Notes for Physics 105: Analytic Mechanics
UC Berkeley, Spring 2019

Aditya Sengupta

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Contents

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 1: Introduction and Motivation

Lecturer: Stuart Bale, Ivan Vasko *22 January* *Aditya Sengupta*

Note: *LaTeX format adapted from template courtesy of UC Berkeley EECS dept.*

1.1 Introduction

This is a course on classical mechanics. Classical mechanics means a system is not quantum and non-relativistic. We can determine whether a system is classical or quantum based on length scales; if the de Broglie wavelength of, for example, a gas is much less than the typical length scale, classical mechanics are okay. We can similarly determine whether or not a system is relativistic based on characteristic time scales. For example, muons have a relativistic effect, through which their lifetime is much longer in the reference frame of the observer than in the muon's:

$$\tau \approx \frac{\tau_\mu}{\sqrt{1 - v^2/c^2}} \tag{1.1}$$

In classical mechanics, time is absolute.

The governing equation of classical mechanics is Newton's second law,

$$m\ddot{\vec{r}} = \vec{F} \tag{1.2}$$

Classical mechanics is a good approximation to quantum and relativistic mechanics. Two major formulations of classical mechanics are currently studied; that of Lagrange, and that of Hamilton. There are actually still many open questions in classical mechanics, such as the mechanics of particles in which small approximations (for example, a particle in an elliptical magnetic field with a radius not much smaller than the radius of curvature of the ellipse) do not hold. This kind of problem deals with *deterministic chaos*.

1.2 Math Review

The high-level view of classical mechanics tells us that we can see systems in 3D space and in absolute time. This means the study of orthogonal transformations, scalars, and vectors will be useful. Usually, 3D space is parameterized by a Cartesian coordinate system, in which a vector to any point in space is given by

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = (x, y, z) \tag{1.3}$$

Suppose we wanted to describe this in some other coordinate system (x', y', z'). Now, the same point has a different expansion for its position vector,

$$\vec{r'} = x'\hat{x}' + y'\hat{y}' + z'\hat{z}' = (x', y', z') \tag{1.4}$$

In more generality, we can say that $\vec{r} = \sum x_i \hat{e}_i$ and $\vec{r'} = \sum x'_i \hat{e}'_i$.

To transform between coordinates, note that for all e'_i,

$$e'_i = \sum_j \lambda_{ij} e_j \tag{1.5}$$

Also, we note that the basis vectors are assumed to be orthogonal, that is, for all i, k:

$$e'_i e'_k = \delta_{ik} \tag{1.6}$$

meaning that when $i = k$, $\vec{e'_i} \cdot \vec{e'_k} = 1$ and otherwise it is 0.

We can use the λ_{ij} coefficients to define a coordinate transformation matrix:

$$\lambda_{ij} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \tag{1.7}$$

For an orthonormal basis, we know that

$$\lambda \cdot \lambda^\dagger = 1 \tag{1.8}$$

for a matrix λ. Also, conventionally, we say that $\lambda^\dagger_{ij} = \lambda_{ji}$

1.2.1 Change of Coordinates

By the definition of the change of coordinates matrix, we can say

$$x'_i \lambda_{ij} = x_j \tag{1.9}$$

We right multiply by λ_{js} (s is just a different index) to get

$$x'_i \lambda_{ij} \lambda_{js} = x_j \lambda_{js} \tag{1.10}$$

$$x_i \delta_{is} = x_j \lambda_{js} \implies x'_s = \lambda_{js} x_j \tag{1.11}$$

Here, we used the property of orthonormality, which suggests that vector length should be preserved. We can confirm this.

1.2.2 Physical Meaning

We have the property that

$$\lambda\lambda^\dagger = \hat{1} \tag{1.12}$$

which can also be written as

$$\lambda_{ij}\lambda_{sj} = \delta_{is} \tag{1.13}$$

Here, we have six equations and nine elements of each matrix, which means there are three unconstrained parameters. It turns out that these are the angles betwen the old and new coordinates.

1.2.3 A Particular Example

Suppose we have a coordinate system (x, y, z) that we rotate around the z axis by some angle ϕ. Since there is no rotation of the z axis, we know that $z' = z$. Trigonometry tells us that $x' = x\cos\phi + y\sin\phi$ and $y' = -x\sin\phi + y\cos\phi$, therefore the matrix is

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1.14}$$

Suppose we do a second coordinate transformation on these coordinates, where we rotate by π about the x' axis.

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{1.15}$$

We can chain together coordinate transformations by multiplying matrices. Suppose there is a transformation λ from S to S', and w from S' to S''. For each coordinate, we can see that

$$x' = \hat{\lambda}x \tag{1.16}$$
$$x'' = \hat{w}x' = w\lambda x \tag{1.17}$$

To check that this is a valid matrix, we verify that it has a unit determinant:

$$w\lambda \cdot (w\lambda)^\dagger = w\lambda \cdot \lambda^\dagger w^\dagger = ww^\dagger = \hat{1} \tag{1.18}$$

Note that here we used the transpose rule $(A \cdot B)^\dagger = B^\dagger A^\dagger$. The set of orthogonal transformations is a group (?).

To check the commutativity of these transformations, we can carry out the transformations in both orders and compare. We get

$$S'' = wS' = w\lambda S \tag{1.19}$$
$$S'' = \lambda S' = \lambda w S \tag{1.20}$$

It turns out that $\lambda \cdot w \neq w \cdot \lambda$. For example, consider two rotation matrices, one about the x axis and the other about the z axis, with the z rotation being applied first:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{1.21}$$

and with the x rotation being applied first:

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \tag{1.22}$$

which can also be seen graphically.

1.2.4 Determinant of a Rotation Matrix

We know that for a transformation to be orthogonal, it must have the property

$$\lambda\lambda^\dagger = \hat{1} \tag{1.23}$$
$$\det(\lambda\lambda^\dagger) = 1 \implies |\,|(|\,\lambda)|^2 = 1 \implies |\lambda| = \pm 1 \tag{1.24}$$

Regular rotations usually have a determinant of $+1$, and inversions have a determinant of -1. for example, the coordinate transformation in which the x coordinate is flipped. The matrix is

$$\lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.25}$$

which has a -1 determinant. If the determinant is -1, we know that an inversion has happened.

1.2.5 Scalar and Vector Fields

A scalar field is one with a scalar output, such as the temperature in a space $T(M) = T(x_i)$ or $T(M) = T(x_i')$ where $x_i' = \lambda_{ij}x_j$. Scalar fields are invariant with respect to (inversions and) rotations. A vector field has a vector output, such as a velocity field which is characterized by coordinates (v_x, v_y, v_z). Vector fields are not invariant with respect to rotations. For example, coordinates $(v_{x'}, v_{y'}, v_{z'})$ would not be numerically the same vector as the velocity vector at the same point in a different coordinate system. We can translate between these velocities with the same change of basis rule:

$$v_{i'} = \lambda_{ij}v_j \tag{1.26}$$

We can apply these in physics to identify whether a quantity is a scalar or a vector, or just unphysical. For example, if an equation gives a vector output that does not follow the invariance rule, it is considered unphysical.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 2: Angular Velocity and Rigid Body Motion

Lecturer: Stuart Bale, Ivan Vasko *24 January* *Aditya Sengupta*

Lecturer: Ivan Vasko

2.1 Coordinate Transformations

Consider a coordinate transformation $\Lambda : S \to S', \hat{e}_i \to \hat{e}'_i$. This is characterized by a matrix, so that $\hat{e}'_i = \Lambda_{ij}\hat{e}_i$. The set of all transformations is a group, which means for any transformations Λ, W, ΛW is an orthogonal transformation, and any orthogonal transformations satisfy $\Lambda\Lambda^\dagger = 1$. This is a noncommutative group, which means $\Lambda W \neq W\Lambda$ (in general, $\Lambda_{ij}W_{jk} \neq W_{ij}\Lambda_{jk}$.)

