQ} |\partial_{x_i} u(y)|\mathrm{d}y\right)\mathrm{d}t \tag{3.111}$$

$$\underbrace{\leq}_{\text{Holder}} \frac{r}{|Q|}\int_0^1 t^{-n}\left(\sum_i \|\partial_{x_i} u\|_{L^p(tQ)}|tQ|^{1/q}\right)\mathrm{d}t \tag{3.112}$$

$$\underbrace{\leq}) |tQ| = t^n r^n \frac{r}{r^n}\int_0^1 t^{-n}\|Du\|_{L^p(\mathbb{R}^n)} t^{n/q} r^{n/q}\mathrm{d}t \tag{3.113}$$

$$\underbrace{=}_{\frac{1}{q}=1-\frac{1}{p}} \frac{r^{1-n/p}}{1-\frac{n}{p}}\|Du\|_{L^p(R^n)} \tag{3.114}$$

Therefore, $|\overline{u} - u(0)| \leq \frac{r^\gamma}{\gamma}\|Du\|_{L^p(\mathbb{R}^n)}$. By translation, this is true for all cubes Q whose sides of length r are parallel to the coordinate axes, so we can make the substitution $0 \to x$. So by the triangle inequality,

$$|u(x) - u(y)| \leq 2\frac{r^\gamma}{\gamma}\|Du\|_{L^p(\mathbb{R}^n)} \;\forall x, y \in Q. \tag{3.115}$$

Given any two $x, y \in \mathbb{R}^n$, there exists a cube Q of side length $r = 2|x-y|$ containing x and y.

$$\frac{|u(x) - u(y)|}{|x-y|^\gamma} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \;\forall x, y \tag{3.116}$$

$$[u]_{C^{0,\gamma}(\mathbb{R}^n)} = \sup_{x,y}\frac{|u(x) - u(y)|}{|x-y|^\gamma} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \tag{3.117}$$

Therefore we've shown the Hölder semi-norm is controlled by $\|Du\|$.
Next, we control the second piece of the Hölder norm, which is the sup norm of $|u|$. Note that any $x \in \mathbb{R}^n$ belongs to a cube of side length 1. So

$$|u(x)| \leq |\overline{u}| + |u - \overline{u}| \leq \int_Q |u(x)|\mathrm{d}x + C\|Du\|_{L^p(\mathbb{R}^n)} \tag{3.118}$$

$$\leq \|u\|_{L^p(\mathbb{R}^n)}\|1\|_{L^q(Q)} + C\|Du\|_{L^p(\mathbb{R}^n)} \tag{3.119}$$

$$\leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{3.120}$$

where C is independent of x. Therefore, $\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$.

Corollary 3.15 (Estimates on $W^{1,p}$, $n < p \leq \infty$)**.** *Suppose $u \in W^{1,p}(U)$ for $U \subset \mathbb{R}^n$ is open and bounded with C^1 boundary. Then there exists $u^* \in C^{0,\gamma}(U)$ with $\gamma = 1 - \frac{n}{p} < 1$ such that $u = u^*$ almost everywhere and $\|u^*\|_{C^{0,\gamma}(U)} \leq C\|u\|_{W^{1,p}(U)}$.*

Proof .
By the extension theorem, there exists $\overline{u} \in W^{1,p}(\mathbb{R}^n)$ with $\overline{u} = u$ almost everywhere on U. Then there exists $(u_j) \subset C_c^\infty(\mathbb{R}^n)$ such that $u_j \to u$ in $W^{1,p}(\mathbb{R}^n)$. This implies $u_j(x) \to \overline{u}(x)$ almost everywhere.
We claim (u_j) is Cauchy in $C^{0,\gamma}(\mathbb{R}^n)$, which holds because $\|u_m - u_j\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u_m - u_j\|_{W^{1,p}(\mathbb{R}^n)}$ by Morrey's inequality. Therefore $u_j \to \overline{u}^*$ in the Banach space $C^{0,\gamma}(\mathbb{R}^n)$ and $u^* = \overline{u}^*\big|_U$ satisfies the conditions of the theorem.

If $U \subset \mathbb{R}^n$ is open and bounded with a C^1 boundary, we have smooth embeddings for $W^{1,p}(U)$ for both the case $p \in [1, n)$ and $p \in (n, \infty)$. The former is $W^{1,p}(U) \hookrightarrow L^{p^*}(U)$ where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, and the latter is $W^{1,p}(U) \hookrightarrow C^{0,\gamma}(U)$ where $\gamma = 1 - \frac{n}{p}$.

If we start with $p < n$, we can use the embedding into L^{p^*} for both u and its derivatives, which implies u lives in another Sobolev space which allows us to use the embedding again.

Example 3.15. For $n = 3$, if $u \in W^{2,2}$ then $u, Du \in W^{1,2}$. For $p = 2$, we get $p^* = \frac{3 \cdot 2}{3-2} = 6$, so we can say $u, Du \in L^6$. This gives us $u \in W^{1,6}$ and $6 > 3 = n$, which implies $\gamma = \frac{1}{2}$. Therefore $u \in C^{0,1/2}$. $\square$

The hope is we can solve a PDE in a Sobolev space and then prove things about their integrability to then get that they are actually regular classical solutions.

With all that set up, let's finally solve some PDEs!

Chapter 4

Second-order elliptic boundary value problems

Contents

We write down second-order elliptic BVPs in one of two standard forms that can be directly manipulated. We find that it is difficult to directly solve them, but we can prove the existence of weak solutions under some reasonable constraints using the machinery of Sobolev spaces. We'll first show solution existence via the Lax-Milgram theorem which makes use of "energy estimates" on a PDE, and then extend it using the Fredholm alternative for compact operators (which requires that we cast the PDE into an equivalent form involving compact operators first). We'll subsequently increase the regularity of weak solutions to get full solutions.

In this entire chapter, we let U be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary.

4.1 Formulating elliptic BVPs

For $u \in C^2(\overline{U})$, we define

$$Lu := -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{4.1}$$

The a^{ij}, b^i, c are given functions on U. Without loss of generality, $a^{ij} = a^{ji}$. Typically we assume these functions are at least $L^\infty(U)$. The above form is called *divergence form*, as it looks like $\boldsymbol{\nabla} \cdot (A\boldsymbol{\nabla} u)$.

If $a^{ij} \in C^1(u)$, then we can write L in *non-divergence form*:

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{j=1}^{n} \tilde{b}^j(x)u_j + c(x). \tag{4.2}$$

The first form is most suited to *energy methods*, while the second form is most suited to the *maximum principle approach*. We'll use energy methods, and the maximum principle approach will be covered in the Lent course on elliptic PDEs.

Definition 4.1. *L is elliptic if $\sum_{i,j} a^{ij}(x)\xi_i\xi_j > 0$ for all $x \in U, \xi \in \mathbb{R}^n \setminus \{0\}$.*

Definition 4.2. *L is uniformly elliptic if $\sum_{i,j} a^{ij}(x)\xi_i\xi_j \geq \theta\|\xi\|^2$ for all $x \in U, \xi \in \mathbb{R}^n$ for some $\theta > 0$ independent of x, ξ.*[1]

Uniform ellipticity is saying $L(\xi) = \xi^\intercal A\xi$ is bounded below.

4.2 Finding weak solutions

Toolkit

In this section, we'll need to know:

[1] "uniform" always means something is independent of position

- *The Riesz representation theorem*. There is a canonical isomorphism between a real Hilbert space H and its dual space H^*. That is, for each continuous $\varphi \in H^*$ there's a unique $x_\varphi \in H$ such that $\varphi(x) = \langle x, x_\varphi \rangle$ identically.
- *Poincaré's inequality*.
- *Young's inequality*, $ab \leq \frac{a^2}{2\epsilon} + \epsilon b^2$ for all $a, b \in \mathbb{R}, \epsilon > 0$. Sometimes referred to as the "Peter-Paul" inequality because you "rob Peter to pay Paul" by how you tune ϵ to make the first/second term respectively big or small.

4.2.1 The Lax-Milgram theorem

Consider the following BVP:

$$\begin{aligned} Lu = f \quad &\text{in} \quad U \\ u = 0 \quad &\text{on} \quad \partial U. \end{aligned} \tag{4.3}$$

Let $f \in L^2(U)$ and $a^{ij}, b^i, c \in L^\infty(U)$.

Suppose we have some function u that is at least $C^2(\overline{U})$ which solves Equation 4.3 pointwise almost everywhere. For any test function $v \in C_c^\infty(U)$, we can multiply this equation by v and integrate by parts.

$$\int_U vf \mathrm{d}x = \int_U (-v(a^{ij}v_{x_i})_{x_j} + vb^j u_{x_j} + vc)\mathrm{d}x \tag{4.4}$$

$$= \underbrace{-\int_{\partial U} va^{ij}u_{x_i}n_j \mathrm{d}S}_{0 \text{ on } \partial U \text{ because } v \text{ is compactly supported}} + \int_U (a^{ij}u_{x_i}v_{x_j} + b^i u_{x_i} v + cuv)\mathrm{d}x \tag{4.5}$$

Therefore

$$\begin{aligned} \int_U vf \mathrm{d}x &= B[u,v] \forall v \in C_c^\infty(U) \\ B[u,v] &= \int_U (a^{ij}u_{x_i}v_{x_j} + b^i u_{x_i} v + cuv)\mathrm{d}x. \end{aligned} \tag{4.6}$$

Therefore, if $u \in C^2(\overline{U})$ solves Equation 4.3, then Equation 4.6 holds. Conversely, if $u \in C^2(\overline{U})$ with $u\big|_{\partial U} = 0$ and Equation 4.6 holds, then by undoing the IBP, we have

$$\int_U (f - Lu)v \mathrm{d}x = 0 \, \forall v \in C_c^\infty(U), \tag{4.7}$$

which implies $Lu = f$ almost everywhere. Note that Equation 4.6 makes sense for $v \in H_0^1(U), u \in H^1(U)$. To encode the boundary condition of u, we place $u \in H_0^1(U)$.

Definition 4.3. *We say $u \in H_0^1(U)$ is a weak solution of the BVP 4.3 for some given function $f \in L^2(U)$ if $B[u,v] = (f,v)_{L^2(U)}$ for all $v \in H_0^1(U)$.*

From here, we want to find a weak solution, and try to extend it by showing that $u \in C^2(\overline{U})$.

As a stepping stone to this, we prove the following theorem.

Theorem 4.1 (Lax-Milgram '54)**.** *Let H be a real Hilbert space with an inner product $(\cdot,\cdot)$. Suppose $B : H \times H \to \mathbb{R}$ is a bilinear map such that there exist constant $\alpha, \beta > 0$ such that*

1. $|B[u,v]| \leq \alpha \|u\| \|v\| \; \forall u, v \in H$ *(boundedness of B)*

2. $\beta \|u\|^2 \leq B[u,u] \; \forall u \in H$ *(coercivity of B over the norm)*

Then, if $f : H \to \mathbb{R}$ is a bounded linear functional ($f \in H^*$) then there exists a unique $u \in H$ such that $\langle f, v \rangle = B[u,v]$ for all $v \in H$.

Boundedness is a fairly normal condition, but coercivity tells us a lot about B on top of what we usually know, because it puts a lower bound on B as well as the usual upper one (which can be seen from condition 1 if we take $u = v$). That is, the norm *coerces* the bilinear form towards being in a particular interval.

It's notable that B is *not necessarily* symmetric: if it were, it would be an alternative inner product for H so the proof would automatically follow from the Riesz representation theorem (canonical isomorphism between H and the dual space H^*.)

Proof of Lax-Milgram.
For each fixed $u \in H$, the map $\varphi_u(v) = B[u,v]$ is a bounded linear functional on H, i.e. $\varphi_u \in H^*$. By the Riesz representation theorem, there exists a unique $w_u \in H$ such that $\varphi_u(v) = (w_u, v) = B[u,v]$ for all $v \in H$. So there is a map $u \to w_u \in H$. Call this map $A : H \to H$; then $Au = w_u$ and $B[u,v] = (Au, v)$.
From here, our plan is to show

1. A is bounded and linear, and therefore continuous.
2. A is injective.
3. The image $A(H)$ is closed.
4. $A(H) = H$, i.e. A is surjective so the inverse A^{-1} exists.
5. Show that $\langle f, v \rangle = B[A^{-1} w_f, v]$ where w_f is the element of H canonically isomorphic to f.

We claim A is a bounded linear operator. If $\lambda_1, \lambda_2 \in \mathbb{R}$, and $u_1, u_2 \in H$, then for each $v \in H$,

$$\begin{aligned}(A(\lambda_1 u_1 + \lambda_2 u_2), v) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] = \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v) \\ &= (\lambda_1 Au_1 + \lambda_2 Au_2, v)\end{aligned} \tag{4.8}$$

and so A is linear since the inner product is non-degenerate.
Now,

$$\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha\|u\|\|Au\|, \tag{4.9}$$

so

$$\|Au\| \leq \alpha\|u\| \; \forall u \in H, \tag{4.10}$$

so A is bounded.
Next, we claim A is injective, and the image $A(H)$ is a closed subspace of H. To do this, we use coercivity to describe a useful expression.

$$\beta\|u\|^2 \leq B[u, u] = (Au, u) \leq \|Au\|\|u\| \quad \text{by Cauchy-Schwarz} \quad , \tag{4.11}$$

and therefore

$$\|u\| \leq \frac{1}{\beta}\|Au\|. \tag{4.12}$$

So if $Au_1 = Au_2$, then

$$\beta\|u_1 - u_2\| \leq \|Au_1 - Au_2\| = 0 \implies u_1 = u_2. \tag{4.13}$$

Next, to show $A(H)$ is closed, we take a sequence $(Au_j)_j \to w \in H$ where $(u_j)_j$ is Cauchy in the complete space H (by 4.12), meaning it converges to some limit $u \in H$. By the continuity of A (as it's bounded and linear),

$$\lim_j A(u_j) = A(\lim_j u_j) = A(u) = w. \tag{4.14}$$

Therefore $A(H)$ is closed.
Now, we claim that $A(H) = H$. Since $A(H)$ is closed in H, we can decompose it as $H = A(H) \oplus A(H)^\perp$. If $A(H) \neq H$, then there exists some $w \in A(H)^\perp$ such that $w \neq 0$. But, looking at the coercivity condition,

$$\beta\|w\|^2 \leq B[w, w] = (Aw, w) \underbrace{=}_{w \text{ in complement, } Aw \text{ in image, orthogonal}} 0, \tag{4.15}$$

and therefore $\|w\| = 0 \implies w = 0$. Therefore $A(H) = H$, This tells us A is bijective, which implies it is invertible.

$$w = Au \iff u = A^{-1}w \tag{4.16}$$

$$\|u\| \leq \frac{1}{\beta}\|Au\| \implies \left\|A^{-1}w\right\| \leq \frac{1}{\beta}\|w\|, \tag{4.17}$$

so $A^{-1} : H \to H$ is linear and bounded.

In summary, we've shown that $A : H \to H$ is bijective, linear, bounded, and that $(Au, v) = B[u, v]$ for all $v \in H$.
Now, we want to solve the problem: given $f \in H^*$ we want to find a u such that $B[u, v] = \langle f, v\rangle$ for all $v \in H$. By the Riesz representation theorem, there exists a unique $w_f \in H$ such that $\langle f, v\rangle = (w_f, v)$ for all $v \in H$. Let $u = A^{-1}(w_f)$. We know this exists because A is bijective. Then

$$B[u, v] = (Au, v) = (w_f, v) = \langle f, v\rangle \; \forall v \in H, \tag{4.18}$$

i.e. $B[u, \cdot] = f(\cdot)$.
Uniqueness remains. Say u, u' both satisfy the condition. Then

$$B[u - u', v] = 0 \; \forall v \in H. \tag{4.19}$$

Set $v = u - u'$. Then

$$\|u - u'\|^2 \leq \frac{1}{\beta} B[u - u', u - u'] = 0 \implies u = u'. \tag{4.20}$$

Explicitly solving PDEs rarely happens in PDE theory, because it's too hard. Instead, we'll more often just come up with estimates like this one that follows from Lax-Milgram:

Corollary 4.2 (Well-posedness)**.** *Let u_1, u_2 be the solutions corresponding to $f_1, f_2 \in H^*$. Then*

$$\|u_1 - u_2\| \leq \frac{1}{\beta} \|f_1 - f_2\|_{H^*} \tag{4.21}$$

This gives us continuous dependence on the data.

Proof of the corollary.
We can actually prove another estimate to get this going. Let u be the unique solution, coming from Lax-Milgram, of $B[u, v] = \langle f, v\rangle$ for all $v \in H$. Then, we know from the coercivity of B that

$$\beta \|u\|^2 \leq B[u, u] = \langle f, u\rangle \leq \|f\|_{H^*} \|u\|, \tag{4.22}$$

which implies

$$\|u\| \leq \frac{1}{\beta} \|f\|_{H^*}. \tag{4.23}$$

Then, $B[u_i, v] = \langle f_i, v\rangle$ for all $v \in H$ implies

$$B[u_1 - u_2, v] = \langle f_1 - f_2, v\rangle \tag{4.24}$$

and we can choose $v = u_1 - u_2$ to get the desired result.

4.2.2 Energy estimates

This is just a statement about Hilbert spaces, so we need to link it to PDEs. We do this by showing that boundedness and coercivity (in some form) hold for the elliptic BVP.

Theorem 4.3 (Energy estimator for B)**.** *Suppose $a^{ij}, b^i, c \in L^\infty(U)$ and L is uniformly elliptic. Then if $B[u,v] := \int_U (a^{ij} v_{x_i} u_{x_j} + b^i v u_{x_i} + cuv)\mathrm{d}x$, then there exist positive constants $\alpha, \beta > 0$ and another constant $\gamma \geq 0$ such that*

(i) $|B[u,v]| \leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)}$

(ii) $\beta \|u\|^2_{H^1(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}$ *for all $u, v \in H^1_0(U)$. (Gårding's inequality)*

Proof .