We know that

$$\det\Lambda\det\Lambda^\dagger = 1 \tag{2.1}$$

$$|\det\Lambda| = \pm 1 \tag{2.2}$$

We claim that all transformations with determinant Λ are rotations, that is, the new coordinate system can be arrived at from the original coordinate system by rotation about some axis. If this is the case, then a zero-rotation transformation is possible:

$$\exists\vec{x}_\lambda : \Lambda\vec{x}_\lambda = \vec{x}_\lambda \tag{2.3}$$

If this is the case, then Λ has eigenvalue 1 (Euler's rotation theorem), which can be shown as follows:

$$(\Lambda - I)\Lambda^\dagger = I - \Lambda^\dagger = (I - \Lambda)^\dagger \tag{2.4}$$

Then, we take a determinant:

$$\left|\Lambda - I\right| = \left|\Lambda - I\right| \cdot \left|\Lambda\right| = \left|(I - \Lambda)^\dagger\right| = \left|I - \Lambda\right| \implies \left|\Lambda - I\right| = 0 \implies |\left|\Lambda\right|| = 1 \tag{2.5}$$

Consider the transformations with determinant -1. These have an odd number of inversions, in which an axis is flipped. For example, a transformation sending $(x, y, z) \to (-x, -y, -z)$ has the matrix representation

$$\begin{bmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{bmatrix} \tag{2.6}$$

This has an odd number of inversions (3) and a determinant of -1.

We claim that any Λ whose determinant is -1 is a combination of rotations and inversions. Consider P, the above inversion matrix (and note that $P^2 = I$ which can be shown explicitly by $P_{ij}P_{jk} = \delta_{ik}$, and Λ, an arbitrary inversion-rotation matrix. Let

$$W = P \cdot \Lambda \tag{2.7}$$

We see that W is a rotation matrix, as it has a determinant of 1. Then, left multiply by P to get $P \cdot W = \Lambda$. This is the decomposition of Λ into rotations (W) and inversions (P).

2.2 Rotations of Vectors in Fixed Coordinates

Consider a transformation $\Lambda : S \to S'$, where $\vec{r} = x_i \hat{e}_i = x'_i \hat{e}'_i$. We know that $x'_i = \Lambda_{ij} x_j$ and $\hat{e}'_i = \Lambda_{ij} \hat{e}_j$. We want to consider vector rotation, which we can first do in 2D.

Consider a 2D space with basis vectors $\hat{e}_1, \hat{e}_2$, rotated by an angle φ. The transformation matrix is

$$\Lambda = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.8}$$

Then, a vector rotation is described by

$$\vec{r'} = \Lambda_\varphi \vec{r} \tag{2.9}$$

or more generally

$$x'_i = \Lambda_{ij} x_j \tag{2.10}$$

Suppose we want to change coordinates from x_i to y_i, and suppose B is the transformation between these:

$$y_i = B_{ij} x_j \tag{2.11}$$

$$y'_i = B_{ij} x'_i \tag{2.12}$$

where $y'_i = \Lambda_{ij} y_j$. Then, we can define a similarity transform (I don't know what that is):

$$\widetilde{\Lambda_y} = B \Lambda B^\dagger \tag{2.13}$$

2.3 The Vector Product

Consider vectors $\vec{a_1}, \vec{a_2}$ in a coordinate system $\hat{e}_1, \hat{e}_2, \hat{e}_3$, where $\vec{a_i} = a_{ik\,k=1,2,3}$. The vector product of these two is

$$|\vec{a_1} \times \vec{a_2}| = |\vec{a_1}| \cdot |\vec{a_2}| \sin\theta \tag{2.14}$$

(Side note: a vector is not just any collection of variables, it is a set of functions that is invariant under inversions and rotations. Consider an equation

$$\frac{d\vec{C}}{dt} = (A_i a_i)\vec{B} \tag{2.15}$$

and apply the inverse to it. A_i and $\vec{B}$ flip their sign, leaving the $\frac{d\vec{C}}{dt}$ term invariant. Since the term is invariant, it cannot be a vector.)

Conventionally, we say

$$\hat{e}_{i+1} \times \hat{e}_{(i+1)\text{mod}3+1} = \hat{e}_{(i+2)\text{mod}3+1} \tag{2.16}$$

In general, this transformation is described by

$$[\hat{e}_i \times \hat{e}_j] = \epsilon_{ijk}\hat{e}_k \tag{2.17}$$

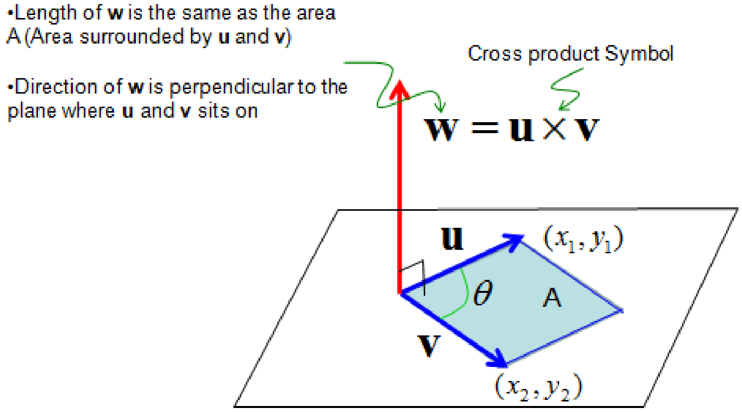
where ϵ_{ijk} is the Levi-Civita tensor. It is a 27-element tensor that is +1 if the permutation (ijk) is even, i.e. there are an even number of transpositions from the order (123), and it is -1 if the permutation is odd. This covers 6 elements, and the remaining 21 are zero. Explicitly,

$$\epsilon_{ijk} = \begin{cases} +1 & (123), (231), (312) \\ -1 & (132), (213), (321) \\ 0 & \text{otherwise} \end{cases} \tag{2.18}$$

In these terms, the vector product is

$$\vec{a_1} \times \vec{a_2} = \sum_{i,j} = a_{1i}a_{2j}|\hat{e}_i \times \hat{e}_j| = \sum a_{1i}a_{2j}\epsilon_{ijk}\hat{e}_k \tag{2.19}$$

The magnitude of the vector product is equivalent to the area of the parallelogram between them:



http://www.sharetechnote.com/html/Handbook_EngMath_Matrix_CrossProduct.html

The vector product can also be represented as a determinant,

$$(\vec{a_1} \times \vec{a_2}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \hat{e}_3 = \det(a) \cdot \hat{e}_3 \tag{2.20}$$

2.4 Vector Product under Coordinate Changes

Consider $\vec{C} = \vec{A} \times \vec{B}$, with $C_k = \epsilon_{kij} A_i B_j$, under an inversion of both A and B. $\vec{C}$ is a pseudovector, not a real vector, as it is not inverted under this process. $\vec{C}$ can be a real vector under just rotation, though, as we can see:

$$A'_i = \Lambda_{ij} A_j \tag{2.21}$$
$$A_s = \Lambda_{is} A'_i, B_e = \Lambda_{ie} B'_i \tag{2.22}$$

Then, $\vec{C}$ in these terms is

$$C_k = \epsilon_{ijk} = \Lambda_{\alpha i} A'_\alpha \Lambda_{\beta j} B'_\beta \tag{2.23}$$
$$\Lambda_{mk} = C'_m = \epsilon_{ijk} \Lambda_{\alpha i} \Lambda_{\beta j} A'_\alpha B''_\beta \tag{2.24}$$