(i) As follows:

$$|B[u,v]| \leq \sum_{i,j} \left\|a^{ij}\right\|_{L^\infty(U)} \int_U |Du||Dv|\mathrm{d}x + \sum_i \left\|b^i\right\|_{L^\infty(U)} \int_U |Du||v|\mathrm{d}x + \|c\|_{L^\infty} \int |u| \int |v|\mathrm{d}x \tag{4.25}$$

$$\underbrace{\leq}_{\text{Cauchy-Schwarz}} C_1 \|Du\|_{L^2} \|Du\|_{L^2} + C_2 \|Du\|_{L^2} \|v\|_{L^2} + C_3 \|u\|_{L^2} \|v\|_{L^2} \tag{4.26}$$

$$\leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)} \quad \text{for some} \quad \alpha > 0 \tag{4.27}$$

(ii) We use uniform ellipticity:

$$\theta \int_U |Du|^2 \mathrm{d}x \leq \int_U \sum a^{ij}(x) u_{x_i} u_{x_j} \mathrm{d}x = B[u,u] - \int_U \left(\sum b^i u_{x_i} u + cu^2\right)\mathrm{d}x \tag{4.28}$$

$$\leq B[u,u] + \sum_i \left\|b^i\right\|_{L^\infty} \int_U |Du||u|\mathrm{d}x + \|e\|_{L^\infty} \int_U |u|^2 \mathrm{d}x \tag{4.29}$$

By Young's inequality with $\epsilon \to \frac{1}{2\epsilon}$

$$\int_U |Du||u|\mathrm{d}x \leq \epsilon \int_U |Du|^2 \mathrm{d}x + \frac{1}{4\epsilon} \int_U |u|^2 \mathrm{d}x. \tag{4.30}$$

Choose ϵ such that $\epsilon \sum_i \left\|b^i\right\|_{L^\infty(U)} \leq \frac{\theta}{2}$ to deduce

$$\frac{\theta}{2} \int_U |Du|^2 \mathrm{d}x \leq B[u,u] + C\|u\|^2_{L^2(U)}. \tag{4.31}$$

Thus we have $\beta\|u\|^2_{H^1(U)} \leq B[u,u] + \gamma\|u\|^2_{L^2(U)}$, which was what we wanted.

If B is a bilinear form corresponding to an operator where $b^i = c = 0$, then we actually have

$$\theta \int_U |Du|^2 \mathrm{d}x \leq B[u,u]. \tag{4.32}$$

Recall Poincaré's inequality, $\|u\|_{L^2(U)} \leq c\|Du\|_{L^2(U)}$ for $u \in H^1_0(U)$. We deduce $\|u\|^2_{H^1(U)} \leq cB[u,u]$ for all $u \in H^1_0(U)$, i.e. Gårding with $\gamma = 0$.

Example 4.16. If $Lu = -\triangle u$ (the Laplacian) we can directly use Lax-Milgram. □

Example 4.17. For $Lu = -\triangle u + cu$ for $c \geq 0$, then $B[u,v] = \int_U (\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + cuv)\mathrm{d}x$, so we have

$$|B[u,v]| \leq c\|u\|\|v\|, \tag{4.33}$$

and

$$|B[u,u]| = \|\boldsymbol{\nabla} u\|^2_{L^2} + c\|u\|^2_{L^2(U)} \underbrace{\geq}_{c\geq 0} \|\boldsymbol{\nabla}_u\|^2_{L^2(U)} \underbrace{\geq}_{\text{Poincaré}} C\|u\|^2_{L^2(U)}, \tag{4.34}$$

for $u \in H^1_0(U)$, so Lax-Milgram applies. □

Theorem 4.4 (Basic existence result for weak solutions)**.** *Let L be as before. There is a $\gamma \geq 0$ such that for any $\mu \geq \gamma$ and any $f \in L^2(U)$, there exists a unique weak solution $u \in H^1_0(U)$ to the BVP*

$$\begin{cases} Lu + \mu u = f & \text{in} \quad U \\ u = 0 & \text{on} \quad \partial U. \end{cases} \tag{4.35}$$

Moreover, there exists a constant $c > 0$ such that

$$\|u\|_{H^1(U)} \leq c\|f\|_{L^2(U)}. \tag{4.36}$$

We've had to change the equation slightly in order to solve it, but on the flip side, we've also found solutions to a whole family of PDEs now.

Proof .
Take γ from the Gårding inequality above. Let $\mu \geq \gamma$, and set

$$B_\mu[u,v] = B[u,v] + \mu(u,v)_{L^2(U)}. \tag{4.37}$$

This is the bilinear map corresponding to $L_\mu u = Lu + \mu u$. It satisfies the conditions needed for Lax-Milgram (we use $\mu \geq \gamma$ to show this.)
Now, we fix $f \in L^2(U)$ and set $\langle f, v\rangle = (f,v)_{L^2(U)}$. This gives a bounded linear functional on $L^2(U)$ and therefore also on H_0^1. Apply Lax-Milgram to find a unique $u \in H = H_0^1(U)$ satisfying

$$B_\mu[u,v] = \langle f, v\rangle = (f,v)_{L^2(U)}\ \forall v \in H_0^1, \tag{4.38}$$

i.e. u is a weak solution of 4.35.
From Gårding,

$$\beta\|u\|^2_{H^1(U)} \leq B_\mu[u,u] = (f,u)_{L^2(U)} \underbrace{\leq}_{\text{C-S}} \|f\|_{L^2}\|u\|_{H^1(U)}. \tag{4.39}$$

Next lecture, we'll extend this back to the original PDE. For now, we'll cover a modified version of the Poincaré inequality, which will be crucial for that.

Lemma 4.5 (Poincaré revisited)**.** *Suppose $u \in H^1(\mathbb{R}^n)$. Let $Q = (\xi_1, \xi_1 + L) \times \cdots \times (\xi_n, \xi_n + L)$ be a cube of side length l. Then we have*

$$\|u\|^2_{L^2(Q)} \leq \frac{1}{|Q|}\left(\int_Q u \mathrm{d}x\right)^2 + \frac{n^2L^2}{2}\|Du\|^2_{L^2(Q)}. \tag{4.40}$$

and

$$\|u - \overline{u}\|_{L^2(Q)} \leq \frac{n^2L^2}{2}\|Du\|^2_{L^2(Q)}, \tag{4.41}$$

where $\overline{u} = \frac{1}{|Q|}\int_Q u(x)\mathrm{d}x$.

If $\overline{u} = 0$, we get back the Poincaré inequality from before. Note that before, we needed to kill of constant functions by restricting u to functions with vanishing trace, $u \in H_0^1$. In this case, having the $\overline{u}$ term kills off constants.

Proof .

(i) Since ∂Q is Lipschitz, $C^\infty(\overline{Q})$ is dense in $H^1(Q)$. For $x, y \in Q$, we use FTC:

$$u(x) - u(y) = \int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \dots, x_n)\mathrm{d}t + \int_{y_2}^{x_2} \frac{\mathrm{d}}{\mathrm{d}t} u(y_1, t, x_3, \dots, x_n) + \cdots + \int_{y_n}^{x_n} \frac{\mathrm{d}}{\mathrm{d}t} u(y_1, \dots, y_{n-1}, t). \tag{4.42}$$

which we should check manually for $n = 3$.

Squaring this identity and applying the inequality $\left(\sum_i a_i\right)^2 \le n\left(\sum_i a_i^2\right)$, we get

$$u(x)^2 + u(y)^2 - 2u(x)u(y) \le n\left(\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \dots, x_n)\mathrm{d}t\right)^2 + \cdots + n\left(\int_{y_n}^{x_n} \frac{\mathrm{d}}{\mathrm{d}t} u(y_1, \dots, y_{n-1}, t)\mathrm{d}t\right)^2. \tag{4.43}$$

Integrating this over $x, y \in Q$, the LHS becomes

$$\int_Q \mathrm{d}x \int_Q \mathrm{d}y (u^2(x) + u^2(y) - 2u(x)u(y)) = 2|Q|\|u\|^2_{L^2(Q)} - 2\left(\int_Q u(x)\mathrm{d}x\right)^2. \tag{4.44}$$

Next, consider $I_1 := \left(\int_{y_1}^{y_2} \frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \dots, x_n)\mathrm{d}t\right)^2$. Then

$$I_1 \le \left(\int_{y_1}^{x_1} I\mathrm{d}t\right) \cdot \int_{y_1}^{x_1} \left(\frac{\mathrm{d}}{\mathrm{d}t} u(-)\right)^2 \mathrm{d}t \tag{4.45}$$

$$\le L \int_{\xi_1}^{\xi_1 + L} \left(\frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \dots, x_n)\right)^2 \mathrm{d}t. \tag{4.46}$$

Therefore, integrating,

$$\int_Q \mathrm{d}x \int_Q \mathrm{d}y I_1 \le L \cdot L|Q|\|D_1 u\|^2_{L^2(Q)}. \tag{4.47}$$

Similarly estimating over the RHS terms, we get

$$2|Q|\|u\|^2_{L^2(Q)} - 2\left(\int_Q u\mathrm{d}x\right)^2 \le L^2 n|Q|\|Du\|^2_{L^2}. \tag{4.48}$$

(ii) This is an application of the proof for part (i). Consider $u - c\eta$ where η is a smooth function where $\eta\big|_Q = 1$, η vanishes outside of a compact set containing Q, and is smooth. Here $c = \frac{1}{|Q|}\int_Q u(x)\mathrm{d}x$.

4.3 Compactness results in PDE

We can often find a bounded sequence of approximate solutions, and we'd like to find a convergent subsequence that actually solves the PDE. This section sets up results that we'll use in establishing the *Fredholm alternative*.

Toolbox

1. A space is separable if it contains a countable dense subset, i.e. there is a sequence $(x_n)_{n=1}^{\infty}$ such that every nonempty open subset of the space contains at least one x_i.
2. The closed unit ball in $\mathbb{R}^n$ is sequentially compact, i.e. for $(x_j) \in B = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, (x_j) has a convergent subsequence.
3. In a metric space, compactness is equivalent to sequential compactness.
4. Banach (X) and Hilbert (H) spaces admit metrics (inner product $\to$ norm $\to$ metric)
5. If X or H is infinite-dimensional, then the closed unit ball $\{x \mid \|x\| \leq 1\}$ is not compact. This isn't what we want, so our norm is too strong and we'd like to weaken the topology.

   Exercise 4.6. *Show $\{f \in C^{0,\alpha}([0,1]), \|f\|_{C^0} \leq 1\}$ is compact. The key point here is that we're using a different norm on our open-ball condition. (This is in the Analysis of Functions lecture notes.)*

6. A Banach space $(X, \|\cdot\|)$ has a strong and a weak topology.
   - For $x_n \in X$, $x_n \to x$ strongly if $\|x_n - x\|_X \to 0$.
   - For $x_n \in X$, $x_n \rightharpoonup x$ weakly if $\langle f, x_n \rangle \to \langle f, x \rangle$ for all $f \in X'$.

   Strong convergence implies weak convergence, but not the reverse.

   Example 4.18. For $1 \leq p < \infty$ consider $(L^p(I), (-,-)_p)$, $I = (0, 2\pi)$. The sequence $f_n(x) = \sin(nx)$ has

   $$(f_n, g)_{L^p} = \int_0^{2\pi} \sin(nx) g(x) \to 0 \quad \text{as} \quad n \to \infty \tag{4.49}$$

   for all $g \in L^p$, so we see $f_n \rightharpoonup 0$. But $\|f_n\|_{L^p(I)} = c(p) > 0$, so $f_n \not\to 0$. □

7. X' is itself a Banach space, with the sup norm

   $$\|f\|_{X'} = \sup_{x \in X, \|x\|_X \leq 1} |f(x)| \tag{4.50}$$

   and it has a strong, a weak, and a weak-* topology.
   - For $f_n \in X'$, $f_n \to f$ if $\|f_n - f\|_{X'} \to 0$
   - $f_n \xrightarrow{w} f$ if $\langle f_n, T \rangle \to \langle f_n, T \rangle$ for all $T \in X''$ (the double dual)

- $f_n \xrightarrow{w^*} f$ if $\langle f_n, x\rangle \to \langle f, x\rangle$ for all $x \in X$.

If X is reflexive, we can identify X with its double dual, and the weak and weak-* topologies on X' are the same.

Theorem 4.7 (Banach-Alaoglu)**.** *Let X be Banach. The closed unit ball in the dual space $\overline{B}' = \{f \in X' \mid \|f\|_{X'} \leq 1\}$ is compact in the weak-* topology of X. If also X is separable then the weak-* topology on $\overline{B}'$ is a metric topology, and so $\overline{B}'$ is sequentially compact. (AoF, L2.2, T2.22)*

Definition 4.4. *Suppose $(H, (-,-))$ is a Hilbert space with $(u_j) \subset H$. We say (u_j) converges weakly to $u \in H$, i.e. $u_j \rightharpoonup u$, if $\lim_j (u_j, w) = (u, w)$ for all $w \in H$.*

A weak limit, if it exists, is unique. (Lecture notes of Minter).

Theorem 4.8 (Banach-Alaoglu for separable Hilbert spaces)**.** *Let H be a separable Hilbert space, and suppose $(u_n)_n \subset H$ is a bounded sequence (i.e. $\|u_n\| \leq K$). Then $(u_n)_n$ has a weakly convergent subsequence.*

We're skipping the proof, it's long and detailed and in the lecture notes but also follows directly from the first version of B-A.

With all of this setup, we can start relating this to PDEs. What's the strongest statement we can make about a bounded sequence in $H^1(U)$?

Theorem 4.9 (Rellich-Kondrachov)**.** *Suppose $U \subset \mathbb{R}^n$, open and bounded, and let $\partial U \in C^1$. Let $(u_m)_m$ be a bounded sequence in $H^1(U)$. Then there exists $u \in H^1(U)$ and a subsequence (u_{m_j}) such that $u_{m_j} \rightharpoonup u$ in $H^1(U)$ (weak in the space with more derivatives) and $u_{m_j} \to u$ in $L^2(U)$ (strong in the space with no derivatives).*

This is an important result. We prove this using B-A and the Poincaré estimates.

Proof .
Use the extension theorem to convince ourselves we can extend each sequence element u_m to some $\overline{u}_m \in H^1(\mathbb{R}^n)$ such that $\operatorname{supp}(\overline{u}_m)$ is compactly contained in some cube Q. We therefore have that the extension map $H^1(U) \to H^1_0(Q)$ is a bounded linear map (why?). This gives us the norm inequality,

$$\|\overline{u}_m\|_{H^1(Q)} \leq c\|u_m\|_{H^1(U)} \leq cK. \tag{4.51}$$

Next, since $H^1_0(Q)$ is a separable Hilbert space (sheet 3), by the separable version of Banach-Alaoglu, there exists some limit point $u \in H^1_0(Q)$ and some weakly convergent subsequence $(\overline{u}_{m_j})$ such that $\overline{u}_{m_j} \rightharpoonup u$ in $H^1_0(Q)$, $\|u\|_{H^1(Q)} \leq cK$.
For convenience, set $w_j := u_{m_j}$. It remains to show that $w_j \to u$ in $L^2(Q)$.
Fix $\delta > 0$. Divide Q into $k(\delta)$ subcubes $\{Q_a\}_{a=1}^k$ of side length $0 < l < \delta$ that intersect only on their faces (sets of measure zero). In particular, we can localise all our integrals to each Q_a. By the Poincaré lemma (4.5), we have

$$\|w_j - u\|^2_{L^2(Q)} = \sum_{a=1}^{k(\delta)} \|w_j - u\|^2_{L^2(Q_a)} \tag{4.52}$$

$$\leq \sum_{a=1}^{k(\delta)} \left[\frac{1}{|Q_a|} \left(\int_{Q_a} (w_j - u) \mathrm{d}x \right)^2 + \frac{n^2\delta^2}{2} \|Dw_j - Du\|^2_{L^2(Q_a)} \right] \tag{4.53}$$

$$= \left[\sum_{a=1}^{k(\delta)} \frac{1}{|Q_a|} \left(\int_{Q_a} (w_j - u) \mathrm{d}x \right)^2 \right] + \frac{n^2\delta^2}{2} \|Dw_j - Du\|^2_{L^2(Q)} \tag{4.54}$$

Let $\epsilon > 0$. Recall the fact that $w_j, u \in H^1_0(Q)$. So applying the triangle inequality on the L^2 norm, we can say $\|Dw_j - Du\|^2_{L^2(Q)} \leq C$ for some constant C.
Fix some small value for $\delta > 0$ such that $\frac{n^2\delta^2}{2}\|Dw_j - Du\|^2_{L^2(Q)} < \frac{\epsilon}{2}$. This also fixes $k(\delta)$. This bounds the second term by $\frac{\epsilon}{2}$,
It remains to control the first term. Note that $f(u) = \int_Q u(x)\mathrm{d}x$ is a bounded linear functional on $H^1(Q)$. Since $w_j \rightharpoonup u$ in $H^1_0(Q)$, we have by the definition of weak convergence that $\langle f, w_j\rangle \to \langle f, u\rangle$, so by linearity and substituting in what f is,

$$\int_{Q_a} (w_j - u)\mathrm{d}x \to 0, \tag{4.55}$$

for all a. Since $k(\delta)$ is some fixed finite number, we can choose j large enough to bound the sum:

$$\sum_{a=1}^{k} \frac{1}{|Q_a|} \left(\int_{Q_a} (w_j - u)\mathrm{d}x \right)^2 < \frac{\epsilon}{2}. \tag{4.56}$$

Therefore, $w_j \to u$ in $L^2(Q)$, so it remains to translate this back to U.
To conclude, consider $\{u_{m_j}\} \subset H^1(U)$. This is bounded, so there is a sub-subsequence m_{j_i} and a limit $v \in H^1(U)$ such that

$$u_{m_{j_i}} \rightharpoonup v \quad \text{in} \quad H^1(U) \tag{4.57}$$

and we can check that $u_{m_{j_i}} \to u, v$ in L^1 implies $u = v$ almost everywhere.

4.4 The Fredholm alternative

Toolbox

Definition 4.5. *Let H be a Hilbert space and $K : H \to H$ a bounded linear operator. The adjoint of K, $K^\dagger : H \to H$ (mimicking the conjugate transpose) is the unique operator such that*

$$(x, K^\dagger y) = (Kx, y) \tag{4.58}$$

for all $x, y \in H$.

Definition 4.6. *K is called compact if for each bounded sequence $(u_j)_j \subset H$ there exists a subsequence $(u_{j_k})_k$ such that the image $(K(u_{j_k}))_k$ converges strongly in H.*

4.4.1 Setting up the Fredholm alternative

Compactness gives us sequential compactness in metric spaces, so we can use the above definition to create the conditions for Rellich-Kondrachov.

Example 4.19. Let $K : L^2(U) \to H^1(U)$ be a bounded linear operator. Since $H^1(U) \hookrightarrow L^2(U)$ (the Sobolev space continuously embeds into L^2), we can think of this as a map $K : L^2(U) \to L^2(U)$. We claim K is compact. Let $(u_j)_j$ be a bounded sequence in $L^2(U)$. Then so is its image under K:

$$\|K(u_j)\|_{H^1(U)} \leq \underbrace{\|K\|}_{\text{operator norm}} \|u_j\|_{L^2U} \leq Ck. \tag{4.59}$$

By Rellich-Kondrachov, there exists a subsequence $(u_{j_k}) \subset H^1(U)$ such that $u_{j_k} \to u$ in $L^2(U)$. That is, $K(u_{j_k})$ converges strongly in $L^2(U)$. $\square$

This is a really important example. For some intuition, consider Poisson's equation $\Delta u = f$. Say we're looking for weak solutions $u \in H^1(U)$, and as usual we're only requiring that $f \in L^2(U)$. So the equation is a map $u \mapsto f, H^1(U) \mapsto L^2(U)$.

The idea with the Fredholm alternative is that we can invert this, and map $L^2(U) \mapsto H^1(U)$. Let's write down a bunch of facts that'll let us do this.

Definition 4.7. *Suppose $A : H \to H$ is a bounded linear operator. The resolvant of A is the open set $\rho(A) = \{\lambda \in \mathbb{R} \mid A - \lambda I \text{is bijective}\}$.*

Definition 4.8. *The real spectrum of A is $\sigma(A) = \mathbb{R} \setminus \rho(A)$.*

Theorem 4.10 (Fredholm alternative for compact operators)**.** *Let H be Hilbert and let $K : H \to H$ be a compact operator. Then*

(i) $\ker(I - K)$ *is finite-dimensional.*

(ii) $\operatorname{im}(I - K)$ *is a closed subspace of H.*

(iii) $\operatorname{im}(I - K) = \ker(I - K^\dagger)^\perp$.

(iv) $\ker(I - K) = \{0\} \iff \operatorname{im}(I - K) = H$

(v) $\dim(\ker(I - K)) = \dim(\ker(I - K^\dagger))$.

This theorem is all about solving the equation $(I - K)u = f$. If we take $\psi \in \ker(I - K^\dagger)$, then we have

$$\langle (I-K)v, \psi \rangle = \langle v, \underbrace{(I-K^\dagger)\psi}_{0} \rangle = \langle f, \psi \rangle \tag{4.60}$$

That is, if the kernel of $I - K^\dagger$ is nontrivial, it is a necessary condition for solvability that f is orthogonal to ψ. What the theorem (i) tells us is that there are finitely many ψs to check, and moreover that the condition $\langle f, \psi \rangle$ is also a sufficient condition.

This machinery is all trivial in one dimension, but we'll see its real power in higher dimensions when we try to handle PDEs. The key to this power is compactness, because that brings the complexity down to evolution in finite dimensions.

In the Fredholm theorem, we have five different results, so let's unpack each one a bit.

(i) $\ker(I - K)$ is finite-dimensional. This is a big deal considering H is infinite-dimensional. We can also use this in the form: $u - Ku = 0$ has only finitely many linearly independent solutions $u \neq 0$.

(ii) $\operatorname{im}(I - K)$ is a closed subspace of H. If $u_j - Ku_j = f_j, f_j \to f$ in H, then there exists $u \in H$ such that $u - Ku = f$.

(iii) $\operatorname{im}(I - K) = \ker(I - K^\dagger)^\perp$. This tells us that $u - Ku = f$ has a solution iff f lives in the kernel of $(I - K)^\perp$. That is, $(f, v) = 0$ for all $v \in H$ such that $v - K^\dagger v = 0$.

(iv) $\ker(I - K) = \{0\} \iff \operatorname{im}(I - K) = H$. "Nothing is missed". The statement "$u = 0$ is the only solution to $u - Ku = 0$" is equivalent to the statement "$u - Ku = f$ has a soluton $u \in H$ for all $f \in H$."

(v) $\dim(\ker(I - K)) = \dim(\ker(I - K^\dagger))$. The number of linearly independent solutions to $u - Ku$ and $v - K^\dagger v = 0$ are the same.

This is referred to as the *alternative* because of (iii) and (iv). We have two options:

(I) either for each $f \in H$, $(I - K)u = f$ has a unique solution $u \in H$;

(II) or the homogeneous equation $(I - K)u = 0$ has nontrivial solutions.

In the second case, the space of homogeneous solutions is finite-dimensional. In this case, when do we have inhomogeneous solutions? This is given by (iv): $(I - K)u = f$ has a solution if and only if $f \in \ker(I - K^\dagger)^\perp$.

This is proved in appendix D.5 of Evans.

The Fredholm alternative is similar to the two possibilities in solving a linear equation $Ax = b$:

(a) either A is invertible and so there is a unique solution $x = A^{-1}b$;

(b) or $\ker A \neq \{0\}$ (that is, the homogeneous equation $Ax = 0$ has nontrivial solutions). Also in finite dimensions, $\operatorname{im} A = (\ker A^\intercal)^\perp$, so $Ax = b$ has a solution iff $y^\intercal b = 0$ for all $y \in \ker A^\intercal$ (or in other words $A^\intercal y = 0$.)

4.4.2 Applying the Fredholm alternative to elliptic BVPs

We're considering a uniformly elliptic PDE on a domain U that is open and bounded with C^1 boundary. Recall that $Lu = -(a^{ij}u_{x_i})_{x_j} + b^i u_{x_i} + cu$ and from that the bilinear form

$$B[u,v] = \int_U (a^{ij}u_{x_i}v_{x_j} + b^i u_{x_i} v + cuv)\mathrm{d}x. \tag{4.61}$$

Before we get to working with elliptic BVPs directly, we need to make the idea of adjoints work with the elliptic BVP formulation we've been using thus far.

Definition 4.9. *The formal adjoint of L is given by*

$$L^\dagger v = -(a^{ij}v_{x_i})_{x_j} - b^i v_{x_i} + \left(c - \sum_{i=1}^n {b^i}_{x_i}\right) v. \tag{4.62}$$

Note that this depends on the x_i derivatives of b^i.

Definition 4.10. *The adjoint bilinear form $B^\dagger : LH_0^1 \to H_0^1$ is given by*

$$B^\dagger[v,u] = B[u,v]. \tag{4.63}$$

If $b^i \in C^1$, then this makes sense by integration by parts.

We say $v \in H_0^1$ is a weak solution of the *adjoint problem* $\begin{cases} L^\dagger v = f & \text{in} \quad U \\ v = 0 & \text{on} \quad \partial U \end{cases}$ if it satisfies $B^\dagger[w,v] = (f,w)_{L^2}$ for all $w \in H_0^1$.

If $b^i \in C^1(\overline{U})$, then $B^\dagger$ is the same as the bilinear form defined by $L^\dagger$.

With all that setup, we're now ready for:

Theorem 4.11 (Fredholm alternative for elliptic BVPs)**.** *Consider* $\begin{cases} Lu = f & \textit{in} \quad U \\ u = 0 & \textit{on} \quad \partial U \end{cases}$.

(a) either for each $f \in L^2$, the BVP admits a unique weak solution $u \in H_0^1$;

(b) or there exists a nonzero weak solution $u \in H_0^1$ to the homogeneous problem (where $f = 0$).

Further, if (b) holds, then $\dim N = \dim N^\dagger < \infty$, where $N \subset H_0^1$ is the set of weak solutions to the homogeneous BVP and $N^\dagger \subset H_0^1$ is the set of weak solutions to the homogeneous adjoint BVP.

Finally, the BVP has a weak solution if and only if $(f,v)_{L^2} = 0$ for all $v \in N^\dagger$.

Proof .
By the basic existence result for weak solutions, there exists $\gamma \geq 0$ such that for any $f \in L^2$ there exists a unique weak solution to $\begin{cases} Lu + \gamma u = f & \text{in} \quad U \\ u = 0 & \text{on} \quad \partial U. \end{cases}$ If $\gamma = 0$, then case (a) holds by the Lax-Milgram theorem, so we may assume $\gamma > 0$.
The corresponding bilinear form therefore satisfies

$$B_\gamma[u,v] = B[u,v] + \gamma(u,v)_{L^2} = (f,v)_{L^2} \; \forall v \in H_0^1 \tag{4.64}$$

and $\|u\|_{H^1} \leq c\|f\|_{L^2}$.
We write $L_\gamma^{-1}(f) = u$ to represent this solution in terms of f. This is a linear map $L^2 \to H_0^1$. Further, $\left\|L_\gamma^{-1}(f)\right\|_{H^1} \leq c\|f\|_{L^2}$, so it's bounded. This implies it is compact as a map $L^2 \to L^2$.
Observe if $g \in L^2$, then $L_\gamma^{-1}(g) = w$ iff $B_\gamma[w,v] = (g,v)$ for all $v \in H_0^1$. Now suppose $u \in H_0^1$ is a weak solution to the BVP, i.e. $B[u,v] = (f,v)_{L^2}$ for all $v \in H_0^1$. Therefore

$$B_\gamma[u,v] = (f + \gamma u, v)_{L^2} \; \forall v \in H_0^1. \tag{4.65}$$

Then u is a weak solution iff $u = L_\gamma^{-1}(f + \gamma u) = L_\gamma^{-1}(f) + \gamma L_\gamma^{-1}(u)$. This is equivalent to solving $u - Ku = h$, where $K = \gamma L_\gamma^{-1}, h = L_\gamma^{-1}(f)$. Observe that $K : L^2 \to L^2$ is compact, so we can apply the Fredholm alternative.
We have

(I) either for all $h \in L^2$, $u - Ku = h$ admits a unique solution $u \in L^2$;

(II) or there exists $u \neq 0$ in L^2 such that $u - Ku = 0$.

Suppose (I) holds. We have established a solution $u \in L^2$, and we got it from a linear combination of outputs from L_γ^{-1}, so it must live in the codomain of L_γ^{-1}, i.e. $u \in H_0^1$ and so it is the required weak solution.
Now, suppose (II) holds. There exists a nonzero $u \in L^2$ such that $u = Ku = \gamma L_\gamma^{-1}(u)$, so similarly we get $u \in H_0^1$. Further, we note that $g = \gamma u$ above, so

$$B[u,v] + \gamma(u,v)_{L^2} = (\gamma u, v)_{L^2} \; \forall v \in H_0^1. \tag{4.66}$$

Both the inner product terms go away by linearity, so we get $B[u,v] = 0 \; \forall v \in H_0^1$. That is, u is a weak solution to the homogeneous BVP, so $u \in N$.
By (i, ii) of the Fredholm alternative, $\dim N = \dim(\ker(I - K)) = \dim(N^\dagger) < \infty$.

We've left the hanging question: if we're in case (II), where $\ker(I - K) \neq \{0\}$, then we can't solve the inhomogeneous BVP for that f. What choices of f do admit a solution?

(directly from a note made on the lecture recording):

We assume now $\ker(I - K) \neq \{0\}$ and ask ourselves: for which f does $B[u,v] = (f,v)_{L^2}$ have a solution u such that that expression holds for all v? This is the same as asking, for which f does $u - Ku = \gamma L_\gamma^{-1}(f)$ have a solution? Which is the same as asking for which f does $L_\gamma^{-1}(f)$ lie in $\operatorname{im}(I - K) = \ker(I - K^\dagger)^\perp$? So to begin with, what are the elements living in $\ker(I - K)^\dagger$? This then is the motivation behind the following claim.

Proposition 4.12. *Let $v \in L^2$. Then*

$$(I - K^\dagger)v = 0 \iff v \in N^\dagger \iff B^\dagger[v, w] = 0 \; \forall w \in H_0^1 \tag{4.67}$$

Proof .
Take $v \neq 0$.

$$v - K^\dagger v = 0 \iff (v, w)_{L^2} = (v, Kw)_{L^2} \; \forall w \in L^2. \tag{4.68}$$

This is the case if and only if

$$(v, w)_{L^2} = (v, \gamma L_\gamma^{-1}(w))_{L^2} \; \forall w \in L^2. \tag{4.69}$$

A weak solution to the corresponding BVP, which is

$$\begin{cases} L_\gamma \overline{w} = \overline{f} & \text{in} \quad U \\ \bar{u} = 0 & \text{on} \quad \partial U \end{cases} \tag{4.70}$$

obeys

$$B[\overline{w}, \varphi] + \gamma(\overline{w}, \varphi)_{L^2} = (\overline{f}, m\varphi) \; \forall \varphi \in H_0^1. \tag{4.71}$$

We can take $\overline{f} = w$, which gives us $\overline{w} = L_\gamma^{-1}(w)$ by definition. We also take $v = \varphi$. This tells us that we have

$$B[L_\gamma^{-1}(w), v] + \gamma\big(L_\gamma^{-1}(w), w\big)_{L^2} = (w, v)_{L^2}. \tag{4.72}$$

Inserting this into the LHS of the original BVP, we see $v - K^\dagger v = 0$ iff

$$\iff B\big[L_\gamma^{-1}(w), v\big] = 0 \; \forall w \in L^2 \tag{4.73}$$
$$\iff B^\dagger[v, L_\gamma^{-1}(w)] = 0 \; \forall w \in L^2. \tag{4.74}$$

To finish the claim, we need $B^\dagger[v, \varphi] = 0$ for all $\varphi \in X$, where X is dense in H_0^1. Because the bilinear map is continuous in each of its entries, this is sufficient.
We only get this fact for elements in the image of $L_\gamma^{-1}(w)$, but in Examples Sheet 3, we show that the image of L_γ^{-1} is in fact dense in H_0^1, so by the continuity of L_γ^{-1}, we have

$$v - K^\dagger v = 0 \iff B^\dagger[v, w] = 0 \; \forall w \in H_0^1, \tag{4.75}$$

which was what we wanted.

This leads to the following conclusion: the original BVP has a weak solution iff $(f, v)_{L^2} = 0$ for all $v \in N^\dagger$.

That is, $B[u,v] = (f,v)_{L^2}\ \forall v \in H_0^1$ if and only if $(I-K)(u) = L_\gamma^{-1}(f)$, which is the same thing as saying $L_\gamma^{-1}(f) \in \operatorname{im}(I-K) = \ker(I-K^\dagger)^\perp$, which is equivalent to $(v, L_\gamma^{-1}(f))_{L^2} = 0$ for all $v \in \ker(I-K^\dagger)$. But for all $v \in \ker(I-K^\dagger)$, we have

$$0 = \left(v, L_\gamma^{-1}(f)\right)_{L^2} = \left(v, \frac{1}{\gamma}K(f)\right)_{L^2} = \frac{1}{\gamma}(K^\dagger v, f)_{L^2} = \frac{1}{\gamma}(v,f)_{L^2}. \tag{4.76}$$

Hence, $(v,f)_{L^2} = 0$ for all $v \in \ker(I-K^\dagger)$, so f is a solution to the homogeneous adjoint problem.

4.4.3 Extended example: the harmonic oscillator

Consider the harmonic oscillator,

$$-\triangle + |x|^2 = H, x \in \mathbb{R}^d. \tag{4.77}$$

To this operator we associate a quadratic form,

$$B(u,v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \mathrm{d}x + \int_{\mathbb{R}^d} |x|^2 uv \mathrm{d}x. \tag{4.78}$$

We also need a Hilbert space of possible solutions:

$$\Sigma = \overline{\rho(\mathbb{R}^d)}^{\|\cdot\|_\Sigma}, \tag{4.79}$$

where

$$\|u\|_\Sigma = \left(\int_{\mathbb{R}^d} |\nabla u|^2 + \int_{\mathbb{R}^d} |x|^2|u|^2\right)^{1/2}. \tag{4.80}$$

This is sort of a physically-determined norm, with a kinetic energy plus a potential energy. It's a useful choice for this kind of problem, but is the Fredholm machinery appropriate? Yes! Let's see why.

We claim that $\Sigma \hookrightarrow H^1(\mathbb{R}^d)$ and that the embedding $Id : \Sigma \mapsto L^2(\mathbb{R}^d)$ is compact.

Proof .
We need to show there exists $c > 0$ such that for all $u \in \rho(\mathbb{R}^d)$, we have the usual norm bound:

$$\int_{|x|\le 1} |u|^2 \le c\left[\int_{\mathbb{R}^d} |\nabla u|^2 + \int_{\mathbb{R}^d} |x|^2|u|^2\right]. \tag{4.81}$$

The first term here is no problem, it's the usual Poincaré kind of bound. The difficulty comes in with the $|x|^2$ dependence: we don't control it on the LHS outside of $|x| \le 1$, so what we need to

show is that the gradient compensates for this. This is usually the kind of thing we have to do by playing with test functions and using integration by parts.
Let $\chi \in C_c^\infty$ such that it is 1 on the unit ball and decays to 0 between $|x| = 1$ and $|x| = 2$. We need to control χu^2.
We do this using the following trick: note that $\nabla \cdot (x\chi) = d\chi + x \cdot \nabla\chi$, and so we can rewrite

$$\chi = \frac{1}{d}(\nabla \cdot (x\chi) - x\nabla\chi) \tag{4.82}$$

and so

$$\int_{\mathbb{R}^d} \chi u^2 = \int_{\mathbb{R}^d} \frac{1}{d}(\nabla \cdot (x\chi) - x\nabla\chi)u^2 \tag{4.83}$$

$$= -\frac{1}{d}\int_{\mathbb{R}^d} (x \cdot \nabla\chi)u^2 - \underbrace{\int_{\mathbb{R}^d} \frac{1}{d} x\chi\nabla(u^2)}_{\text{IBP on the first term}} \tag{4.84}$$

$$= -\frac{1}{d}\int_{\mathbb{R}^d} (x \cdot \nabla\chi)u^2 - \frac{2}{d}\int ux\chi\nabla u. \tag{4.85}$$

Now, we try to bound $\chi|u|^2$ by the assumptions we took on χ. $\nabla\chi$ is zero for $|x| \geq 2$, so we can bound the first term by $|x|^2|u|^2$. For the second term we use Cauchy-Schwarz/Young's inequality, so we can get

$$\int_{\mathbb{R}^d} \chi|u|^2 \leq C_1 \int_{\mathbb{R}^d} |x|^2|u|^2 + C_2\left(\epsilon \int_{\mathbb{R}^d} \chi|u|^2 + \frac{1}{\epsilon}\int_{\mathbb{R}^d} |\nabla u|^2\right). \tag{4.86}$$

If we take an appropriate ϵ we can make the constants match up and we can absorb the blue term back into the LHS, and what remains is a bound just in terms of the Σ norm, which was what we wanted.
It remains to show that the embedding $Id : (\Sigma, \langle\cdot,\cdot\rangle_\Sigma) \mapsto (L^2, \langle\cdot,\cdot\rangle_{L^2})$ is compact. Between Hilbert spaces, a compact operator is one that transforms weakly convergent sequences into strongly convergent ones, i.e. $T : H_1 \to H_2$ is compact iff for all $x_i \rightharpoonup 0$ in H_1, $Tx_i \to 0$ in H_2. (Something about metrizability here that I'm just going to trust will always work out.)
We're looking at the map that goes from the Σ norm to the L^2 norm, I think. We want to estimate $\int |x_n|^2$, which we split at some R into

$$\int_{\mathbb{R}^d} |u_n|^2 = \int_{|x|<R} |u_n|^2 + \int_{|x|\geq R} |u_n|^2. \tag{4.87}$$

For the $|x| \geq R$ term, we can say

$$|u_n|^2 \leq \frac{|x|^2}{R^2}|u_n| \tag{4.88}$$

$$\int_{|x|\geq R} |u_n|^2 \leq \frac{1}{R^2}\int_{\mathbb{R}^d} |x^2||u_n|^2 \leq \frac{1}{R^2}|u_n|_\Sigma^2 \leq \frac{C}{R^2}. \tag{4.89}$$

Since R is free, this is equivalent to bounding by any desired $\epsilon > 0$. This is true for all $n \geq 1$. Further, for any R, the first term must go to 0 as $n \to \infty$ by Rellich-Kondrachov (expanding out the definition of the L^2 norm and restricting the domain to $|x| \geq R$.) Therefore, the integral goes to 0 strongly in L^2, which was what we wanted.