Substituting in the previous expression for C here,

$$\Lambda_{mk} \cdot \epsilon\alpha\beta m A'_\alpha B'_\beta = \epsilon_{ijk} \Lambda_{\alpha i} \Lambda_{\beta j} A'_\alpha B'_\beta \tag{2.25}$$

or

$$(\Lambda_{mk} \epsilon_{\alpha\beta m - \epsilon_{ijk} \Lambda_{\alpha i} \Lambda_{\beta j}}) \times A'_\alpha B'_\beta \tag{2.26}$$

Applying the same coordinate transformation again,

$$\Lambda_{sk}\Lambda_{mk}\epsilon_{\alpha\beta m} = \epsilon_{ijk}\Lambda_{\alpha i}\Lambda_{\beta j}\Lambda_{sk} \tag{2.27}$$

We can use this to prove the following general transformation law,

$$\epsilon_{\alpha\beta\gamma} = \epsilon_{ijk}\Lambda_{\alpha i}\Lambda_{\beta j}\Lambda_{\gamma k} \tag{2.28}$$

or

$$||A||\epsilon\alpha\beta\gamma = \epsilon_{ijk}A_{\alpha i}A_{\beta j}A_{\gamma k} \tag{2.29}$$

2.5 Scalar Triple Product

Consider the product

$$\vec{a_3}\cdot(\vec{a_1}\times\vec{a_2}) = \mathbb{V} = ||a_{ij}|| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{2.30}$$

Explicitly, we can say that

$$a_{3i}\hat{e}_i\cdot(\vec{a_1}\times\vec{a_2})_k\hat{e}_k = a_{3k}\epsilon_{kij}a_{1i}a_{2j} \tag{2.31}$$

which we set equal to the determinant,

$$||a|| = \epsilon_{ijk}a_{1i}a_{2j}a_{3k} \implies ||a||\epsilon_{\alpha\beta\gamma} = \epsilon_{ijk}a_{\alpha i}a_{\beta j}a_{\gamma k} \implies ||a|| = \frac{1}{6}\epsilon_{\alpha\beta\gamma}\epsilon_{ijk} \tag{2.32}$$

Note that a known formula is

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b}) \tag{2.33}$$

which leads to some other stuff I don't get

2.6 Angular Velocity

Consider a coordinate system with a rotation Λ. We can say that

$$\Lambda_{ij} = \delta_{ij} + \delta\phi_{ij} \tag{2.34}$$

where $\delta\phi_{ij} << 1$. Then we can say that $\Lambda = \vec{1} + \delta\hat{\varphi}$, an infinitesimal operator. $\delta\varphi$ is given in matrix form by

$$\delta\varphi = \begin{bmatrix} 0 & -\delta\varphi_3 & \delta\varphi_2 \\ \delta\varphi_3 & 0 & -\delta\varphi_1 \\ -\delta\varphi_2 & \delta\varphi_1 & 0 \end{bmatrix} \tag{2.35}$$

We apply the identity plus this infinitesimal operator to an arbitrary vector $\vec{x}$,

$$\vec{x}' = (\hat{1} + \delta\hat{\varphi})\vec{x} \implies \vec{x}' - \vec{x}\delta\hat{\varphi}\vec{x} \tag{2.36}$$

or

$$\delta\vec{x} = (\delta\vec{\varphi} \times \vec{x}) \tag{2.37}$$

This is an infinitesimal rotation for some reason. Differentiate this in time to get

$$\frac{d\vec{r}}{dt} = [\vec{\Omega} \times \vec{r}] \tag{2.38}$$

Ω is a vector with respect to rotations, but not inversions.

We introduce the time derivative of position as usual,

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \tag{2.39}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx_i}{dt}\hat{e}_i = v_i\hat{e}_i \tag{2.40}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{2.41}$$

In terms of a coordinate system (ρ, φ, z), we see

$$\vec{r} = \rho\hat{e}^{\rho} + z\hat{e}_z \tag{2.42}$$

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{e}_{\rho} + \dot{z}\hat{e}_z + \rho\dot{e}_{rho} \tag{2.43}$$

2.7 Velocity Transformation

In changing the coordinate system of a rigid body, we get

$$\frac{d\vec{r}}{dt}_S = \frac{d\vec{r}}{dt}_{S'} + (\vec{\Omega} \times \vec{r}) \tag{2.44}$$

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 3: Calculus of Variations

Lecturer: Stuart Bale, Ivan Vasko *29 January* *Aditya Sengupta*

Homework 1 is due Friday, Feb 8, and it consists of problems 6.1, 6.2, 6.11, 6.17, 6.22, 6.24 from Taylor.

General OH: Campbell 355, Tuesday 11-12.

3.1 The Idea of Lagrangian Mechanics

There are two major problems with using Newtonian mechanics for everything. One is it is not very general, dealing only with the motion of particles, and the other is it usually deals well only with orthogonal Cartesian coordinates. The calculus of variations is the mathematical foundation for dealing with mechanics in generalized coordinates $q_i (i = 1, \ldots, N)$ or in Hamiltonian mechanics generalized momenta p_i. These q_is define a *configuration space*, and when they are coupled with the p_is, it creates *phase space*. Instead of a path parameterized by x, y, z, we have a path in the configuration space.

We know from Newtonian mechanics that

$$\vec{F} = m\ddot{q} \tag{3.1}$$

We need two initial or boundary conditions to completely specify q here. For example, let $\ddot{q} = g$, considering the case of free fall in a gravitational field:

$$\vec{F} = mg \tag{3.2}$$

$$\ddot{q} = g \tag{3.3}$$

$$\dot{q} = gt + v_0 \text{ where } v_0 = \dot{q}(t = 0) \tag{3.4}$$

$$q = \frac{1}{2}gt^2 + v_0 t + q_0 \text{ where } q_0 = q(t = 0) \tag{3.5}$$

Here, we need the initial conditions; what the particle is doing, in terms of its velocity and position, at $t = 0$. Lagrangian mechanics deals with boundary conditions instead, starting from $q(t_1)$ and $q(t_2)$. Lagrangian mechanics starts with these two data points, and asks "what path in q space does the system follow from $q(t_1)$ to $q(t_2)$?".

3.2 Calculus of Variations

Consider the example of a plane flying between SFO and EWR. Airlines want to save money, so other than paying their workers very poorly, they can minimize fuel. The path that minimizes fuel may not be the shortest path in terms of distance, for example, if there are strong head winds over the main path. It may be around those on a path that is longer. We express this idea with a *functional*,

$$F(\vec{x}) = \int_{t_1}^{t_2} f(\vec{x}, \dot{\vec{x}}, t) dt \tag{3.6}$$

where we want to minimize F, which in this example might be fuel usage.

The usual canonical example of a problem that can be solved quickly with the calculus of variations is the *brachistochrone problem*. Consider two points A and B in a plane, with gravity directed downward. What trajectory would a particle take from A to B to minimize its travel time? This is not a straight line between them, because of the force of gravity.

We can write the travel time as an integral over the path,

$$t = \int_{(x_A, y_A)}^{(x_B, y_B)} \frac{ds}{v} \tag{3.7}$$

which is the same sort of functional expression, where we want to minimize a quantity that is expressed as an integral. (This problem will be solved later after we have built the machinery of variational calculus.)