With all of this set up, we can start using the Fredholm machinery.

Choose $f \in L^2(\mathbb{R}^d)$ and define the linear form

$$L_f(v) = \langle f, v\rangle_{L^2}, \tag{4.90}$$

which exists by Lax-Milgram. We claim that L_f is continuous on Σ; it is linear, so we just need to show boundedness, which follows from

$$|L_f(v)| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_{\Sigma}. \tag{4.91}$$

By the Riesz representation theorem, there exists $T(f) \in \Sigma$ such that $L_f(v) = \langle T(f), v\rangle_\Sigma = \langle f, v\rangle_{L^2}$ for all $v \in \Sigma$. This defines a map $T : L^2 \mapsto \Sigma, f \to T(f)$, such that (expanding out the Σ norm and the L^2 norm),

$$\int \nabla T(f) \cdot \nabla v + \int |x^2| T(f) v = \int_{L^2} f v \; \forall v \in \Sigma. \tag{4.92}$$

In particular, we can let v be any test function, so $T(f)$ is a weak solution to the PDE. We can also take $v = T(f)$ to find

$$\|T(f)\|_\Sigma^2 \leq \|f\|_{L^2}\|T(f)\|_\Sigma \implies \|T(f)\|_\Sigma \leq \|f\|_{L^2}. \tag{4.93}$$

This means $T : L^2 \mapsto \Sigma$ is continuous, and we know the injection $\Sigma \mapsto L^2$ is compact, so the *resolvant map* $T : L^2 \to L^2$ is compact.

Exercise 4.13. *Show $T : L^2 \mapsto L^2$ is self-adjoint.*

This implies T is diagonalizable in a Hlbertian basis of $L^2(\mathbb{R}^d)$. In the case of the harmonic oscillator, this tells us that solving $T\psi_n = \lambda_n^{-1}\psi_n$ is equivalent to solving $H\psi_n = \lambda_n \psi_n$.

Can we understand what this Hilbertian basis is? We can explicitly diagonalise H,

$$u = e^{-\frac{-|x|^2}{2}} v \tag{4.94}$$

$$Hu = e^{-\frac{-|x|^2}{2}} \left[\overbrace{(-\triangle + (d+1) + 2x \cdot \nabla)}^{\tilde{H}} v \right]. \tag{4.95}$$

Then, $Hu = \lambda u \iff \tilde{H}v = \lambda v$.This is easy in dimension 1, which gives us $\tilde{H}v_0 = 2v_0$, and induction on n lets us construct

$$\tilde{H}P_n = (2n+2)P_n, \tag{4.96}$$

where the P_n are the *Hermite polynomials*. This is the Hilbertian basis in which the resolvant T is diagonal. The abstract theorem that tells us that self-adjoint compact operators on Hilbert spaces are diagonalizable in this case admits a concrete representation.

We can further analyse this by finding eigenvalues by the variational method.

In real life, though, we often have an extra potential:

$$\Big(-\triangle + |x|^2 + V(x)\Big)\psi = \lambda\psi. \tag{4.97}$$

For small perturbations $V \to \epsilon V$, we can run the standard machinery of perturbation theory from quantum mechanics (take $\psi = \psi_0 + \epsilon\psi_1, \lambda = \lambda_0 + \epsilon\lambda_1$, and expand) to get

$$(H_0 - \lambda_0)\psi_1 = \lambda_1\psi_0 - V\psi_0 + \epsilon(\lambda_1\psi_1 - \epsilon V\psi_1) := f. \tag{4.98}$$

In QM, we would take the ϵ to 0 and get a correction to first order, but here we can treat this as its own PDE.

We previously solved the PDE involving $H_0 = -\triangle + |x|^2, T = (-\triangle + |x|^2)^{-1}$, and now we have

$$H_0\psi_1 = \lambda_0\psi_1 + f. \tag{4.99}$$

Therefore,

$$\psi_1 = T(\lambda_0\psi_1 + f) \tag{4.100}$$

or equivalently

$$(I - \lambda_0 T)\psi_1 = Tf. \tag{4.101}$$

This is exactly the Fredholm structure! Therefore, we can invert this operator if and only if we have orthogonality with f. We know $H_0 - \lambda_0$ has a nontrivial kernel because ψ_0 is in it. We have to precisely know the kernel, and in this case it is the span of ψ_0. In such a case, $(H_0 - \lambda_0)u = f$ admits a solution if and only if $\langle f, \psi_0\rangle L^2 = 0$.

Looking at the form of f, we need to impose

$$\langle \lambda_1\psi_0 - V\psi_0 + \epsilon\lambda_1\psi_0\rangle_{L^2} = 0 \tag{4.102}$$

$$\lambda_1 = \frac{\langle V\psi_0, \psi_0\rangle - \epsilon\langle \lambda_1\psi_1 + V\psi_1, \psi_0\rangle}{\|\psi_0\|_{L^2}^2}. \tag{4.103}$$

With this choice of λ_1, we can invert $H_0 - \lambda_0$ to get ψ_1.

4.5 The spectra of elliptic PDEs

Toolbox

Definition 4.11. *Let H be a real Hilbert space. Suppose $A : H \to H$ is a bounded linear operator.*

- *The resolvant of A is defined as $\{\lambda \in \mathbb{R} \mid A - \lambda I$ is invertible$\}$ and denoted $\rho(A)$.*
- *The real spectrum of A is defined as $\sigma(A) = \mathbb{R} \setminus \rho(A)$.*
- *$\eta \in \sigma(A)$ is said to belong to the point spectrum of A, $\delta_p(A)$, if $ker(A - \eta I) \neq \{0\}$. That is, there exists $\omega \neq 0$ such that $A\omega = \eta\omega$, and we call ω an eigenvector.*
- *The adjoint $A^\dagger : H \to H$ is defined such that $(Ax, y) = (x, A^\dagger y)$ for all $x, y \in H$.*
- *A is self-adjoint if $A = A^\dagger$.*

This next theorem is from Part II Linear Analysis.

Theorem 4.14 (Spectrum of a compact operator)**.** *Assume H is a separable Hilbert space with $K : H \to H$ compact. Then*

1. *$0 \in \delta(K)$, i.e. compact operators are not invertible.*
2. *$\delta(K) \setminus \{0\} = \delta_p(K) \setminus \{0\}$*
3. *$\delta(K) \setminus \{0\}$ is at most countable, i.e. it can be represented as $\{\lambda_i\}_{i=1}^\infty$, and if it is infinite then $\lambda_i \to 0$*
4. *If K is self-adjoint, then there exists a countable orthonormal basis for H consisting of eigenvectors of K (the spectral theorem)*

These are proved in appendix D.5 of Evans, as theorems 6 and 7.

4.5.1 Characterising the spectrum

Now, why are we doing all of this? We're trying to solve a problem of the form

$$\begin{cases} Lu = f & \text{in} U \\ u = 0 & \text{on} \partial U \end{cases} \tag{4.104}$$

by associating a bilinear form $B[u, v]$ and applying Garding's inequality, that there exist $\beta > 0, \gamma \geq 0$ such that $B\|u\|_{H^1}^2 \leq B[u, u] + \gamma\|u\|_{L^2}^2$. Further, we know we can weakly solve $Lu + \mu u = f$ for $\mu \geq \gamma$ such that the solution is bounded by the data.

When proving the Fredholm alternative, we set $L_\mu = L + \mu I$ and used this to define a map $u = L_\mu^{-1}(f)$, and $L_\mu^{-1} : L^2 \to H_0^1 \hookrightarrow L^2$, so by Rellich-Kondrachov, we could show $L_\mu^{-1} : L^2 \to L^2$ is compact.

We can therefore see that solving $Lu = f$ can be reduced to

$$\begin{aligned} Lu = f &\iff Lu + \mu u = f + \mu u \\ &\iff L_\mu u = f + \mu u \\ &\iff u = L_\mu^{-1}(f + \mu u) \\ &\iff u - \mu L_\gamma^{-1}(u) = L_\mu^{-1}(f) \\ &\iff u - Ku = h, \end{aligned} \tag{4.105}$$

where $K = \mu L_\mu^{-1}$ is compact.

Theorem 4.15 (Spectrum of L)**.** *Under the assumptions of the Fredholm alternative,*

(i) there exists an at most countable set $\Sigma \subset \mathbb{R}^n$ (the "bad set") such that the BVP $Lu = \lambda u + f$ in U and $u = 0$ on ∂U has a weak solution for all $f \in L^2$, if and only if $\lambda \notin \Sigma$;

(ii) if Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^\infty$ and (after reordering) we have $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \ldots$ with $\lambda_k \to \infty$.

(iii) To each $\lambda \in \Sigma$ there is a finite-dimensional space,

$$\mathscr{E}(\lambda) \triangleq \left\{ u \in H_0^1(U) \mid u \text{ is a weak solution of} \begin{cases} Lu = \lambda u & U \\ u = 0 & \partial U \end{cases} \right\}. \tag{4.106}$$

We say $\lambda \in \Sigma$ is an eigenvalue of L and $u \in \mathscr{E}(\lambda)$ is a corresponding eigenfunction.

In the case where $L = -\triangle$, we get the Helmholtz equation $-\triangle u = \lambda u$.

Proof .
Pick $\gamma > 0$ as in the Fredholm alternative and set $\mu \geq \gamma$. Then $L_\mu = L + \mu I$ is invertible and $L_\mu^{-1} : L^2 \to L^2$ is compact. Now $Lu = \lambda u + f \iff Lu - \lambda u = f$, so if $\mu = -\lambda \geq \gamma$, i.e. if $\lambda \leq -\gamma$, then the problem $\begin{cases} Lu + \mu u = f & U \\ u = 0 & \partial U \end{cases}$ admits a unique weak solution for all $f \in L^2$. That is, Σ lies somewhere in $(-\gamma, \infty)$; below γ, by this property, we can't have any valid $\lambda \in \Sigma$ values.
Consider $\lambda > -\gamma$. solving the BVP is equivalent to solving $\begin{cases} Lu - \lambda u = f, & U \\ u = 0, & \partial U \end{cases}$. Applying the Fredholm alternative to $L - \lambda I$, we see that $u \equiv 0$ is the only solution to $\begin{cases} (L - \lambda I)u = 0, & U \\ u = 0, & \partial U \end{cases}$.
That is, case (b) in the Fredholm alternative does not occur. This is equivalent to saying $u \equiv 0$ is the only solution to $\begin{cases} Lu + \gamma u = (\gamma + \lambda)u, & U \\ u = 0, & \partial U \end{cases}$.
In turn, we get that $u \equiv 0$ is the only solution to $L_\gamma^{-1}((\gamma + \lambda)u) = \frac{\gamma+\lambda}{\gamma} K(u)$, i.e. $K(u) = \frac{\gamma}{\gamma+\lambda} u$ only has $u \equiv 0$ as a solution. That is, $\frac{\gamma}{\gamma+\lambda}$ is not an eigenvalue of K.
Therefore, $\lambda \in \Sigma$ if and only if $\frac{\gamma}{\gamma+\lambda}$ is an eigenvalue of K. By Theorem 4.14, the set of eigenvalues of K consists of a finite set or else the values of a sequence tend to 0. Let this sequence be $\{\mu_k\}$.

Then $\mu_k \to 0$. Therefore $\lambda_k = \frac{\gamma}{\mu_k} - -\gamma \to \infty$. The fact that $\mathscr{E}(\lambda)$ is finite-dimensional follows from the Fredholm alternative.

If $\lambda \notin \Sigma$, then there exists $c > 0$ such that $\|u\|_{L^2} \leq c\|f\|_{L^2}$, and $c \to \infty$ if λ tends to an eigenvalue.

4.5.2 Self-adjoint positive operators

Definition 4.12. *The operator L is said to be formally self-adjoint if $L = L^\dagger$.*

Exercise 4.16. *Check that this is equivalent to $b^i \equiv 0$, which implies $B[u,v] = B[v,u]$.*

Definition 4.13. *The operator L is positive if there exists $B > 0$ such that $\beta\|u\|^2_{H^1} \leq B[u,u]$ for all $u \in H^1_0$.*

This is coercivity of the H^1 norm, so Lax-Milgram applies.

Theorem 4.17 (Eigenvalues of symmetric self-adjoint elliptic operators)**.** *Let L be a uniformly elliptic, formally self-adjoint, positive operator on U. Then we can represent the eigenvalues of L as $0 < \lambda_1 \leq \lambda_2 \leq \dots$ where each eigenvalue appears according to its multiplicity $\dim(\mathscr{E}(\lambda))$, and there exists an orthonormal basis $\{w_k\}_{k=1}^\infty$ for $L^2(U)$ of eigenfunctions, $w_k \in H^1_0$, i.e. w_k is a weak solution to $Lw_k = \lambda_k w_k$ on U and $w_k = 0$ on ∂U.*

Proof .
By the positivity and Lax-Milgram, we have that L is invertible, and $L^{-1} : L^2 \to H^1_0 \hookrightarrow L^2$. Denote $S := L^{-1} : L^2 \to L^2$. By Rellich-Kondrachov, S is compact. We claim S is self-adjoint. To show this, we pick $f, g \in L^2$. Then $S(f) = u$ means $u \in H^1_0$ is the unique weak solution to $Lu = f$ on U, $u = 0$ on ∂U. This also holds for $S(g) = v$. Therefore

$$B[u,w] = (f,w)\forall w, B[v,\varphi] = (g,\varphi)\forall\varphi. \tag{4.107}$$

Then

$$(S(f),g)_{L^2} = (u,g) \underbrace{=}_{\varphi=u} B[v,u], \tag{4.108}$$

and

$$(f,S(g))_{L^2} = (f,v)_{L^2} = B[u,v]. \tag{4.109}$$

Since L is self-adjoint, $B[u,v] = B[v,u]$, so

$$(f,S(g))_{L^2} = (S(f),g)_{L^2}\forall f,g \in L^2. \tag{4.110}$$

By 4.14 (iv), there exists $(\mu_k)_k \subset \mathbb{R}$ such that $\mu_k \to 0$ and there exists $w_k \in L^2(U)$ such that $(w_k)_k$ is an orthonormal basis of L^2 with $Sw_k = \mu_k w_k \implies L^{-1}w_k = \mu_k w_k \in H^1_0$. Therefore $Lw_k = \lambda_k w_k, \lambda_k = \frac{1}{\mu_k}$. The positivity of eigenvalues comes from the positivity of L.

4.6 Elliptic regularity

We've talked a lot about weak solutions, i.e. $u \in H_0^1(U)$. Now, we want to improve their regularity to $u \in C^2(\bar{U})$ so that we can make sense of $Lu = f$ pointwise almost everywhere. The idea here is that improving the regularity of f implies we can improve the regularity of u.

Example 4.20. Suppose $u \in C_c^\infty(\mathbb{R}^n)$ solves Poisson's equation, $-\triangle u = f$. Then

$$\int_{\mathbb{R}^n} f^2 \mathrm{d}x = \int_{\mathbb{R}^n} (\triangle u)^2 \mathrm{d}x = \sum_{i,j} \int_{\mathbb{R}^n} (D_i D_i u)(D_j D_j u) \tag{4.111}$$

$$\underbrace{=}_{\text{IBP}} -\sum_{i,j} \int (D_j D_i D_i u)(D_j u) \tag{4.112}$$

$$\underbrace{=}_{\text{IBP}} \sum_{i,j} (D_i D_j u)(D_i D_j u) = \int \sum_{i,j} |D_i D_j u|^2 \tag{4.113}$$

$$= \left\|D^2 u\right\|_{L^2(\mathbb{R}^n)}^2. \tag{4.114}$$

Therefore $\left\|D^2 u\right\|_{L^2} \leq \|\triangle u\|_{L^2}$: we can control all the second derivatives (mixed ones as well) just from the Laplacian. □

An Aside

We can make sense of $B[u,v] = \langle f, v\rangle$ for $f \in (H_0^1(U))' = H^{-1}(U)$ where $\langle \cdot, \cdot \rangle$ denotes the pairing between H_0^1 and its dual H^{-1}. If $f \in H^{-1}(U)$ then there exists $f_0, \ldots, f_u \in L^2(U)$ such that

$$\langle f, v\rangle \underbrace{=}_{\text{Riesz}} (w, v)_{H_0^1} = (f^0, v)_{L^2} + (f', v_{x_1})_{L^2}, \tag{4.115}$$

for all $v \in H_0^1$.

We could consider $Lu = f, f \in H^{-1}(U)$. Suppose $u \in H^1(U)$ is a weak solution to $Lu = f$ on U and $u = g$ on ∂U. We can make sense of this in the sense of traces: $g = \mathrm{Tr}(w), w \in H^1(U)$, and we subtract off the data by taking $\bar{u} = u - g \in H_0^1(U)$, and we solve $\begin{cases} L\bar{u} = \bar{f} & U \\ \bar{u} = 0 & \partial U \end{cases}$, where $\bar{f} = f - Lu \in H^{-1}(U)$.

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The takeaway here is that dual space methods in the context of elliptic PDEs are kind of nasty.

If $A : H \to H$ is a bounded linear operator, we say A is a Fredholm operator if $\ker A$ is finite-dimensional, if $\operatorname{im} A$ is closed such that $\dim \operatorname{coker}(\operatorname{im}(A)) = \dim(H/\operatorname{im} A) < \infty$.

The *index* of A is $\operatorname{ind} A = \dim\ker A - \dim\operatorname{coker} A$.

Example 4.21. Consider a compact operator $K : H \to H$. $\ker(I - K)$ is finite-dimensional, $\operatorname{im}(I - K)$ is closed, and $\dim \operatorname{coker}(I - K) = \dim \operatorname{im}((I - K)^{\perp}) \underbrace{=}_{\text{Fredholm}} \dim \ker(I - K^{\dagger}) < \infty$.

Therefore compact operators have an index of 0. $\square$

The Atiyah-Singer index theorem links this with topological ideas: if $U \to$ a closed Riemann surface Σ_g, and $A \to d + \delta : \Omega^1 \to \Omega^2 \oplus \Omega^0$, then $\operatorname{ind}(d + \delta) = 2g - 2 = -\chi(\Sigma_g)$.

If L is uniformly elliptic and a self-adjoint positive operator, then there exists an orthonormal basis $\{w_k\}$ of $L^2(U)$ where $w_k \in H_0^1(U)$ and $\begin{cases} Lw_k = \lambda w_k & U \\ w_k = 0 & \partial U \end{cases}$. We states $\lambda_k \to \infty$. If $L = -\triangle$, the rate at which these go to infinity are given by the Weyl asymptotics:

$$\lambda_k \sim \frac{\sqrt{2\pi}}{\operatorname{Vol}(B_1(0))^{2/n}} \operatorname{vol}(U)^{2/n} k^{2/n}. \tag{4.116}$$

If $\mu \neq \lambda_k$, then $Lu - \mu u = f$ has a unique solution, and

$$u = \sum_{k \geq 1} \frac{(f, w_k)_{L^2}}{\lambda_k - \mu} w_k \quad \text{converges in} \quad L^2. \tag{4.117}$$

If $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ and we have two eigenvalues for $L = -\triangle$. If the eigenvalues are equal, are Ω_1, Ω_2 isometric? (Can you hear the shape of a drum?) This is unsolved in general.

Our motivating example is finding $u \in C_c^{\infty}(\mathbb{R}^n)$ solving $-\triangle u = f$, $\int_{\mathbb{R}^n} f^2 \mathrm{d}x = \int (\triangle u)^2 = \int u_{x_i x_i} u_{x_j x_j}$. Last time, we applied integration by parts to show that

$$\left\| D^2 u \right\|_{L^2} \leq \|\triangle u\|_{L^2}; \tag{4.118}$$

the Laplacian controls all the second-order derivatives.

In this section, we let $U \subset \mathbb{R}^n$ be open and bounded, and $V \Subset U$.

Definition 4.14. *For $0 < |h| < \operatorname{dist}(V, \partial U)$, we define the difference quotient $\triangle_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}$, and $\triangle^h u = (\triangle_1^h u, \ldots, \triangle_n^h u)$.*

The idea here is we want to mimic the classical derivative in L^p spaces.

If $u \in L^2(U)$, then we have $\triangle^h u \in L^2(V)$.

$D(\triangle^h u) = \triangle^h(Du)$, so if $u \in H^1(U)$ then $\triangle^h u \in H^1(v)$. The derivative of the difference quotient will start to look like $D^2 u$.

Lemma 4.18 (Difference quotients are useful)**.** *Suppose $u \in L^2(U)$. Then $u \in H^1(V)$ if and only if for all h with $0 < |h| < \frac{1}{2} \operatorname{dist}(V, \partial U)$, we have*

$$\left\|\triangle^h u\right\|_{L^2(V)} \leq C \tag{4.119}$$

where $C > 0$ is independent of h.

Moreover, there exists a constant $\tilde{C}$ also independent of h such that

$$\frac{1}{\tilde{c}}\|Du\|_{L^2(V)} \leq \left\|\triangle^h u\right\|_{L^2(V)} \leq \tilde{c}\|Du\|_{L^2(V)}. \tag{4.120}$$

Theorem 4.19 (Interior regularity)**.** *Suppose L is a uniformly elliptic operator on U and assume that $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$ and $f \in L^2(U)$. Suppose further that $u \in H^1(U)$ (no zero subscript here: u doesn't necessarily solve the BVP) satisfies the equation*

$$B[u,v] = (f,v)\ \forall v \in H_0^1(U). \tag{4.121}$$

Then $u \in H^2_{loc}(U)$, and for each $V \subset\subset U$, we have

$$\|u\|_{H^2(V)} \leq C\Big(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\Big), \tag{4.122}$$

where C is a function of V, U, a^{ij}, b^i, c, n but is independent of f, u.

This result means that we can get two weak derivatives of U, which is an improvement. It is also useful to write the inequality in the theorem as

$$\|u\|_{H^2(V)} \leq c\Big(\|Lu\|_{L^2(U)} + \|u\|_{L^2(U)}\Big). \tag{4.123}$$

(cf. with the Laplacian where we had $\left\|D^2 u\right\|_{L^2} \leq \|\triangle u\|_{L^2}$).

Proof .

1. Fix $V \subset\subset U$ and assume W is compact such that $V \subset\subset W \subset\subset U$. Take a cutoff function $\xi \in C_c^\infty(W)$, $0 \leq \xi \leq 1$ such that $\xi 1_V = 1$ and $\xi 1_{\partial W} = 0$. We rewrite $B[u,v] = (f,v)$ as

$$\sum_{i,j=1} \int_U a^{ij} u_{x_i} v_{x_j}\,\mathrm{d}x = \int_U \tilde{f} v\ \forall v \in H_0^1(U), \tag{4.124}$$

where $\tilde{f} = f - b^i u_{x_i} - cu$ is in $L^2(U)$.

Choose $v = -\triangle_k^{-h}(\xi^2 \triangle_k^h u)$ for some $k = 1, \ldots, n$ fixed and $0 < |h| < \frac{1}{2}\operatorname{dist}(W, \partial U)$. Note that $v \in H_0^1(W)$ and think of v as approximating $D^2 u$.

Set $A = \sum_{i,j} \int_U A^{ij} u_{x_j} v_{x_i}$, $B = \int_U \tilde{f} v \mathrm{d}x$, with this particular choice of v. Now, we observe a couple of properties about difference quotients.

For $\psi, \phi \in L^2(U)$ that are supported in W, we have the following algebraic property,

$$\int_U \psi(x)(\triangle_k^{-h}\phi)(x)\mathrm{d}x = -\int_U (\triangle_k^h \psi)(x)\phi(x)\mathrm{d}x, \tag{4.125}$$

and we call this integration by parts for difference quotients.

We also have a kind of Leibniz rule:

$$\triangle_k^h(\psi \cdot \phi)(x) = \frac{(\psi(x+e_k h)\phi(x+e_k h) - \psi(x)\phi(x))}{h} = (\tau_k^h \psi)(x)\triangle_k^h \phi(x) + (\triangle_k^h \psi)(x)\phi(x), \tag{4.126}$$

where $\tau_k^h \psi(x) = \psi(x + he_k)$ is the translation operator.

2. Next, let's bound A. If we expand it out using the particular choice of v we made, we get the following:

$$\begin{aligned} A &= -\int_U a^{ij} u_{x_i} \triangle_k^{-h}(\xi^2 \triangle_k^h u)_{x_j} \mathrm{d}x && (4.127) \\ &= \int_U \triangle_k^h(a^{ij} u_{x_i})(\xi^2 \triangle_k^h u)_{x_j} \mathrm{d}x && (4.128) \\ &= \int_U \left[(\tau_k^h a^{ij})\triangle_k^h u_{x_i} + (\triangle_k^h a^{ij})u_{x_i}\right] \times \left[\xi^2 \triangle_k^h u_{x_j} + 2\xi\xi_{x_j}\triangle_k^h u\right]\mathrm{d}x && (4.129) \\ &= A_1 + A_2, && (4.130) \end{aligned}$$

where

$$A_1 = \int_U \xi^2 (\tau_k^h a^{ij})(\triangle_k^h u_{x_i})(\triangle_k^h u_{x_j}), \tag{4.131}$$

and $A_2 = A - A_1$, which we don't need to explicitly write down yet.

By uniform ellipticity of the operator L,

$$\sum (\tau_k^h a^{ij}(x))\eta_i \eta_j \geq \theta |\eta|^2 \ \forall \eta \in \mathbb{R}^n, \forall x \in W, \theta > 0. \tag{4.132}$$

Applying this with $\eta_2 = \triangle_k^h u_{x_i}$, we get

$$A_1 \geq \theta \int_U \xi^2 \left|\triangle_k^h(Du)\right|^2 \mathrm{d}x. \tag{4.133}$$

Next, we write down A_2, or we just look at where Paul Minter did it. This gives us a bound,

$$|A_2| \leq \epsilon \int_W \xi^2 \left|\triangle^h (Du)\right|^2 \mathrm{d}x + \frac{C}{\epsilon} \int_W (|Du|^2 + \left|\triangle_k^h u\right|^2) \mathrm{d}x. \tag{4.134}$$

Applying the quotient-difference lemma, we can further write down

$$|A_2| \leq \int_W \xi^2 \left|\triangle^h (Du)\right|^2 \mathrm{d}x + \frac{C}{\epsilon} \int_W |Du|^2 \mathrm{d}x. \tag{4.135}$$

We make a good choice of ϵ. Set $\epsilon = \frac{\theta}{2}$ and use $A_2 \geq -|A_2|$ to find

$$A = A_1 + A_2 \geq A_1 - |A_2| \geq \frac{\theta}{2} \int_U \xi^2 \left|\triangle^h (Du)\right|^2 - C \int_W |Du|^2. \tag{4.136}$$

3. Next, we bound B:

$$\tilde{f} = f - b^i u_{x_i} - cu \tag{4.137}$$

$$|B| = \left|\int_W \tilde{f} u\right| \leq C \int_W \{|f| + |Du| + |u|\} \left|\triangle_k^{-h} (\xi^2 \triangle_k^h u)\right|. \tag{4.138}$$

By Lemma 4.14,

$$\int_W \left|\triangle_k^{-h} (\xi^2 \triangle_k^h u)\right|^2 \mathrm{d}x \leq C \int_W \left|D(\xi^2 \triangle_k^h u)\right|^2 \mathrm{d}x \tag{4.139}$$

$$= C \int_W |\xi|^2 |D\xi|^2 \left|\triangle_k^h u\right|^2 + \xi^2 \left|\triangle_k^h (Du)\right|^2 \mathrm{d}x \tag{4.140}$$

$$\leq C \int_W |Du|^2 + c \int_W \xi^2 \left|\triangle_k^h (Du)\right|^2 \mathrm{d}x. \tag{4.141}$$

By Young's inequality,

$$|B| \leq \epsilon \int_U \xi^2 \left|\triangle_k^h (Du)\right|^2 + \frac{C}{\epsilon} \int_W (f^2 + u^2 + |Du|^2). \tag{4.142}$$

4. The condition that $B[u, v] = (f, v)$ for all v implies that $A = B$, so $|A| = |B|$. So in particular, we can write out the sandwich of inequalities we've shown:

$$\frac{\theta}{2} \int_U \xi^2 \left|\triangle_k^h (Du)\right|^2 - C \int_W |Du|^2 \leq |A| = |B| \tag{4.143}$$

$$\leq \frac{\theta}{4} \int_U \xi^2 \left|\triangle_k^h (Du)\right|^2 + \int_W (f^2 + u^2 + |Du|^2) \tag{4.144}$$

$$\tag{4.145}$$

and so

$$\int_U \xi^2 \left|\triangle_k^h(Du)\right|^2 \leq C \int_W f^2 + u^2 + |Du|^2. \tag{4.146}$$

Since $\xi\big|_V = 1$, we get that if $u \in H^1(U)$ solves $B[u,v] = (f,v)$, then

$$\int_V \left|\triangle_k^h(Du)\right|^2 \leq \underbrace{C}_{\text{indep of} h} \int_W (f^2 + u^2 + |Du|^2), \tag{4.147}$$

which implies by Lemma 4.14 that $Du \in H^1(U) \implies u \in H^2_{\text{loc}}(U)$ with

$$\left\|D^2 u\right\|_{L^2(U)} \leq C\Big(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}\Big). \tag{4.148}$$

5. We show that the $\|Du\|_{L^2(W)}$ term is unnecessary (in the $H^1(W)$ norm on u) is unnecessary. Let $\xi \in C_c^\infty(U)$ with $\xi\big|_W = 1$. Set $v = \xi^2 u$ in $B[u,v] = (f,v)$ to get

$$\int_U (a^{ij} u_{x_i} (\xi^2 u)_{x_j} + b^i u_{x_i} \xi^2 u + c u^2 \xi^2) \mathrm{d}x = \int_U \xi^2 f u \mathrm{d}x. \tag{4.149}$$

Then, repeating the proof of Garding's inequality,

$$\|Du\|^2_{L^2(W)} \leq C\Big(B[u,u] + \gamma \|u\|^2_{L^2(W)}\Big) \leq c\Big(\|f\|^2_{L^2(W)} + \|u\|^2_{L^2(W)}\Big). \tag{4.150}$$

and therefore

$$\|u\|_{H^1(W)} \leq C\Big(\|f\|_{L^2(W)} + \|u\|_{L^2(W)}\Big) \tag{4.151}$$

$$\left\|D^2 u\right\|_{L^2(V)} \leq C\Big(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\Big). \tag{4.152}$$

This is a local result: to have $u \in H^2(V)$, it is enough to have $f \in L^2(W)$ where $V \subset\subset W$. Therefore singularities/losses of regularity don't propagate from the boundary. So if somehow $f \notin L^2$ near ∂U, we don't see that in our estimates.

We claim that $Lu = f$ almost everywhere in U. We see this since $(Lu - f, v)_{L^2(V)} = 0$, so $Lu - f = 0$ almost everywhere in V, but since $V \subset\subset U$ is arbitrary, it holds in U.

Theorem 4.20 (Improved interior regularity). *If $a^{ij}, b^i, c \in C^{m+1}(U)$ and $f \in H^m(U)$, for some $m \in \mathbb{N}$, we have $u \in H^{m+2}_{loc}(U)$ and for all $V \subset\subset W \subset\subset U$,*

$$\|u\|_{H^{m+2}(V)} \leq C\Big(\|f\|_{H^m(W)} + \|u\|_{L^2(W)}\Big). \tag{4.153}$$

Proof .
By induction, on sheet 4.

Holder regularity roughly says that if $f \in C^{m,\alpha}(U)$, then $u \in C^{m+2,\alpha}(U)$ for all $0 < \alpha < 1$.

We can combine the above theorem with Sobolev embedding results. In particular, if m is large (specifically if $m > \frac{n}{p} = \frac{n}{2}$) then $u \in H^{m+2}_{\text{loc}}(U) \hookrightarrow C^2_{\text{loc}}(U)$. If $f \in C^\infty(U)$ then so is u.

Having talked about interior regularity, let's look at boundary regularity.

Theorem 4.21 (Boundary H^2 regularity)**.** *Assume $a^{ij} \in C^1(\bar{U}), b^i, c \in L^\infty(U), f \in L^2(U)$, and $\partial U \in C^2$. Suppose we have a weak solution $u \in H^1_0(U)$ of $\begin{cases} Lu = f & U \\ u = 0 & \partial U \end{cases}$.*

Then $u \in H^2(U) \cap H^1_0(U)$ and $\|u\|_{H^2(U)} \le C\Big(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\Big)$.

Moreover, if u is the unique weak solution to the BVP, then $\|u\|_{H^2(U)} \le C\|f\|_{L^2(U)} = C\|Lu\|_{L^2(U)}$.

Proof sketch.
We sketch the main idea for $U = B_1(0) \cap \{x_n > 0\}$. Let $V = B_{1/2}(0) \cap \{x_n > 0\}$ and choose $\xi \in C^\infty_c(B_1(0))$ with $\xi\big|_V = 1, 0 \le \xi \le 1$. That is, the support of ξ is between V and U and it's 1 on V.
Since u is a weak solution to the PDE, we have

$$\sum_{i,j=1}^n \int_U a^{ij} u_{x_i} v_{x_j} = \int_U \tilde{f} v \;\forall v \in H^1_0(U). \tag{4.154}$$

Let $0 < |h| \le \frac{1}{4}\operatorname{dist}(\operatorname{supp}\xi, \partial B_1(0))$, so that if we shift we still stay away from ∂U.
As before, take $v = -\triangle_k^{-h}(\xi^2 \triangle_k^h u)$ for fixed $k = 1, \dots, n-1$ (working tangential to $\{x_n = 0\}$ so that v stays inside the domain).
We claim $v \in H^1_0(U)$, which we can show because

$$\begin{aligned} v(x) &= -\frac{1}{h}\triangle_k^{-h}\big(\xi^2(x)(u(x+he_k) - u(x))\big) \\ &= \frac{1}{h^2}\big[\xi^2(x-he_k)(u(x) - u(x-he_k)) + \xi^2(x)(u(x+he_k) - u(x))\big] \end{aligned}$$

for $x \in U$. Since the translation is horizontal and $\operatorname{Tr}(u)\big|_{x_n=0} = 0$, we have $\operatorname{Tr}(u(x \pm he_k))\big|_{x_n=0} = 0$ for all $|x| < 1-h$. For $x_n = 0$, $|x| \ge 1-h$ have $\xi^2(x) = 0, \xi(x - he_k) = 0$. So as in the proof of Theorem 4.20, we deduce

$$\int_V \left|\triangle_k^h(Du)\right|^2 \le C\int_U \Big(f^2 + u^2 + |Du|^2\Big). \tag{4.155}$$

Therefore, we can controll all second-order derivatives of the form $D_k D_i u$ with $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, n-1\}$ via

$$\|D_k D_i u\|_{L^2(V)} \leq C\Big(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\Big). \tag{4.156}$$

To control $u_{x_n x_n}$ we use the weak formulation, IBP, and a previous remark to find

$$-(a^{ij}u_{x_i})_{x_j} + b^i u_{x_i} + cu = f \text{ a.e. in } U \tag{4.157}$$

and so

$$a^{nn}u_{x_n x_n} = F = -\sum_{i<n,j<n} a^{ij}u_{x_i x_j} + \tilde{b}^i u_{x_i} + cu - f \text{ a.e. in } U. \tag{4.158}$$

So $F \in L^2(V)$, and using the previous results

$$\|F\|_{L^2(V)} \leq C\Big(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\Big). \tag{4.159}$$

Next, we use the uniform ellipticity condition on L to show $a^{nn}(x) \geq \theta|\xi|^2 = \theta > 0$ by substituting in $\xi = (0, \ldots, 0, 1)$.
Since $a^{nn} \in C^1(\overline{U})$, we see $a^{nn} \geq C > 0$, so we can divide by a^{nn} and get $u_{x_n x_n} \in L^2(V)$ with the same bound as for F. This implies

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}). \tag{4.160}$$

Again using the proof of Garding's inequality, we can replace the $\|\cdot\|_{H^1(U)}$ on the RHS by $\|\cdot\|_{L^2(U)}$.

Higher regularity results can still be shown. For example, if a^{ij}, b^i, c are in $C^{m+1}(\overline{U})$, $f \in H^m(U)$, and $\partial U \in C^{m+2}$, and we have $u \in H^1_0$ as our weak solution, then $u \in H^{m+2}(U)$ with the corresponding data bound

$$\|u\|_{H^{m+2}(U)} \leq c\Big(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}\Big). \tag{4.161}$$

In particular, if everything is C^∞, then $u \in C^\infty$ as well, and we have a classical solution. In doing this, we implicitly make use of the Sobolev embedding theorem, to bump us up from weak to regular derivatives.

An example of where this might come up is in the eigenvalue problem $Lu = \lambda u$. $L - \lambda I$ is uniformly elliptic (if L is), and so we have $(L - \lambda)u = 0 = f \in C^\infty(U)$: an infinitely differentiable L admits infinitely differentiable eigenfunctions.