3.3 The Machinery of Variational Calculus

Again, the generalized statement of a functional is

$$F(\vec{x}) = \int_{t_1}^{t_2} f(\vec{x}, \dot{\vec{x}}, t) dt \tag{3.8}$$

By definition of the optimal path, any perturbation away from that path will increase the value of F. Let this perturbation be $\eta(t)$ and add the requirement that $\eta(t_1) = \eta(t_2) = 0$.

$$x(t) = \bar{x}(t) + \eta(t) \approx \bar{x}(t) \tag{3.9}$$

Then, we can rewrite the functional expression. By definition, $F(\bar{\vec{x}}) \leq F(\vec{x})$, where

$$F(\vec{x}) = \int_{t_1}^{t_2} f(\bar{\vec{x}} + \eta, \dot{\bar{\vec{x}}} + \dot{\eta}, t) dt \tag{3.10}$$

We expand this as a Taylor series,

$$F(\vec{x}) = \int_{t_1}^{t_2} \left(f(\bar{x}, \dot{\bar{x}}, t) + \frac{\partial f}{\partial x} \cdot \eta + \frac{\partial f}{\partial \dot{x}} \cdot \dot{\eta} + \dots \right) dt \tag{3.11}$$

Therefore the difference between the two paths is

$$F(\vec{x}) - F(\vec{\bar{x}}) = \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial \dot{x}}\dot{\eta} \right) dt \tag{3.12}$$

We apply integration by parts to the second part of this,

$$\int_{t_1}^{t_2} \frac{\partial f}{\partial \dot{x}}\dot{\eta} dt = \left. \frac{\partial f}{\partial \dot{x}}\eta \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\eta dt \tag{3.13}$$

The first term drops out, because we stipulated initially that a deviation would not change the end points of a path, i.e. $\eta(t_1) = \eta(t_2) = 0$. Therefore overall we get

$$F(x) = F(\bar{x}) + \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} \right) \cdot \eta dt \tag{3.14}$$

If we consider the deviation to be small, it follows that the integral must be zero, i.e.

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = \frac{\partial f}{\partial x} \tag{3.15}$$

This is the *Euler-Lagrange* equation.

In general, this will have to be applied as many times as there are dimensions in the configuration space.

3.4 Applying Variational Calculus

Consider two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$. We can use the E-L equations to find that the shortest path between them is a line. The functional expression is

$$L = \int_{(x_1,y_1)}^{(x_2,y_2)} ds = \int_{x_1}^{x_2} \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{3.16}$$

$$L = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx \tag{3.17}$$

This is the same form as the general functional setup for a variational calculus problem, $f(y, y', x) = \sqrt{1 + y'^2}$, therefore we can apply the E-L equation:

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \tag{3.18}$$

where $\frac{\partial f}{\partial y}$ is trivially zero. We get

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0 \tag{3.19}$$

or

$$\begin{aligned} y' &= c\sqrt{1+y'^2} && (3.20)\\ y' &= \text{const} && (3.21)\\ \frac{dy}{dx} &= m && (3.22)\\ y &= mx + b && (3.23) \end{aligned}$$

as we expected.

3.5 Brachistochrone Problem

We previously wrote the time equation for the brachistochrone problem. Let point A be the origin for convenience. We know that $m\ddot{x} = 0$ and $m\ddot{y} = mg$. From conservation of energy, we can find the velocity (as is required in the time equation),

$$\frac{1}{2}mv^2 = mgy \implies v = \sqrt{2gy} \tag{3.24}$$

and the differential element ds is the same as in the straight-line example. Therefore we get

$$t = \int_0^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx \tag{3.25}$$

This gives us the function under the integral that is required as input to the E-L equation,

$$f(y, y', x) = \frac{\sqrt{1+y'^2}}{\sqrt{y}} \tag{3.26}$$

to which we can then apply the equations and get the path. However, there is a shortcut using the Hamiltonian, which we will see in a few months. Take H as follows:

$$H = y'\frac{\partial f}{\partial y'} - f \tag{3.27}$$

Then, take an x derivative:

$$\frac{dH}{dx} = y'' \frac{\partial f}{\partial y'} + y' \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - y' \frac{\partial f}{\partial y} - y'' \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial x} \tag{3.28}$$

which can be simplified to

$$\frac{dH}{dx} = y' \left(\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\right) - \frac{\partial f}{\partial x} \tag{3.29}$$

which just gives us $\frac{dH}{dx} = -\frac{\partial f}{\partial x}$, because the first part is exactly the E-L equation and is therefore zero. This is an alternative statement of conservation of energy: when there is no time dependence, H is conserved.

Returning to the brachistochrone problem, we have

$$f = \sqrt{\frac{1 + y'^2}{y}} \tag{3.30}$$

and so the Hamiltonian is

$$\frac{-1}{\sqrt{y(1 + y'^2)}} = C \tag{3.31}$$

(as we just saw, the Hamiltonian is a constant)

Solving and setting $\sqrt{y} = \sqrt{C} \sin\theta$, we get

$$x = \frac{C}{2}(2\theta - \sin 2\theta) \tag{3.32}$$

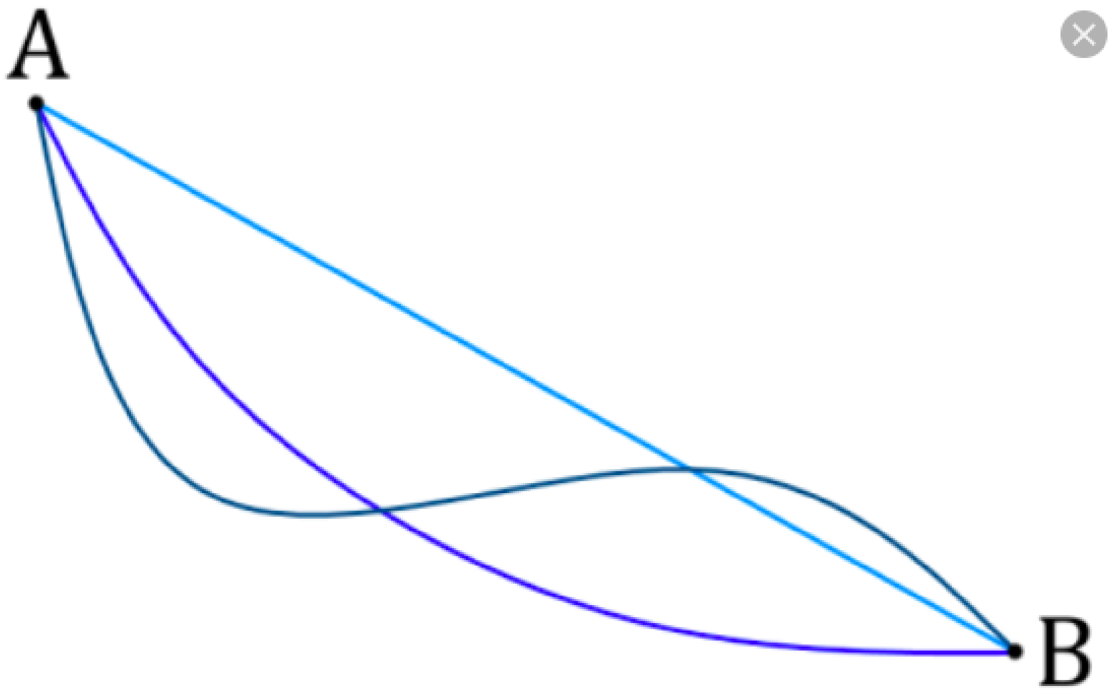
$$y = C\sin^2\theta = C(1 - \cos^2 2\theta) \tag{3.33}$$

Therefore, setting $\varphi = 2\theta$, we get

$$x = C(\varphi - \sin\varphi) \tag{3.34}$$

$$y = C(1 - \cos\varphi) \tag{3.35}$$

This is the equation for a cycloid.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Physics 105: Analytic Mechanics **Spring 2019**

Lecture 4: Lagrangian Mechanics and Variational Calc. Applications

Lecturer: Stuart Bale, Ivan Vasko *31 January* *Aditya Sengupta*

4.1 Recap

We can optimize a functional of the form

$$F(\vec{x}) = \int_{t_1}^{t_2} f(\vec{x}, \dot{\vec{x}}, t) dt \tag{4.1}$$

using the Euler-Lagrange equation,

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{x}} - \frac{\partial f}{\partial x} = 0 \tag{4.2}$$

We can also optimize it using the Hamiltonian,

$$H = y'\frac{\partial f}{\partial y'} - f \tag{4.3}$$

$$\frac{dH}{dx} = -\frac{\partial f}{\partial x} \tag{4.4}$$

4.2 Plateau's Problem

Imagine two hoops that are fixed and rigid, with radius R, separated by some distance D. We want to find the minimum energy state for this system if, for example, a soap film extends between the two. There is some optimal surface that sweeps out the two rings and minimizes the surface area and therefore the surface tension of the soap film.

Consider a differential element of the volume, with some length dx spanning the radial space $r(x)$. The area of the boundary is

$$2\pi r(x)\sqrt{dr^2 + dx^2} = 2\pi r(x)\sqrt{1 + r'(x)^2}dx \tag{4.5}$$

Therefore, the total area is found by integrating this over x:

$$A = 2\pi \int_{-D/2}^{D/2} r(x)\sqrt{1 + r'(x)^2}dx \tag{4.6}$$

This gives us $f(r, r', x) = r(x)\sqrt{1 + r'(x)^2}$. Since this is independent of x, we can use the Hamiltonian:

$$H = r'\frac{\partial f}{\partial r'} - f = \frac{rr'^2}{\sqrt{1+r'^2}} - r\sqrt{1+r'^2} = -\frac{r}{\sqrt{1+r'^2}} = K \tag{4.7}$$

This can be rearranged to get

$$\frac{dr}{dx} = \pm\sqrt{\left(\frac{r}{H}\right)^2 - 1} \tag{4.8}$$

$$\int \frac{dr}{\sqrt{\left(\frac{r}{H}\right)^2 - 1}} = \int dx = x + \alpha \tag{4.9}$$

This has a solution

$$\frac{r}{H} = \cosh\psi, dr = H\sinh\psi d\psi \tag{4.10}$$

and therefore

$$\frac{r^2}{H^2} - 1 = \sinh^2\psi \tag{4.11}$$

If we assume $r(-x) = r(x)$, i.e. the solution is symmetric, we find that $\alpha = 0$, and therefore

$$r(x) = H\cosh\frac{x}{H} \tag{4.12}$$

We can apply boundary conditions to this: $x = \pm D/2 \implies r = R$. This gives us

$$R = H\cosh\frac{D}{2H} \tag{4.13}$$

Let $\chi = \frac{H}{D}$, then

$$R = \chi D\cosh\frac{1}{2\chi} \tag{4.14}$$

$$\frac{R}{D} = \chi\cosh\frac{1}{2\chi} \tag{4.15}$$

where χ is a specific number.

Note that the cosh function means there is no solution for $\frac{R}{D} < 0.75$, which physically corresponds to the soap film breaking. For $\frac{R}{D} = 0.75$, there is one solution, and for $\frac{R}{D} > 0.75$, there are two solutions.

4.3 Connecting Quantum Mechanics and Lagrangian Mechanics

In quantum mechanics, we have a transition amplitude, representing the probability of a transition between a state $q_1(t_1)$ and $q_2(t_2)$. This is written as

$$\langle q_2(t) \mid q_1(t) | q_2(t) \mid q_1(t)\rangle \tag{4.16}$$

and let $t_2 = t_1 + \Delta t$. The transition is then characterized by

$$\langle q_2(t+\Delta t)|q_1(t)|q_2(t+\Delta t)|q_1(t)\rangle = \left\langle q_2|e^{-i\Delta H}|q_1\middle|q_2|e^{-i\Delta H}|q_1\right\rangle \tag{4.17}$$

or

$$e^{i\Delta tH} \sim e^{-i\Delta t\frac{p^2}{2m}} e^{i\Delta tV} \tag{4.18}$$

Then the action is given by

$$S(x(t)) = \int_{t_1}^{t_2} dt \left(\frac{p^2}{2m} - V\right) \tag{4.19}$$

The amplitude functional is

$$A[x(t)] = e^{iS/\hbar} \tag{4.20}$$

and the probability is

$$|A|^2 = \int_{allpaths} x(t) e^{iS(x(t))/\hbar} \tag{4.21}$$

We want to find the stationary values of S, as this gives us a stationary phase and therefore the most probable state. Hamilton's Principle states

$$\delta S(x(t)) = \delta \int dt \left(\frac{p^2}{2m} - V\right) = 0 \tag{4.22}$$

We define the Lagrangian $L = T - V$. Then, we have

$$\delta \int_{t_1}^{t_2} dt L(q, \dot{q}, t) = 0 \tag{4.23}$$

Using the Euler-Lagrange equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{4.24}$$

If we consider a mass m and a potential $V(q)$, the Lagrangian is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{q}^2 \tag{4.25}$$

$$L = T - V = \frac{1}{2}m\dot{q}^2 - V(q) \tag{4.26}$$

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \tag{4.27}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = m\ddot{q} \implies m\ddot{q} = -\frac{\partial V}{\partial q} \tag{4.28}$$

which is just the statement that $ma = F$.

4.4 Spherical pendulum

Consider a pendulum being described in 3D space by angles θ and φ, with a constant length l, and under a constant gravitational field g. We write the position vector,

$$\vec{r} = l\cos\varphi\sin\theta\hat{x} + l\sin\varphi\sin\theta\hat{y} + l\cos\theta\hat{z} \tag{4.29}$$

$$\dot{\vec{r}} = (-l\dot{\varphi}\sin\varphi\sin\theta + l\cos\varphi\dot{\theta}\cos\theta)\hat{x} + (l\dot{\varphi}\cos\varphi\sin\theta + l\sin\varphi\dot{\theta}\cos\theta)\hat{y} - l\dot{\theta}\sin\theta\hat{z} \tag{4.30}$$

We dot this with itself and find

$$\dot{r}^2 = l^2\dot{\varphi}^2(\sin^2\varphi + \cos^2\varphi)\sin^2\theta + l^2\dot{\theta}^2(\sin^2\varphi + \cos^2\varphi)\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta = l^2\dot{\varphi}^2\sin^2\theta + l^2\dot{\theta}^2 \tag{4.31}$$

Therefore the kinetic energy is

$$T = \frac{1}{2}m\dot{r}^2 = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\varphi}^2\sin^2\theta \tag{4.32}$$

and the potential energy is just $V = mgl\cos\theta$.

Therefore the Lagrangian is

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\varphi}^2\sin^2\theta - mgl\cos\theta \tag{4.33}$$

We can write E-L equations for both coordinates,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \tag{4.34}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} \tag{4.35}$$

In φ, we get

$$\frac{\partial L}{\partial \dot{\varphi}} = p_\varphi \tag{4.36}$$

which is a constant, so

$$\frac{\partial L}{\partial \dot{\varphi}} = ml^2\dot{\varphi}^2 \sin^2\theta = p_\varphi \tag{4.37}$$

Now in θ we get

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \implies \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta} \tag{4.38}$$

$$\frac{\partial L}{\partial \theta} = mgl\sin\theta + ml^2\dot{\phi}^2\sin\theta\cos\theta \tag{4.39}$$

Therefore after some simplification we get

$$\ddot{\theta} = \frac{g}{l}\sin\theta + \dot{\varphi}^2\sin\theta\cos\theta \tag{4.40}$$

$$\ddot{\theta} = \frac{g}{l}\sin\theta + \frac{p_\varphi^2}{m^2l^4}\frac{\cos\theta}{\sin^3\theta} \tag{4.41}$$