Chapter 5

Second-order linear hyperbolic equations

Contents

5.1 Defining hyperbolicity

These are sometimes known as "wave equations", but really, the wave equation is only a particular case.

Definition 5.1. *A second-order partial differential equation*

$$\sum_{i,j=1}^{n+1} \left(a^{ij}(y)u_{y_i}\right)_{y_j} + \sum_{i=1}^{n} a^i(y)u_{y_i} + a(y)u = f, \tag{5.1}$$

with $y \in \mathbb{R}^{n+1}$, $a^{ij} = a^{ji}$, $a^i, a \in C^\infty(\mathbb{R}^{n+1})$ is said to be hyperbolic if the quadratic form $q(\xi) = \sum_{i,j=1}^{n+1} a^{ij}(y)\xi_i\xi_j$, the principal symbol, has signature $(+, -, \dots, -)$ for all $y \in \mathbb{R}^{n+1}$. This means that at each point y (possibly after a change of basis), we can write

$$q(\xi) = \lambda_{n+1}^2 \xi_{n+1}^2 - \sum_{i=1}^{n} \lambda_i^2 \xi_i^2, \tag{5.2}$$

where $\lambda_k(y) > 0$ for all $k = 1, \dots, n+1$.

Another way of saying this is that the matrix $(a^{ij}(y))_{ij}$ is diagonalizable with exactly one positive eigenvalue.

By a coordinate transformation, we can transform Equation 5.1 into a more familiar form, as long as we're working locally.

$$u_{tt} - \sum \left(a^{ij}(x,t)u_{x_i}\right)_{x_j} + \sum b^i(x,t)u_{x_i} + c(x,t)u = f. \tag{5.3}$$

Here, we've undergone a relabelling $(x_1, \dots, x_n, t) = (y_1, \dots, y_{n+1})$. In general, we can patch together local solutions to make global ones when we're working with hyperbolic PDEs, so this is a good result to have.

Note that assuming $\sum a^{ij}\xi_i\xi_j \geq \theta|\xi|^2$,m we see that $\{(x,t), t = 0\}$ is a non-characteristic (good) surface of the PDE, so we can apply the Cauchy-Kovalevskaya theorem to solve the PDE if we have analytic data $u\big|_{t=0}$, $u_t\big|_{t=0}$. We could do this, but we'd like to use the structure of the PDE to solve the equation under weaker assumptions.

5.2 Hyperbolic initial boundary value problems

Suppose $U \subset \mathbb{R}^n$ is open and bounded, with $\partial U \in C^1$, and we define the following notation: $U_T = (0,T) \times U$ (so this is open), $\Sigma_t = U \times \{t\}$, and $\partial^* U_T = [0,T] \times \partial U$. If U is a circle, U_T is an open cylinder, Σ_t is a particular cross-section of the cylinder, and $\partial^* U_T$ is the curved boundary, *missing* the caps of the cylinder (intentionally).

So $\partial(U_T) = \Sigma_0 \cup \Sigma_T \cup \partial^* U_T$, and these sets are disjoint.

Having defined the domain of interest, we can look at an IBVP in this domain. Let $u \in C^2(U_T)$ satisfy the IBVP (the wave equation)

$$\begin{cases} u_{tt} - \triangle u = 0 & \text{in} \quad U_T \\ u = \psi_0 & \text{on} \quad \Sigma_0 \\ u_t = \psi_1 & \text{on} \quad \Sigma_0 \\ u = 0 & \text{on} \quad \partial^* U_T. \end{cases} \tag{5.4}$$

The specification of u, u_t on Σ_0 is the initial value part, and the specification of $u = 0$ on the curved surface specifies the boundary values. In this case u cannot escape from the sides of the cylinder. The wave equation usually deals with energy propagation, so this is essentially requiring conservation of energy.

Now, we perform an energy estimate as we've done for elliptic equation. Multiply the PDE by u_t and integrate by parts over U_t to get

$$0 = \int_{U_t} (u_{tt}u_t - u_t \triangle u) \mathrm{d}x\mathrm{d}t \tag{5.5}$$

$$= \int_{U_t} \left(\frac{1}{2} \partial_t \big((u_t)^2 \big) - \mathrm{div}_x (u_t Du) + Du_t \cdot Du \right) \mathrm{d}x\mathrm{d}t, \tag{5.6}$$

where in the second step we used the identity $\nabla(g\nabla h) = \nabla g \nabla h + g\triangle h$. Continuing to simplify and using the divergence theorem, we get

$$0 = \int_{U_t} \left(\frac{1}{2} \partial_t \Big((u_t^2) + |Du|^2 \Big) - \mathrm{div}_x (u_t Du) \right) \mathrm{d}x\mathrm{d}t \tag{5.7}$$

$$= \frac{1}{2} \int_{\Sigma_t} (u_t^2 + |Du|^2) \mathrm{d}x - \frac{1}{2} \int_{\Sigma_0} \Big(u_t^2 + |Du|^2 \Big) \mathrm{d}x - \underbrace{\int_0^t \int_{\partial U} u_t Du \cdot \vec{n} \mathrm{d}S}_{=0 \text{ since } u \equiv 0 \text{ on } \partial^* U_T \implies u_t \equiv 0 \text{ on } \partial^* U_T} . \tag{5.8}$$

Therefore we get an energy conservation law,

$$\int_{\Sigma_t} (u_t)^2 + |Du|^2 \mathrm{d}x = \int_{\Sigma_0} \Big((\psi_1)^2 + |D\psi_0|^2 \Big) \mathrm{d}x. \tag{5.9}$$

We'll sometimes use the weaker result where the = is a $\leq$. We call this an *a priori estimate*, taking on a similar role to the Garding inequality in elliptic theory. Let's see how we can use this estimate in practice. Let $v, \overline{v} \in C^2$ be two solutions with data $\phi_i, \overline{\phi}_i$, with $i = 0, 1$. Let $u = v - \overline{v}, \psi_i = \phi_i - \overline{\phi}_i$. Then there exists $c > 0$ such that

$$\sup_{t \in [0,T]} \Big(\|u(\cdot, t)\|_{H^1(U)}^2 + \|u_t(\cdot, t)\|_{L^2(U)}^2 \Big) \leq c(\|\psi_0\|) \tag{5.10}$$

This comes out of the Poincaré inequality via $u = 0$ on $\partial^* U_T$. This shows us uniqueness, and continuous dependence on initial data.

We proceed similarly to how we established elliptic theory, by defining an operator L:

$$Lu = -\sum \left(a^{ij}(x,t)u_{x_i}\right)_{x_j} + \sum b^i(x,t)u_{x_i} + b(x,t)u_t + c(x,t)u. \tag{5.11}$$

Note that the $b(x,t)$ term is new. We take $a^{ij} = a^{ji}, b^i, b, c \in C^1(\overline{U}_T)$. We assume the principal part is uniformly elliptic, i.e. there exists $\theta > 0$ such that $\sum a^{ij}(x,t)\xi_i\xi_j \geq \theta\|xi\|^2$ for all $(x,t) \in U_T, k\xi \in \mathbb{R}^n$. We consider the IBVP

$$\begin{cases} u_{tt} + Lu = f & \text{in} \quad U_T \\ u = \psi_0 & \text{on} \quad \Sigma_0 \\ u_t = \psi_1 & \text{on} \quad \Sigma_0 \\ u = 0 & \text{on} \quad \partial^* U_T. \end{cases} \tag{5.12}$$

We aim to find a weak formulation for this problem. Suppose $u \in C^2(\overline{U}_T)$ is a solution to this equation. Multiply the PDE by some $v \in C^2(\overline{U}_T)$ such that $v = 0$ on the sides $\partial^* U_T$ and the top Σ_t, but not necessarily on the bottom. We get

$$\int_{U_T} fv \mathrm{d}x\mathrm{d}t = \int_{U_T} (u_{tt}v + Lu \cdot v)\mathrm{d}x\mathrm{d}t \tag{5.13}$$

and we expand out L and integrate by parts,

$$\int_{U_T} fv \mathrm{d}x\mathrm{d}t = \int_{U_T} \left(-u_t v_t + a^{ij}u_{x_i}v_{x_j} + b^i u_{x_i} v + bu_t v + cuv\right)\mathrm{d}x\mathrm{d}t + \left[\int_U u_t v \mathrm{d}x\right]_{t=0}^{t=T} - \int_0^T \int_{\partial U} a^{ij}\vec{n}_i u_{x_j} v \mathrm{d}S\mathrm{d}t \tag{5.14}$$

$$\int_{U_T} fv \mathrm{d}x\mathrm{d}t = \int_{U_T} \left(-u_t v_t + a^{ij}u_{x_i}v_{x_j} + b^i u_{x_i} v + bu_t v + cuv\right)\mathrm{d}x\mathrm{d}t - \int_{\Sigma_0} \psi_1(x)v(x,0)\mathrm{d}x, \tag{5.15}$$

and $u\big|_{\Sigma_0} = \psi_0$, $u\big|_{\partial^* U_T} = 0$. We'd like to treat this as an alternative version of the PDE, which requires that we can undo this and get back the PDE. Suppose the above holds for all $v \in C^2(\overline{U}_T)$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$. If v has compact support on U_T, then undoing the IBP gets us

$$0 = \int_{U_T} (u_{tt} + Lu - f)v. \tag{5.16}$$

Since v is arbitrary, we get $u_{tt} + Lu - f = 0$ on U_T.

Next, we drop the compact support assumption and instead say v is C^∞. We get

$$\int_{U_T} (u_{tt} + Lu - f)v \mathrm{d}x\mathrm{d}t = \int_{\Sigma_0} (\psi_1 - u_t)v \mathrm{d}x, \tag{5.17}$$

but the LHS vanishes by the reasoning above, so we get

$$\int_{\Sigma_0} (\psi_1 - u_t)v\mathrm{d}x = 0 \; \forall v \in C^\infty(\overline{U}_T),\; v\Big|_{\partial^* U_T \cup \Sigma_T} = 0. \tag{5.18}$$

We separate variables: let $v(x,t) = \chi(t)\varphi(x)$ with $\chi \in C^\infty([0,T])$ such that $\chi = 1$ near $t = 0$ and $\chi = 0$ near $t = T$. This tells us that $v\big|_{\Sigma_0} = \varphi$, where φ is any compactly supported function on Σ_0. From here, we say

$$\int_{\Sigma_0} (\psi_1(x) - u_t(x,0))\varphi(x) \; \forall \varphi \in C_c^\infty(\Sigma_0) \tag{5.19}$$

$$\implies \psi_1 = u_t \quad \text{on} \quad \Sigma_0. \tag{5.20}$$

Having established this equivalence, we can formally define what it means to be a weak solution to the IBVP.

Definition 5.2. *Suppose $f \in L^2(U_T), \psi_0 \in H_0^1(\Sigma_0), \psi_1 \in L^2(\Sigma_0)$, $a^{ij} = a^{ji}, b^i, b, c \in C^1(\overline{U}_T)$ where a^{ij} are uniformly elliptic. We say $u \in H^1(U_T)$ is a weak solution to the hyperbolic initial boundary value problem*

$$\begin{cases} u_{tt} + Lu = f & \textit{in} \quad U_T \\ u = \psi_0, u_t = \psi_1 & \textit{on} \quad \Sigma_0 \\ u = 0 & \textit{on} \quad \partial^* U_T. \end{cases} \tag{5.21}$$

if $u\big|_{\Sigma_0}, u_{\partial^* U_T} = 0$ (in the trace sense) and

$$\int_{U_T} (-u_t v_t + a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + b u_t v + cuv)\mathrm{d}x\mathrm{d}t - \int_{\Sigma_0} \psi_1(x) v(x,0)\mathrm{d}x = \int_{U_T} fv\mathrm{d}x\mathrm{d}t \tag{5.22}$$

holds for all test functions $v \in H^1(U_T)$ with $v\big|_{pd^* U_T \cup \Sigma_T}$ (in a trace sense).

Theorem 5.1. *A weak solution to the hyperbolic IBVP is unique.*

Proof .
If $w, \bar{w}$ are two weak solutions to the IBVP with the same initial data, then $u = w - \bar{w}$ is a weak solution with $f \equiv 0, u(x,0) = 0, u_t(x,0) = 0$. The idea from here is to use the energy method to get a norm on u, show it vanishes, and hence show that u itself vanishes. We would like to pick $v = u_t$ in the weak solution condition, because we'd then get something similar to what we got for the wave equation. The issue with this is v may not be in $H^1(U_T)$, and v may not vanish on Σ_T. So we need to find a workaround.
Define $v(x,t) = \int_t^T e^{-\lambda s} u(x,s)\mathrm{d}s$ for some $\lambda > 0$ that we'll fix later. This has the desired properties, $v \in H^1(U_T)$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$. Also we can check $v_t = -e^{\lambda t} u(x,t) \in H^1(U_T)$.
We can take this v and substitute it into the weak-equation condition, to get a long integral expression:

$$\int_{U_T} (u_t u e^{-\lambda t} - e^{\lambda t} a_{ij} v_{tx_j} v_{x_i} + b^i u_{x_i} v + b u_t u + (c-1)uv - e^{\lambda t} v v_t)\mathrm{d}x\mathrm{d}t = 0. \tag{5.23}$$

Integrating by parts on the b^i, b terms and the $e^{\lambda t}vv_t$ term, we get something even worse:

$$\int_{U_T}(u_tue^{-\lambda t} - e^{\lambda t}a^{ij}v_{tx_j}v_{x_i} + \underbrace{(b^iuv)_{x_i}}_{a} + \underbrace{(buv)_t}_{b} - (b^i_{x_i}uv + b^iuv_{x_i} + b_tuv + buv_t) \\ + (c-1)uv - \frac{1}{2}\partial_t(v^2e^{\lambda t}) + \frac{1}{2}\lambda v^2e^{\lambda t})\mathrm{d}x\mathrm{d}t = 0 \tag{5.24}$$

$\int a = 0$ since $u = v = 0$ on ∂^*U_T. $\int_b = 0$ since $v = 0$ on $\Sigma_T, u = \psi_0 = 0$ on Σ_0.
We group what's left over into one total derivative in time:

$$A = B \tag{5.25}$$

$$A = \int_0^T\int_{\Sigma_T}\frac{1}{2}\partial t\big(u^2e^{-\lambda t} - a^{ij}v_{x_i}v_{x_j}e^{\lambda t} - v^2e^{\lambda t}\big)\mathrm{d}x\mathrm{d}t + \int_{U_T}\frac{\lambda}{2}(u^2e^{-\lambda t} + a^{ij}e^{\lambda t}v_{x_i}v_{x_j} + v^2e^{\lambda t})\mathrm{d}x\mathrm{d}t \tag{5.26}$$

$$B = \int_{U_T}\left(\frac{1}{2}a^{ij}_tv_{x_i}v_{x_j}e^{\lambda t} + (b^ix_i + b_t + 1 - c)uv + b^iv_{x_i}u + buv_t\right)\mathrm{d}x\mathrm{d}t \tag{5.27}$$

We're going to estimate A and B separately and then put those estimates together. Note that uniform ellipticity doesn't apply to B because we don't know that a^{ij}_t is uniformly elliptic. We use it for A:

$$A = e^{\lambda T}\int_{\Sigma_T}\frac{1}{2}u^2\mathrm{d}x + \frac{1}{2}\int_{\Sigma_0}(a^{ij}v_{x_i}v_{x_j} + v^2)\mathrm{d}x + \frac{\lambda}{2}\int_{U_T}(\dots) \tag{5.28}$$

and since the first two terms are nonnegative, we can say

$$A \geq \frac{\lambda}{2}\int_{U_T}(u^2e^{-\lambda t} + \theta|Dv|^2e^{\lambda t} + v^2e^{\lambda t})\mathrm{d}x\mathrm{d}t. \tag{5.29}$$

Next, we apply Young's inequality with $ab = (ae^{-\lambda t})(e^{\lambda t}b)$ to lower-bound B:

$$B \leq C(a^{ij}_t)\int_{U_T}e^{\lambda t}|Dv|^2 + C(b, b^i, c)\int_{U_T}|u||v| + C(b^i)\int_{U_T}|u||Dv| + C(b)\int_{U_T}u^2e^{-\lambda t} \tag{5.30}$$

$$\leq \frac{C}{\theta}int_{U_T}e^{\lambda t}\theta|Dv|^2 + C\int_{U_T}\left(e^{-\lambda t}|u|^2 + e^{\lambda t}(|v|^2 + |Dv|^2)\right). \tag{5.31}$$

We did all this to ensure the two integrands would match up, so we can say (prefactor of A's lower bound minus prefactor of B's upper bound) times the integral is less than or equal to 0:

$$\left(\frac{\lambda}{2} - C\right)\int_{U_T}(u^2e^{-\lambda t} + \theta|Dv|^2e^{\lambda t} + v^2e^{\lambda t})\mathrm{d}x\mathrm{d}t \leq 0. \tag{5.32}$$

Pick $\lambda > 2C$ to get

$$\int_{U_T} e^{-\lambda t} u^2 \mathrm{d}x\mathrm{d}t = 0, \tag{5.33}$$

and this is the vanishing norm expression we wanted, so $u \equiv 0$ almost everywhere on U_T.