For a fixed $\theta = \theta_0$, we can get

$$\left(\frac{g}{l} + \dot{\varphi}^2\cos\theta_0\right)\sin\theta_0 = 0 \tag{4.42}$$

or

$$\dot{\varphi}^2\cos\theta_0 = -\frac{g}{l} \implies \dot{\varphi}^2 > \frac{g}{l} - \omega_0^2 \tag{4.43}$$

4.5 Another example

Consider a pendulum attached to a spring along the x axis, $x = a\cos\omega t$. We can write the Lagrangian by finding T,

$$\dot{r}^2 = \dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l\cos\theta \tag{4.44}$$

and so

$$L = T - V = \frac{1}{2}\dot{m}(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l\cos\theta) + mgl\cos\theta \tag{4.45}$$

Then, applying the E-L equations gets us

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{a\omega^2}{l}\cos\omega t \tag{4.46}$$

This is a driven harmonic oscillator. We find that this differential equation has a general solution $\theta(t) = A\cos\omega_0 t + B\sin\omega_0 t$, which we solve to get

$$\theta(t) = -\frac{F}{\omega^2 - \omega_0^2}\cos\omega t \tag{4.47}$$

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 5: Applying Lagrangian Mechanics, Noether's Theorem

Lecturer: Stuart Bale, Ivan Vasko *5 February* *Aditya Sengupta*

5.1 Angular Momentum

Consider a particle of mass m constrained to move on the inner surface of a cone of radius r and cone angle α, under a gravitational field. We know from the geometry that

$$\frac{r}{z} = \tan\alpha \tag{5.1}$$

We construct a position vector for the particle,

$$\vec{r} = r\cos\theta\hat{x} + r\sin\theta\hat{y} + z\hat{z} \tag{5.2}$$

and we take a derivative,

$$\dot{\vec{r}} = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\hat{x} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{y} + \dot{z}\hat{z} \tag{5.3}$$

We want to find the kinetic energy, for which we dot this vector with itself,

$$\dot{r}^2 = \dot{\vec{r}}\cdot\dot{\vec{r}} = \dot{r}^2\cos^2\theta - 2r\dot{r}\sin\theta\cos\theta + r^2\dot{\theta}^2\sin^2\theta + \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\cos^2\theta + \dot{z}^2 \tag{5.4}$$

Lots of terms cancel and we get

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) \tag{5.5}$$

We can write this using the geometric constraint to eliminate z, by $z = r\cot\theta \implies \dot{z} = \dot{r}\cot\theta$,

$$T = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2\cot^2\alpha\right) = \frac{1}{2}m\left(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2\right) \tag{5.6}$$

The potential energy is just $V = mgz = mgr\cot\alpha$. This means we can write a Lagrangian,

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2\right) - mgr\cot\alpha \tag{5.7}$$

We see that $\frac{\partial \mathcal{L}}{\partial \theta} = 0$. When the Lagrangian does not depend on a spatial coordinate, we say that that coordinate is cyclic. Here, θ is a cyclic coordinate. From the Euler-Lagrange equation, we get

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{5.8}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \text{const} = p_\theta \tag{5.9}$$

We can take the derivative of the Lagrangian and find that

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} = p_\theta \tag{5.10}$$

This is exactly the statement of conservation of angular momentum.

We can apply the other Euler-Lagrange equation on the second spatial coordinate, to get an equation of motion,

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - mg\cot\alpha \tag{5.11}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}\csc^2\alpha \implies \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial r}\right) = m\ddot{r}\csc^2\alpha \tag{5.12}$$

Then, the E-L equation gives us

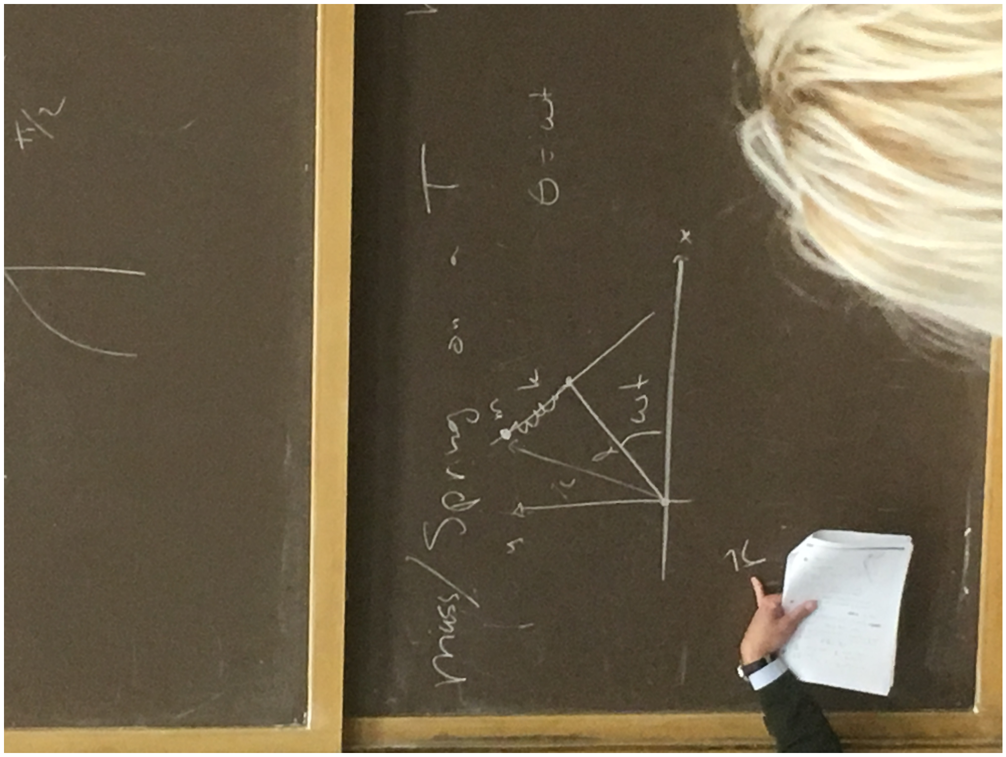
$$\ddot{r} = r\dot{\theta}^2\sin^2\alpha - g\sin\alpha\cos\alpha \tag{5.13}$$

We can consider points with no acceleration in r,

$$r\dot{\theta}^2\sin\alpha = g\cos\alpha \implies \dot{\theta}^2\tan\alpha = \frac{g}{r_0} = \omega_0^2 \tag{5.14}$$

5.2 Mass/Spring on a T

Consider a mass and a spring in the following arrangement,



In terms of the length radially outwards l, and the length along the part orthogonal to that radial length ρ, we can write a position vector,

$$\vec{r} = (l\cos\omega t - \rho\sin\omega t)\hat{x} + (l\sin\omega t + \rho\cos\omega t)\hat{y} \tag{5.15}$$

$$\dot{\vec{r}} = (\omega(-l\sin\omega t - \rho\cos\omega t) - \dot{\rho}\sin\omega t)\,\hat{x} + (\omega(l\cos\omega t - r\sin\omega t) + \dot{\rho}\cos\omega t)\,\hat{y} \tag{5.16}$$

Squaring this and substituting into the Lagrangian, along with the spring equation for the potential energy, we get

$$\mathcal{L} = \frac{1}{2}m\left(\omega^2(l^2+\rho^2) + \dot{\rho}^2 + 2\omega l\dot{\rho}\right) - \frac{1}{2}k\rho^2 \tag{5.17}$$

Now, we can apply the Euler-Lagrange equation to the system with the coordinate ρ,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\rho}}\right) = \frac{\partial\mathcal{L}}{\partial\rho} \tag{5.18}$$

We get

$$\ddot{\rho} = \left(\frac{k}{m} - \omega^2\right)\rho = 0 \tag{5.19}$$

There are three possibilities based on the value of ω. In the case where $\omega < \sqrt{\frac{k}{m}}$, we get simple harmonic motion, and $\rho(t) = A\cos(\omega_0 t + \theta_0)$. In the case $\omega > \sqrt{\frac{k}{m}}$, we get a sum of exponentials, $\rho(t) = Be^{\alpha t} + Ce^{-\alpha t}$, where $\alpha = \sqrt{\omega^2 - \frac{k}{m}}$. In case of equality, we get resonance, and $\rho(t) \propto t$.