Theorem 5.2 (Existence of solutions to the hyperbolic IBVP). *Given $\psi_0 \in H_0^1(\Sigma_0), \psi_1 \in L^2(\Sigma_0), f \in L^2(U_T)$, then there exists a weak solution $u \in H^1(U_T)$ of the IBVP 5.1 with*

$$\|u\|_{H^1(U_T)} \leq C\Big(\|\psi_0\|_{H_1(U)} + \|\psi_1\|_{L^2(U)} + \|f\|_{L^2(U_T)}\Big). \tag{5.34}$$

Proof .
We'll use Galerkin's method, which is also used for parabolic equations (which is a bit easier than this) - see Evans for that application. The idea is we're going to project everything onto some finite-dimensional subspace of $H_0^1 \times L^2$, then we're going to get a uniform bound then argue that we have a weak solution by Banach-Alaoglu.
First, recall that the eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$ of the Laplacian $L = -\triangle$ with Dirichlet boundary conditions form an orthonormal basis of $L^2(U)$. We have that $\varphi_k \in H_0^1(U)$, and in fact by elliptic regularity, $\varphi_k \in C^{\infty}(U)$. Also recall that $(\varphi_k, \varphi_l)_{L^2(U)} = \delta_{kl}$.
Therefore, we can expand any $u \in L^2(U)$ in this basis:

$$u = \sum_{k\geq 1} (u, \varphi_k)_{L^2(U)} \varphi_k \tag{5.35}$$

with convergence in $L^2(U)$.
We're going to do a finite-dimensional approximation onto the span of N of these eigenfunctions. By density, we can consider our data to be smooth and compactly supported, and likewise for our sourcing term: $\psi_0, \psi_1 \in C_c^{\infty}(\Sigma_0), f \in C_c^{\infty}(U_T)$. Define an approximate solution with a finite number of terms by separation of variables, $u^N(x,t) = \sum_{k=1}^N u_k^N(t)\varphi_k(x)$. Later, we may drop the superscript N on each component function for simplicity.
Assume $u_k(t) \in C^2([0,T])$ and that $u^N(x,t)$ is a weak solution to the IBVP. We can figure out an ODE system that u_k must satisfy. Take as our test functional $v(x,t) = \rho(t)\varphi_l(x)$, where $\rho \in C_c^{\infty}((0,T))$ arbitrary. Put this into the test-function formulation of the IBVP to get

$$\int_{U_T} \big(-u_t^N \dot{\rho}\varphi_l + a^{ij}(u^N)_{x_j}(\varphi_l)_{x_i} + b^i(u^N)_{x_i}\rho\varphi_l + b(u^N)_t\rho\varphi_l + cu\rho\varphi_l - f\rho\varphi_l\big)\mathrm{d}x\mathrm{d}t = 0. \tag{5.36}$$

Here, the $\dot{\rho}$ indicates a time derivative. Every term carries a factor of ρ except the first one, so we integrate that by parts:

$$\int_{U_T} -(u^N)_t \dot{\rho}\varphi_l \mathrm{d}x\mathrm{d}t = \int_{U_T} (u^N)_{tt}\rho\varphi_l \mathrm{d}x\mathrm{d}t \tag{5.37}$$

If we put all the ρ dependent terms, including the one we just found using IBP, into a function G, we get $\int_0^T \int_U G(x)\rho(t)\mathrm{d}x\mathrm{d}t = 0$ for all ρ. Therefore $\int_U G(x)\mathrm{d}x = 0$, and expanding out what G is gives using

$$(u^N_{tt}, \varphi_l)_{L^2(U)} + \int_U (a^{ij}(u^N)_{x_j}(\varphi_l)_{x_i} + b^i(u^N)_{x_i}\varphi_l + b(u^N)_t\varphi_l + cu^N\varphi_l)\mathrm{d}x = (f, \varphi_L)_{L^2(U)}. \tag{5.38}$$

We're going to derive an ODE system from this equation, using orthonormality:

$$(u^N_{tt}, \varphi_l)_{L^2(U)} = \sum_{k=1}^N (\ddot{u}_k(t)\varphi_k, \varphi_l)_{L^2(U)} = \ddot{u}_l(t). \tag{5.39}$$

In this way we obtain the following ODE system, for $l = 1 \dots N$,

$$\ddot{u}_l(t) + \sum_{k=1}^N \alpha_{l,k}(t)u_k(t) + \beta_{l,k}(t)\dot{u}_k(t) = f_l(t) = \int_U f(x,t)\varphi_l(x)\mathrm{d}x, \tag{5.40}$$

where

$$\alpha_{l,k}(t) = \int_U (a^{ij}(\varphi_l)_{x_i}(\varphi_k)_{x_j} + b^i(\varphi_l)_{x_i}\varphi_k + c\varphi_l\varphi_k)\mathrm{d}x \tag{5.41}$$

$$\beta_{l,k}(t) = \int_U b(x,t)\varphi_l\varphi_k\mathrm{d}x. \tag{5.42}$$

This is a system of N second-order ODEs linear in u with coefficients that are uniformly bounded in $t \in [0,T]$, so by Picard-Lindelöf, there exists a unique solution $u_k \in C^2([0,T])$. Note that we can't always do the "stitching" argument that lets us go from a P-L solution in $[0,\epsilon]$ to one over an entire desired time interval, but we can here because we have uniformly bounded constants. Because we're taking finite combinations, we can show that $u^N \in H^1(U_T)$, $\partial_t u^N \in H^1(U_T)$.
Next, we want to use our *a priori* estimates to get $\left\|u^N\right\|_{H^1(U_T)} \le C$ for all N, so that we can use Banach-Alaoglu.
We take our expression for $(f,\varphi_l)_{L^2(U)}$ and multiply it by $e^{-\lambda t}\dot{u}_l(t)$, sum over $l = 1 \dots N$, and integrate over $[0,\tau]$ for $\tau \in [0,T]$. We find

$$\int_{U_T} \big((u^N)_{tt}(u^N)_t + a^{ij}(u^N)_{x_i}(u^N)_{tx_j} + b^i(u^N)_{x_i}(u^N)_t + b(u^N)_t^2 + cu^N(u^N)_t\big)e^{-\lambda t}\mathrm{d}x\mathrm{d}t \tag{5.43}$$

$$= \int_0^\tau \int_U f(u^N)_t e^{-\lambda t}\mathrm{d}x\mathrm{d}t. \tag{5.44}$$

We're going to rewrite this in a smart way.

$$\tilde{A} = \tilde{B} \tag{5.45}$$

$$\tilde{A} = \int_{U_T} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(Q_a e^{-\lambda t}\right)\mathrm{d}x\mathrm{d}t + \int_{U_T} \frac{\lambda}{2} Q_a e^{-\lambda t}\mathrm{d}x\mathrm{d}t \tag{5.46}$$

$$\tilde{B} = \int_{U_T} \left[\frac{1}{2}(a^{ij})_t (u^N)_{x_i}(u^N)_{x_j} - b^i (u^N)_{x_i}(u^N)_t - b(u^N)_t^2 + (1-c)u^N(u^N)_t + f(u^N)_t\right] e^{-\lambda t}\mathrm{d}x\mathrm{d}t, \tag{5.47}$$

where

$$Q_a = (u^N)_t^2 + a^{ij}(u^N)_{x_i}(u^N)_{x_j} + (u^N)^2. \tag{5.48}$$

Let $Q_\theta = (u^N)_t^2 + \theta\left|Du^N\right|^2 + (u^N)^2$. Using uniform ellipticity, Young's inequality and $e^{-\lambda t} \leq 1$, we can show that

$$\tilde{B} \leq C \int_{U_T} Q_\theta e^{-\lambda t}\mathrm{d}x\mathrm{d}t + \|f\|^2_{L^2(U_T)} \tag{5.49}$$

and that

$$\tilde{A} \geq \frac{e^{-\lambda\tau}2}{\int}{}_{\Sigma_T} Q_\theta \mathrm{d}x - \frac{1}{2}\int_{\Sigma_0} Q_a \mathrm{d}x + \frac{\lambda}{2}\int_{U_T} Q_\theta e^{-\lambda t}\mathrm{d}x\mathrm{d}t. \tag{5.50}$$

Therefore, for $\frac{\lambda}{2} - C \geq \frac{1}{2}$, we get

$$\begin{aligned} &e^{-\lambda\tau}\int_{\Sigma_\tau} Q_\theta \mathrm{d}x + \int_0^\tau \int_U Q_\theta e^{-\lambda t}\mathrm{d}x\mathrm{d}t \\ &\leq \int_{\Sigma_0} Q_a \mathrm{d}x + C\|f\|^2_{L^2(U_\tau)} \\ &\leq C\Big(\left\|u^N(\cdot,0)\right\|^2_{H^1(U)} + \left\|\dot{u}^N(\cdot,0)\right\|^2_{L^2(U)} + \|f\|^2_{L^2(U_T)}\Big). \end{aligned} \tag{5.51}$$

This is true for all $\tau \in [0,T]$, so we get

$$\begin{aligned} &\sup_{\tau\in[0,T]}\Big(\left\|u^N(\cdot,\tau)\right\|^2_{H^1(U)} + \left\|\dot{u}^N(\cdot,\tau)\right\|^2_{L^2(U)}\Big) + \left\|u^N\right\|^2_{H^1(U_\tau)} \\ &\leq Ce^{\lambda T}\Big(\left\|u^N(\cdot,0)\right\|^2_{H^1(U)} + \left\|\dot{u}^N(\cdot,0)\right\|^2_{L^2(U)}\Big) + \|f\|^2_{L^2(U_T)}, \end{aligned} \tag{5.52}$$

where the constant C is independent of N.

Since $u^N(0) = \sum_{k=1}^N (\varphi_0,\varphi_k)\varphi_k \xrightarrow{N} \psi_0$ in $H^1(U)$, we can put a norm bound: if $\psi_0 \neq 0$ then for large N, $\left\|u^N(0)\right\|_{H^1(U)} \leq 2\|\psi_0\|_{H^1(U)}$ If $\psi_0 = 0$, then $u^N(0) = 0$. Similarly, $\left\|\dot{u}^N\right\|_{L^2(U)} \leq 2\|\psi_1\|_{L^2(U)}$ Therefore, we can place a uniform bound in N on u^N:

$$\left\|u^N\right\|_{H^1(U_T)} \leq C\Big(\|\psi_0\|_{H^1(U)} + \|\psi_1\|_{L^2(U)} + \|f\|_{L^2(U_T)}\Big) = C_1. \tag{5.53}$$

Note that $u^N \in \tilde{H}^1(U_T) := \{\varphi \in H^1(U_T) \mid \varphi\big|_{\partial^* U_T} = 0\}$ is a closed subspace of $H^1(U_T)$. This implies weak compactness, so there exists a $(u^{N_i})_i \rightharpoonup u$ in $\tilde{H}^1(U_T)$ for some $u \in \tilde{H}^1(U_T)$. Also,

$$\|u\|_{H^1(U_T)} \leq \liminf_{i\to\infty} \left\|u^{N_i}\right\|_{H^1(U_T)} \leq C_1. \tag{5.54}$$

Now, we want to show that u is a weak solution to the original IBVP. Relabel $u^{N_i} \to u_N$ and fix m. Consider $v = \sum_{k=1}^m v_k(t)\varphi_k(x)$ with $v_k \in H^1((0,T)), v_k(T) = 0$. Note that v is a test function for the weak formulation. Consider the Galerkin system (4) for $N > m$. Multiply the kth equation in (4) by $v_k(t)$ and sum over $k = 1, 2, \ldots, N$, where we take $v_{m+1} = \cdots = v_N = 0$. This tells us

$$(u^N_{tt}, v)_{L^2(U)} + \int_{\Sigma_t} \left(a^{ij}(u^N)_{x_i} v_{x_i} + b^i (u^N)_{x_i} v + b u^N_t v + cuv\right) = (f, v)_{L^2(U)}. \tag{5.55}$$

Integrate over $[0, T]$, use IBP, and use $v(T) = 0$. This tells us

$$-\int_{\Sigma_0} \dot{u}^N v \mathrm{d}x + \int_{U_T} u^N_t v_t + a^{ij}(u^N)_{x_i} v_{x_j} + b^i(u^N)_{x_i} v + b u^N_t v + c u^N v \mathrm{d}x\mathrm{d}t = \int_{U_T} f v \mathrm{d}x\mathrm{d}t. \tag{5.56}$$

Since $N > m$, we can show using Parseval's theorem that $\int_{\Sigma_0} (u^N)_t v \mathrm{d}x = \int_{\Sigma_0} \psi_1 v \mathrm{d}x$. Passing to the weak limit in $H^1(U_T)$, we get

$$\int_{\Sigma_0} -\psi_1 v \mathrm{d}x + \int_{U_T} (-u_t v_t + a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + b u_t v + cuv) \mathrm{d}x\mathrm{d}t = \int_{U_T} f v \mathrm{d}x\mathrm{d}t. \tag{5.57}$$

Therefore, for the specific choice of vs we made above, u satisfies the weak solution condition. It still remains to check some more stuff.
We have that $u\big|_{\partial^* U_T} = 0$ by $u \in \tilde{H}^1(U_T)$, but we still need to check that $u\big|_{\Sigma_0} = \psi_0$. For each fixed $k = 1, \ldots, m$, we can show that the map $\Phi_k : H^1(U_T) \to \mathbb{R}, \omega \mapsto \int_{\Sigma_0} \omega\varphi_k \mathrm{d}x$ is a functional. (what's the definition being used here?)
Brief proof of this using Cauchy-Schwarz on the integral $|\Phi_k(\omega)| \leq \|\omega\|_{L^2(\Sigma_0)} \|\varphi_k\|_{L^2(\Sigma_0)} \leq \|\mathrm{Tr}(\omega)\|_{L^2(\partial U_T)}$, and by the trace theorem, this is bounded by $c\|w\|_{H^1(U_T)}$.
By weak convergence, $\Phi_k(u^N) \to \Phi_k(u)$, so $\int_{\Sigma_0} \psi_0 \varphi_k \mathrm{d}x = \int_{\Sigma_0} u^N(x,0)\varphi_k(x)\mathrm{d}x \to \int u(x,0)\varphi_k \mathrm{d}x$. Then

$$\int_{\Sigma_0} (\psi_0 - u(x,0))\varphi_k \mathrm{d}x = 0 \;\forall k, \tag{5.58}$$

but m is arbitrary, so $u = \psi_0$ on Σ_0.
Now, the linear space $\{v \in \sum_{k=1}^m v_k(t)\varphi_k(x), v_k \in H^1((0,T)), v_k(T) = 0\}$ is dense in $\tilde{H}^1(U_T)$ (exercise), so u is a weak solution as the equation holds for all v.

Definition 5.3. *If X is a Banach space, then we denote*

$$\begin{cases} \|u\|_{L^p((0,T);X)} := \left(\int_0^T \|u\|_X^p \mathrm{d}t\right)^{1/p} & 1 \le p < \infty \\ \operatorname{ess\,sup}_{t\in(0,T)} \|u(t)\|_X & p = \infty, \end{cases} \tag{5.59}$$

and $L^p((0,T);X) := \{u : (0,T) \to X \mid \|u\|_{L^p((0,T),X)} < \infty\}$.

In the proof, we showed $\|u\|_{H^1(U_T)} \le C_1$. In fact, the weak solution satisfies

$$\|u\|_{L^\infty((0,T),H^1(U))} + \|u_t\|_{L^\infty((0,T),L^2(U))} \le C_1. \tag{5.60}$$

5.3 Finite Speed of Propagation

A crucial feature of hyperbolic equations is that information can only travel at a finite speed.

Definition 5.4. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a zero-set of some function F. That is, $\Sigma = \{(x,t) \mid F(x,t) = 0\}$. Define $w(F_{x_1}, \ldots, F_{x_n}, F_t) = (F_t)^2 - \sum_{i,j=1}^n a^{ij} F_{x_i} F_{x_j}$. We say Σ is spacelike if $w > 0$, timelike if $w < 0$, and characteristic (null if we were in a GR class) if $w = 0$.*

Example 5.22.

1. The plane $t = 0$ is spacelike.
2. The cylinder $F = |x - x_0|^2 - R^2$ is timelike.
3. Let $S_0 \subset U$ be an open set with smooth boundary. Let $\tau : S_0 \to (0,T)$ be a smooth function such that $\tau\big|_{\partial S_0} = 0$. Let S^1 be the graph of τ. Then $F(x_1, \ldots, x_n, t) = t - \tau(x)$, and so S^1 is spacelike if $1 - \sum a^{ij}\tau_{x_i}\tau_{x_j} > 0 \iff \sum_{i,j=1}^n a^{ij}(x)\tau_{x_i}\tau_{x_j} < 1$ for all $x \in S_0$. Let $D = \{(x,t) \in U_T \mid x \in S_0, 0 < t < \tau(x)\}$. Some sketch.

□

Theorem 5.3 (Domain of dependence)**.** *If S^1 is spacelike and and u is a weak solution to (2), then $u\big|_D$ depends only on the values of ψ_0, $\psi_1\big|_{S_0}$, $f\big|_D$.*

Proof .
We return to the uniqueness proof (of Theorem 5.1). By linearity it is sufficient to prove that $u\big|_D = 0$ if $\psi_0\big|_{S_0} = \psi_1\big|_{S_0} = f\big|_D = 0$. Take the test function

$$v = \int_t^{\tau(x)} e^{-\lambda s} u(x,s)\mathrm{d}s, (x,t) \in D, \tag{5.61}$$

and set $v = 0$ otherwise. As an exercise, we can show that $v \in H^1(U_T)$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$

and

$$\begin{aligned} v_{x_i} &= \tau_{x_i} e^{-\lambda\tau(x)} u(x,\tau(x)) + \int_t^{\tau(x)} e^{-\lambda s} u_{x_i}(x,s)\mathrm{d}s \quad \text{in} \quad D \\ v_t &= -e^{-\lambda t} u(x,t) \quad \text{in} \quad D \end{aligned} \tag{5.62}$$

and otherwise $v_{x_i} = v_t = 0$ on $U_T \setminus D$.
Inserting this into the definition of a weak solution, we obtain

$$\begin{aligned} &\int_D \frac{1}{2}\frac{\partial}{\partial t}\big(u^2 e^{-\lambda t} - a^{ij} v_{x_i} v_{x_j} e^{\lambda t} - v^2 e^{\lambda t}\big)\mathrm{d}x\mathrm{d}t + \frac{\lambda}{2}\int_D \big(u^2 e^{-\lambda t} + a^{ij} v_{x_i} v_{x_j} e^{\lambda t} + v^2 e^{\lambda t}\big)\mathrm{d}x\mathrm{d}t \\ &= \int_D \left(-\frac{1}{2}(a^{ij})_t v_{x_i} v_{x_j} e^{\lambda t} + (b^i_{x_i} + b_t + 1 - c)uv + b^i v_{x_i} u + buv_t\right)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{5.63}$$

By Fubini's theorem, $\int_D \cdot \mathrm{d}x\mathrm{d}t = \int_{S_0} \mathrm{d}x\Big(\int_0^{\tau(x)} \mathrm{d}t\Big)$, so we can integrate the $\frac{\partial}{\partial t}$ term. We also use $v\big|_{S'} = 0$ and $v_{x_i}\big|_{S'} = \tau_{x_j} u(x,\tau(x)) e^{-\lambda\tau(x)}$.

$$\overline{A} = \frac{1}{2}\int_{S_0} u^2(x,\tau(x)) e^{-\lambda\tau(x)}(1 - a^{ij}\tau_{x_i}\tau_{x_j})\mathrm{d}x + \frac{1}{2}\int_{S_0} (a^{ij} v_{x_i} v_{x_j} + v^2)\Big|_{t=0} \mathrm{d}x \tag{5.64}$$

and each individual term is ≥ 0; the $1 - a^{ij}\tau_{x_i}\tau_{x_j}$ is nonnegative because S' is spacelike, and $a^{ij} v_{x_i} v_{x_j} \geq 0$ by uniform ellipticity.
As in the proof of uniqueness, we obtain

$$\left(\frac{\lambda}{2} - C\right)\int_D \Big(u^2 e^{-\lambda t} + \theta |Du|^2 e^{\lambda t} + v^2 e^{\lambda t}\Big)\mathrm{d}x\mathrm{d}t \leq 0. \tag{5.65}$$

So by taking λ sufficiently large, we conclude $u\big|_D = 0$.