One more case that can be considered is what happens when gravity is added. The term $mg(l\sin\omega t + \rho\cos\omega t)$ is added to the potential energy, which gives us the following slightly altered equation of motion,

$$\ddot{\rho} + \omega_0^2 \rho = -g\cos\omega t \tag{5.20}$$

$$\rho(t) \sim A\cos(\omega_0 t + \theta_0) + \frac{g}{2\omega^2 - \frac{k}{m}}\cos\omega t \tag{5.21}$$

5.3 Symmetries of the Lagrangian

A symmetry of the Lagrangian is when a change to $\mathcal{L}$, via a change in the coordinates from q_i to $\widetilde{q_i}$, does not perturb the Lagrangian.

For example, let $\widetilde{x} = x + \epsilon$, where ϵ is a small constant. $\dot{\widetilde{x}} = \dot{x}$. Then, the Lagrangian is altered as follows:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x) \tag{5.22}$$

$$\widetilde{\mathcal{L}} = \frac{1}{2}m\dot{\widetilde{x}}^2 - V(\widetilde{x}) \tag{5.23}$$

Therefore, the Lagrangian does not vary if $V(x) = V(x + \epsilon)$. We can Taylor expand,

$$V(x + \epsilon) = V(x) + \epsilon\frac{\partial V}{\partial x} + \dots \tag{5.24}$$

Therefore, for the required condition to be true, $\frac{\partial V}{\partial x}$ must be 0. This is equivalent to saying the force in the x direction is zero, or alternatively

$$\frac{d}{dt}(m\dot{x}) = 0 \tag{5.25}$$

$$m\dot{x} = p_x \tag{5.26}$$

where p_x is a constant, the linear momentum.

In general, when we have a coordinate that is slightly shifted to first order in time, the Lagrangian on that coordinate becomes

$$\mathcal{L}(q + \epsilon\kappa, \dot{q} + \epsilon\dot{\kappa}) = L(q, \dot{q}) = \epsilon\sum_i \dot{\kappa}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \epsilon\sum_i \kappa\frac{\partial L}{\partial q_i} + \dots \tag{5.27}$$

If we require that the perturbation is zero, we get *Noether's Theorem*,

$$\sum_i \dot{\kappa}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \sum_i \kappa_i \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{5.28}$$

This can be written in product rule form as follows:

$$\frac{d}{dt}\left(\sum_i k_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \text{const} \tag{5.29}$$

5.4 Pendulum Motion

The Lagrangian for a pendulum of length l and mass m with an angle from the normal θ is

$$\mathcal{L} = \frac{1}{2}m\dot{l}^2 + \frac{1}{2}ml^2\dot{\theta}^2 \tag{5.30}$$

We perturb θ slightly, and from Noether's theorem we get

$$\mathcal{L}(\theta + \epsilon, \dot{\theta}) = \mathcal{L}(\theta, \dot{\theta}) + \epsilon \frac{\partial \mathcal{L}}{\partial \theta} \tag{5.31}$$

which gives us the conservation law

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{5.32}$$

5.5 The Hamiltonian

Consider

$$H = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} \tag{5.33}$$

We can take the time derivative of H,

$$\frac{dH}{dt} = \frac{d}{dt}\left(\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{d\mathcal{L}}{dt} \tag{5.34}$$

$$\frac{dH}{dt} = \ddot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} + \dot{q}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \ddot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \dot{q}\frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{L}}{\partial t} \tag{5.35}$$

More simply, this is

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{5.36}$$

We see that if $\mathcal{L}$ does not explicitly depend on time, then H is a constant with time.

In the one-dimensional case, consider a generic Lagrangian, and write the corresponding Hamiltonian,

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x) \tag{5.37}$$

$$H = \dot{x}\frac{\partial L}{\partial \dot{x}} - \mathcal{L} = m\dot{x}^2 - \left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = \frac{1}{2}m\dot{x}^2 + V(x) \tag{5.38}$$

which is just the total energy.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 6: Small-Amplitude Oscillations

Lecturer: Stuart Bale, Ivan Vasko *7 February* *Aditya Sengupta*

Note that a property of the Lagrangian is it is invariant under constant multipliers; if $\mathcal{L}$ is a valid Lagrangian, then $\widetilde{\mathcal{L}} = c\mathcal{L}$ is also valid.

6.1 Pendulum

Conside a pendulum with mass m under a gravitational field g, with an angle θ from the normal. The Lagrangian is

$$\mathcal{L} = T - V \tag{6.1}$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{6.2}$$

where $x = l\sin\theta, y = -l\cos\theta$, and the potential energy is

$$V = -mgl\cos\theta \tag{6.3}$$

Therefore, we can drop some constant factors and write as a valid Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\theta}^2 + \frac{g}{l}\cos\theta = \frac{1}{2}\dot{\theta}^2 + \omega_0^2\cos\theta \tag{6.4}$$

where $\omega_0^2 = \frac{g}{l}$. We apply the E-L equation,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta} \tag{6.5}$$

Therefore

$$\ddot{\theta} = -\omega_0^2\sin\theta = -\frac{\partial V}{\partial\theta} \tag{6.6}$$

Look at the specific case where $\ddot{\theta} = 0$. Then we get $\sin\theta = 0$, so $\theta = 0$ or $\theta = \pi$. There are two solutions to this; $\theta = 0$, with the pendulum pointing straight down, and $\theta = \pi$, with it pointing straight up. Only $\theta = 0$ is a stable solution; under a slight deflection $\delta\theta$, only $\theta = 0$ will remain around that point as time goes on. We can see this formally.

$$\theta = \theta_0 + \delta\theta \tag{6.7}$$
$$\delta\ddot{\theta} = -\omega_0^2 \sin(\theta_0 + \delta\theta) = -\omega_0^2(\sin\theta_0 \cos\delta\theta + \sin\delta\theta\cos\theta_0) \tag{6.8}$$

We know by definition that $\sin\theta_0 = 0$, so

$$\delta\ddot{\theta} = -\omega_0^2 \cos\theta_0 \delta\theta \tag{6.9}$$

Here, if $\theta_0 = 0$, we get

$$\delta\ddot{\theta} = -\omega_0^2 \delta\theta \tag{6.10}$$

This is a linear differential equation that is of the familiar form of an oscillator. Formally, we assume $\delta\theta = Ae^{pt}$ is a solution, and we get $p^2 = -\omega_0^2$. So a formal solution is

$$\delta\theta = A_1 e^{-i\omega_0 t} + A_2 e^{i\omega_0 t} \tag{6.11}$$

which has a small magnitude for all times. (We assume that because $\delta\theta$ is small, A_1 and A_2 must also be small.)

However, for the case $\theta_0 = \pi$, we get

$$\delta\ddot{\theta} = \omega_0^2 \delta\theta \tag{6.12}$$
$$\delta\theta = A_1 e^{-\omega_0 t} + A_2 e^{\omega_0 t} \tag{6.13}$$

Therefore, even due to the small magnitudes of the coefficients, the solution grows exponentially with time. Unless we have initial conditions that let $A_2 = 0$, we will have an unstable system. We can try to find these.

$$\omega_0 \left.\delta\theta\right|_{t=0} = (A_1 + A_2)\omega_0 \tag{6.14}$$
$$\left.\delta\dot{\theta}\right|_{t=0} = (-A_1 + A_2)\omega_0 \tag{6.15}$$

Therefore

$$\omega_0 \delta\theta(t=0) + \delta\dot{\theta}(t=0) = 2\omega_0 A_2 = 0 \tag{6.16}$$

Therefore

$$\delta\dot{\theta}(t=0) = -\omega_0 \delta\theta(t=0) \tag{6.17}$$

Therefore we need a velocity that slows down and stops as θ approaches π, which is unphysical.