This implies that no signal can travel faster than a certain speed. For instance, let $x_0 \in U$ and let S_0 be some ball about x_0. The data outside S_0 does not determine $u(x_0,t)$ provided $(x_0,t) \in D$, i.e. $0 \leq t \leq \tau(x_0)$. Only after $t \geq \tau(x_0)$ will the solution $u(x,t)$ be determined by data outside S_0.

Everything is local in a hyperbolic PDE.

5.4 Hyperbolic regularity

So far we have established the existence of a weak solution to Equation 5.1. provided $\psi_0 \in H^1_0(U), \psi_1 \in L^2(U), f \in L^2(U_T)$. Moreover,

$$\|u_t\|_{L^\infty((0,T);L^2(U))} + \|u\|_{L^\infty((0,T);H^1(U))} + \|u\|_{H^1(U_T)} \leq C\Big(\|\psi_0\|_{H^1(\Sigma_0)} + \|\psi_1\|_{L^2(\Sigma_0)} + \|f\|_{L^2(U_T)}\Big). \tag{5.66}$$

We have no gain in x regularity. We want to improve the regularity of u and bump it up to $H^2(U_T)$ if possible.

Example 5.23. This is a motivating example for improving regularity. Let $L = -\triangle$ in the usual hyperbolic IBVP and suppose $u \in C^\infty(U_T)$ solves it. Set $w = u_t$. Then w is a solution to

$$\begin{cases} w_t t - \triangle w = 0 & U_T \\ w = \psi_1, w_t = \triangle\psi_0 & \Sigma_0 \\ w = 0 & \partial^* U_T \end{cases} \tag{5.67}$$

where the second boundary condition comes from $w_t = u_t t = \Delta u = \Delta\psi_0$. The bound on the solution from the data translates to

$$\|w\|_{L^\infty((0,T);H^1(U))} + \|w\|_{L^\infty((0,T);L^2(U))} \leq C\Big(\|\psi_1\|_{H^1(U)} + \|\triangle\psi_0\|_{L^2(U)} + \|f\|_{H^1(U_T)}\Big). \tag{5.68}$$

This gives us control on u_{tt}, u_{tx_i} in $L^2(U)$ in terms of the initial data. To recover the $u_{x_i x_i}$ derivative, we note that for every $\Sigma_t = U \times \{t\}$ we have $\triangle u = u_{tt}$ and $u = 0$ on $\partial\Sigma_t$. By elliptic regularity with $f = u_{tt}$, we get

$$\|u\|_{H^2(U)} \leq C\|f\|_{L^2(\Sigma_t)} = C\|u_{tt}\|_{L^2(\Sigma_t)} \tag{5.69}$$

(I think the f is correct but ni the notes it's a u, double check)

This implies

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{H^2(U)} \leq C\Big(\|\psi_1\|_{H^1(U)} + \|\psi_0\|_{H^2(U)}\Big) \tag{5.70}$$

(exercise, repeat with source term f, Evans 7.2.3) $\square$

Theorem 5.4 (Hyperbolic regularity). *Suppose $a^{ij}, b^i, b, c \in C^2(\overline{U}_T)$, $\partial U \in C^2$. Then for*

$$\psi_0 \in H^2(U) \cap H_0^1(U), \psi_1 \in H_0^1(U), f, f_t \in L^2(U_T)$$

the unique weak solution to the IBVP in fact satisfies

$$\begin{aligned} u &\in H^2(U_T) \cap L^\infty((0,T); H^2(U)) \\ u_t &\in L^\infty((0,T); H_0^1(U)) \\ u_{tt} &\in L^\infty((0,T); L^2(U)). \end{aligned}$$

Proof .
By approximation we assume $f \in C^\infty(\overline{U}_T)$, and that ψ_0, ψ_1 are compactly supported in U. Returning to the Galerkin approximation, if $u^N(x,t) = \sum_k u_k(t)\varphi_k(x)$ solves the hyperbolic initial boundary value problem where φ_k are eigenfunctions of the Dirichlet Laplacian, then

$$(u^N_{tt}, \varphi_k)_{L^2(U)} = -\int_U (a^{ij}(u_N)_{x_i}(\varphi_u)_{x_j} + b_i(u^N)_{x_i}\varphi_k + b(u^N)_t\varphi_k + cu^N\varphi_k)\mathrm{d}x + (f, \varphi_k)_{L^2(U)}$$

for $k = 1, \dots, N$.
This is a linear second-order ODE with coefficients $C^2([0,T])$, so $u^N \in C^3([0,T])$. As in the proof for existence, for λ large we have

$$\begin{aligned}&\sup_{t\in[0,T]} \Big(\big\|u^N(\cdot,t)\big\|^2_{H^1(U)} + \big\|(u^N)_t(\cdot,t)\big\|^2_{L^2(U)}\Big) + \big\|u^N\big\|^2_{H^1(U_T)} \\ &\leq e^{\lambda T}C\Big(\|\psi_0\|^2_{H^1(\Sigma_0)} + \|\psi_1\|^2_{L^2(\Sigma_0)} + \|f\|^2_{L^2(U_T)}\Big).\end{aligned} \tag{5.71}$$

Now, since u^N is C^3, we can differentiate Equation 5.71 with respect to t:

$$\begin{aligned}&(u^N_{ttt}, \varphi_u)_{L^2(U)} + \int_{\Sigma_t} \big(a^{ij}(u^N)_{tx_i}(\varphi_k)_{x_j} + b^i(u^N)_{tx_i}\varphi_k + b(u^N)_{tt}\varphi_k + c(u^N)_t\varphi_k\big)\mathrm{d}x \\ &= (f_t, \varphi_k)_{L^2(U)} - \int_{\Sigma_t} \Big((a^{ij})_t(u^N)_{x_i}(\varphi_k)_{x_j} + \dot{b}^i(u^N)_{x_i}\varphi_k + \dot{b}(u^N)_t\varphi_k + \cdot cu^n\varphi_k\Big)\mathrm{d}x.\end{aligned} \tag{5.72}$$

Multiply this expression by $\ddot{u}_k e^{-\lambda t}$, sum over $k = 1, \dots, N$, and integrate over $\int_0^\tau \mathrm{d}t$ for $t \in [0,T]$, and pick λ large to get

$$\sup_{t\in[0,T]} \Big(\big\|(u^N)_t(\cdot,t)\big\|^2_{H^1(U)} + \big\|(u^N)_{tt}(\cdot,t)\big\|^2_{L^2(U)}\Big) + \big\|u^N\big\|^2_{H^1(U_T)} \tag{5.73}$$

$$\leq e^{\lambda T}C\left(\|\psi_0\|^2_{H^1(\Sigma_0)} + \|\psi_1\|^2_{L^2(\Sigma_0)} + \|f\|^2_{L^2(U_T)} + \Bigg\|\underbrace{u^N_t}_{a}\Bigg\|^2_{H^1(\Sigma_0)} + \Bigg\|\underbrace{(u^N)_{tt}}_{b}\Bigg\|^2_{L^2(\Sigma_0)} + \|f_t\|^2_{L^2(U_T)}\right) \tag{5.74}$$

We control a using a basis φ_k of L^2:

$$u^N_t\Big|_{t=0} = \sum(\psi_1, \varphi_k)_{L^2(\Sigma_0)}\varphi_k \implies \big\|u^N_t\big\|_{H^1(\Sigma_0)} \leq \|\psi_1\|_{H^1(\Sigma_0)}. \tag{5.75}$$

For b, we use Equation 5.71 on the slice $t = 0$. Multiply by $\ddot{u}_k$ and sum over $k = 1, \dots, N$ (note no time integral) to get

$$
\begin{aligned}
\left\|(u^N)_{tt}\right\|^2_{L^2(\Sigma_0)} &= -\int_{\Sigma_0} \left(a^{ij}(u^N)_{x_i}(u^N)_{ttx_j} + b^i(u^N)_{x_i}(u^N)_{tt} + b(u^N)_t(u^N)_{tt} + cu^N(u^N)_{tt}\right)\mathrm{d}x \\
&\quad + (f,(u^N)_{tt})_{L^2(\Sigma_0)} \\
&\underbrace{=}_{\text{IBP}} \int_{\Sigma_0} \left(-(a^{ij}(u^N)_{x_j})_{x_i}(u^N)_{tt} + b^i(u^N)_{x_i}(u^N)_{tt} + b(u^N)_t(u^N)_{tt} + cu^N(u^N)_{tt}\right)\mathrm{d}x \\
&\quad + (f,(u^N)_{tt})_{L^2(\Sigma_0)}
\end{aligned} \tag{5.76}
$$

Using C-S, we get

$$
\left\|(u^N)_{tt}\right\|_{L^2(\Sigma_0)} \leq C\Big(\left\|u^N\right\|_{H^2(\Sigma_0)} + \left\|u^N_t\right\|_{L^2(\Sigma_0)} + \|f\|_{L^2(\Sigma_0)}\Big) \tag{5.77}
$$

and we can further show that $\|f\|_{L^2(\Sigma_0)} \leq \|f\|_{L^2(U_T)} + \|f_t\|_{L^2(U_T)}$.
From here, our goal is to control $\left\|(u^N)_{tt}\right\|_{L^2(\Sigma_0)}$ uniformly in N. It remains to prove we can control $\left\|u^N\right\|_{H^2(\Sigma_0)}$ uniformly in N. Note that $\triangle$ is self-adjoint with respect to $L^2(\Sigma_0)$, so we take the inner product,

$$
\left(\triangle u^N, \triangle u^N\right)_{L^2(\Sigma_0)} = \left(u^N \triangle^2 u^N\right)_{L^2(\Sigma_0)} = \left(\psi_0, \triangle^2 u^N\right)_{L^2(\Sigma_0)} = \left(\triangle\psi_0, \triangle u^N\right)_{L^2(\Sigma_0)}, \tag{5.78}
$$

where the relation to ψ_0 comes in from $\triangle\varphi_k\big|_{\partial U} = 0$ and u^N being a finite sum of φ_ks.
By Cauchy-Schwarz,

$$
\left\|\triangle u^N\right\|^2_{L^2(\Sigma_0)} \leq \|\triangle\psi_0\|_{L^2(\Sigma_0)}\left\|\triangle u^N\right\|_{L^2(\Sigma_0)} \tag{5.79}
$$

$$
\left\|\triangle u^N\right\|_{L^2(\Sigma_0)} \leq \|\triangle\psi_0\|_{L^2(\Sigma_0)} \leq \|\psi_0\|_{H^2(\Sigma_0)}. \tag{5.80}
$$

By elliptic estimates, this gives us $\left\|u^N\right\|_{H^2(\Sigma_0)} \leq c\|\psi_0\|_{H^2(\Sigma_0)}$.
Finally, we bring this all together.

$$
\left\|(u^N)_t\right\|_{L^\infty((0,T);H^1(U))} + \left\|(u^N)_{tt}\right\|_{L^\infty((0,T);L^2(U))} + \left\|u^N\right\|_{H^1(U_T)} \tag{5.81}
$$

$$
\leq C\Big(\|\psi_0\|_{H^1(\Sigma_0)} + \|psi_1\|_{L^2(\Sigma_0)} + \|f\|_{L^2(U_T)} + \left\|u^N_t\right\|_{H^1(\Sigma_0)} + \left\|(u^N)_{tt}\right\|_{L^2(\Sigma_0)} + \|f_t\|_{L^2(U_T)}\Big) \tag{5.82}
$$

$$
\leq C\Big(\|\psi_0\|_{H^2(\Sigma_0)} + \|\psi_1\|_{H^1(\Sigma_0)} + \|f\|_{L^2(U_T)} + \|f_t\|_{L^2(U_T)}\Big) \tag{5.83}
$$

The RHS is independent of N, so by Banach-Alaoglu we can pass to a subsequence to get

$$
\begin{aligned}
u_t &\in H^1(U_T) \\
u_t &\in L^\infty((0,T);H^1_0(U)) \\
u_{tt} &\in L^\infty((0,T);H^2(U)).
\end{aligned}
$$

Finally, we want something about spatial derivatives. $u_{tt} + Lu = f$ for almost all times. So $u(\tau)$ is a weak solution of

$$-(a^{ij}u_{x_i})_{x_j} = -f - u_{tt} - b^i u_{x_i} - bu_t - cu =: \tilde{f} \tag{5.84}$$

in Σ_τ with $u = 0$ on $\partial\Sigma_\tau$.
Now $\tilde{f} \in L^2(U)$ implies $u(\tau) \in H^2(U)$ with

$$\begin{aligned} \|u(\tau)\|^2_{H^2(U)} &\leq C\left\|\tilde{f}\right\|^2_{L^2(U)} \\ &\leq C\Big(\|\psi_0\|^2_{H^2(U)} + \|\psi_1\|^2_{H^1(U)} + \|f\|^2_{L^2(U_T)} + \|f_t\|^2_{L^2(U_T)}\Big) \end{aligned} \tag{5.85}$$

and so $u \in L^\infty((0,T); H^2(U))$.

We study the solutions of

$$\begin{cases} u_t + \sum_{j=1}^n B_j u_j = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{5.86}$$

where $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^m, (x,t) \mapsto (u_1, u_2, \ldots, u_m)$.

B_j, f vary with the point (x,t); we have as givens $B_j : \mathbb{R}^m \times [0,\infty) \to M(m)$ ($m \times m$ matrices), $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^m$, and as data $g : \mathbb{R}^n \to \mathbb{R}^m$.

So far we haven't introduced hyperbolicity at all.

Definition 5.5. *We say the system 5.86 is hyperbolic if $\tilde{B}(x,t;y) = \sum_{j=1}^n y_j B_j$ is diagonalizable for all $x, y, t \geq 0$.*

There exist eigenvalues and eigenvectors spanning $\mathbb{R}^m$. It may be useful to assume the B_j are symmetric without loss of generality, but we won't need to do this for now.

So why have we done this setup?

Assume $f \equiv 0$ and B_j are constant, and look for plane wave solutions, i.e. $u(x,t) = v(y \cdot x - \sigma t)$ for some v, y, σ. This tells us

$$(-\sigma I + \tilde{B}(y))v' = 0, \tag{5.87}$$

so if hyperbolicity holds, we have

$$(y \cdot x - \lambda_k(y)t)\gamma_k(y) = 0 \; \forall k \in \{1, \ldots, m\}, \tag{5.88}$$

i.e. an eigenvector equation.

Another piece of motivation is given by the following exercise.

Exercise 5.5. *Suppose $v_{tt} - a^{ij}v_{x_i x_j} = 0$, where $a^{ij} = a^{ji}$. Write $u = (u_1, \ldots, u_m) = (v_{x_1}, \ldots, v_{x_{m-1}}, v_t)$ as a system of the form $B_0 u_t + \sum_{j=1}^n B_j u_j = 0$ where B_0 is positive definite.*

Finally, we can look at the Einstein field equations as a first-order system of hyperbolic PDEs, using the ADM formalism.

Recall that for well-posedness, we need to show existence, uniqueness, continuous dependence on the data, and regularity. We'll first look at existence in the case $f \equiv 0$ and B_j is constant.

Theorem 5.6. *Suppose $g \in H^s(\mathbb{R}^n; \mathbb{R}^m)$ for some $s > \frac{n}{2} + m$. Then there exists a unique solution $u \in C^1([0,\infty), \mathbb{R}^m)$ of 5.86.*

Proof .
Assume $u = (u_1, \ldots, u_m)$ is smooth, and $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_m)$ is its spatial Fourier transform. We have

$$\hat{u}_t + i \sum_{j=1}^{n} y_j B_j \hat{u} = 0. \tag{5.89}$$

We can then solve the PDE for fixed y:

$$\hat{u}(y,t) = e^{-itB(y)} \hat{g}(y), \tag{5.90}$$

and we can show this implies

$$U(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot y} e^{-itB(y)} \hat{g}(y) \mathrm{d}y. \tag{5.91}$$

This is actually a solution of the original PDE. Fix $y \in \mathbb{R}^n$, and say all the eigenvalues have a maximum absolute value r. They then all lie within $B(0,r)$. Then

$$e^{-itB(y)} = \frac{1}{2\pi i} \overbrace{\int_{\partial B(0,r)} e^{-itz} (zI - B(y))^{-1} \mathrm{d}z}^{A(y,t)}. \tag{5.92}$$

The proof of the above is as follows. Fix $x \in \mathbb{R}^n$.

$$\begin{aligned} B(y)A(y,t)x &= \frac{1}{2\pi i} \int_{\partial B(0,r)} e^{-itz} B(y)(zI - B(y))^{-1} x \mathrm{d}z &(5.93)\\ &= \frac{1}{2\pi i} \int_{\partial B(0,r)} e^{-itz} \left[z(zI - B(y))^{-1} x - x \right] \mathrm{d}z &(5.94)\\ &= -\frac{1}{i} \frac{\mathrm{d}}{\mathrm{d}t} A(t,y)x. &(5.95) \end{aligned}$$

This is a first-order differential equation:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + iB(y) \right) A(t,y) = 0. \tag{5.96}$$

We get

$$A(0,y)x = \frac{1}{2\pi i}\int (zI - B(y))^{-1}x\mathrm{d}z \tag{5.97}$$
$$= \frac{1}{2\pi i}\int z^{-1}[x + B(y)(zI - B(y))^{-1}x]\mathrm{d}z \tag{5.98}$$
$$= x + \frac{1}{2\pi i}\int B(y)(zI - B(y))^{-1}x\frac{\mathrm{d}z}{z} \to 0. \tag{5.99}$$

Now, define a new region Δ to be the union of small open balls encircling each eigenvalue, and change the region of integration to this. We can bound the integral over this region by bounding the two parts of the integrand:

$$\left\|(zI - B(y))^{-1}\right\| \leq \left\|\frac{\operatorname{cof}(zI - B)^{\intercal}}{|(| zI - B)}\right\| \leq C(1 + |z|^{m-1} + \|B(y)\|^{m-1}) \leq C(1 + |y|^{m-1}), \tag{5.100}$$

and

$$\left\|e^{-itB(y)}\right\| \leq Ce^{t}(1 + |y|^{m-1}). \tag{5.101}$$

Now, $g \in H^s(\mathbb{R}^n, \mathbb{R}^m)$ implies there exists $f \in L^2(\mathbb{R}^n, \mathbb{R}^m)$ such that $|\hat{g}(y)| \leq C(1 + |y|^s)^{-1}|f(y)|$. Putting these together, we get

$$\int_{\mathbb{R}^n} \left|e^{ix\cdot y}e^{-itB(y)}\hat{g}(y)\right| \leq C\|f\|_{L^2}\left(\int_{\mathbb{R}^n} \frac{\mathrm{d}y}{1 + |y|^{2(s-m+1)}}\right)^{1/2}. \tag{5.102}$$