6.2 Coordinate Deviations

Consider a Lagrangian in a coordinate variable q,

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - V(q) \tag{6.18}$$

We perturb this slightly, $q = q_0 + \delta q$, and we get

$$\mathcal{L} = \frac{1}{2}(\delta\dot{q})^2 + V(q_0 + \delta q) = \frac{1}{2}\delta\dot{q}^2 - V(q_0 + \delta q) = \frac{1}{2}\delta\dot{q}^2 - V(q_0) - \left(\frac{\partial V}{\partial q}\right)_{q=q_0}\delta q - \frac{1}{2}\left(\frac{\partial^2 V}{\partial q^2}_{q=q_0}\delta q^2\right) \tag{6.19}$$

Simplifying and dropping constants,

$$\mathcal{L} = \frac{1}{2}\delta\dot{q}^2 - \frac{1}{2}\left(\frac{\partial^2 V}{\partial q^2}\right)\delta q^2 \tag{6.20}$$

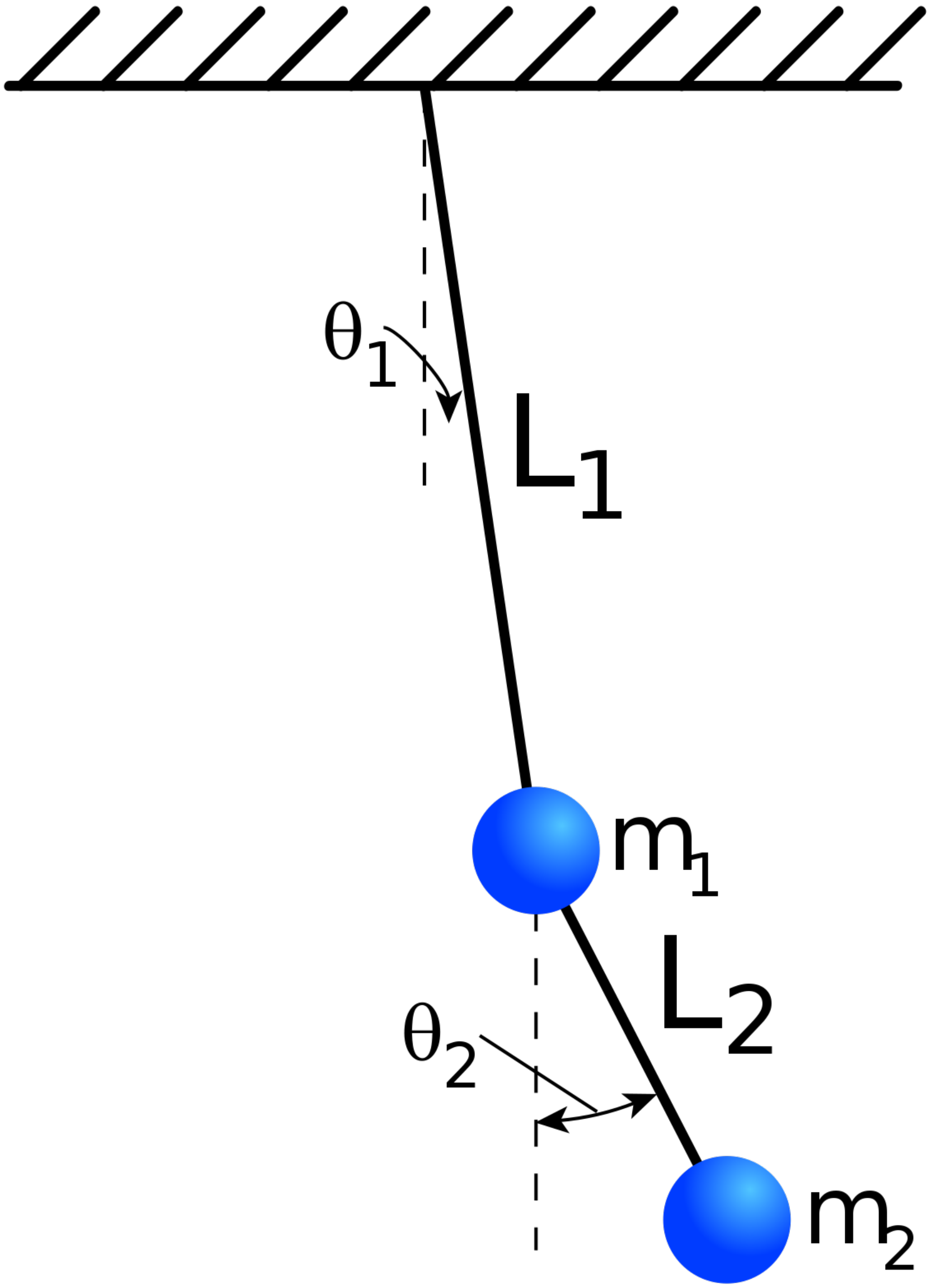
Therefore

$$\delta\ddot{q} = -\left(\frac{\partial^2 V}{\partial q^2}_0\right)\delta q \tag{6.21}$$

or

$$\omega^2 = \left(\frac{\partial^2 V}{\partial q^2} \geq 0\right) \tag{6.22}$$

6.3 Double Pendulum



We can set up the Lagrangian for this system,

$$U = -m_1 g l_1 \cos\theta_1 - m_2 g(l_1 \cos\theta_1 + l_2 \cos\theta_2) \tag{6.23}$$
$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \tag{6.24}$$

We can write the Cartesian coordinates in terms of lengths and angles, $x_1 = l_1 \cos\theta_1, y_1 = -l_1 \cos\theta_1$, and $x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2, y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$. To substitute into the Lagrangian, we can take derivatives. Eventually, we get the following for the kinetic energy:

$$T = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 \left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\right) \tag{6.25}$$

Therefore, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 \left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\right) + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2 \tag{6.26}$$

Then, we can apply the Euler-Lagrange equations on both θ_1 and θ_2. Consider the equilibrium case, where $\theta_1^{(0)} = \theta_2^{(0)} = 0$.

$$\theta_1 = \theta_1^{(0)} + \delta\theta_1 \tag{6.27}$$

We linearize the Lagrangian around the equilibrium point to first order, and drop some constants,

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2}(m_1 + m_2) g l_1 \theta_1^2 + \frac{1}{2}m_2 g l_2 \theta_2^2 \tag{6.28}$$

We introduce a coefficient for the mass ratio and the length, as well as the equilibrium angular frequency, to get

$$\mathcal{L} = \frac{1}{2}\dot{\theta}_1^2 + \frac{1}{2}\mu l^2 \dot{\theta}_2^2 + \mu l \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2}\omega_0^2 \theta_1^2 + \frac{1}{2}\mu l \omega_0^2 \theta_2^2 \tag{6.29}$$

where $\mu = \frac{m_2}{m_1 + m_2}$, $l = \frac{l_2}{l_1}$, and $\omega_0^2 = \frac{g}{l_1}$. We apply the Euler-Lagrange equations to θ_1 and get

$$\ddot{\theta}_1 + \mu l \dot{\theta}_2 = -\omega_0^2 \theta_1 \tag{6.30}$$
$$l\ddot{\theta}_2 + \ddot{\theta}_1 = -\omega_0^2 \theta_2 \tag{6.31}$$

This is a system of linear differential equations. We assume an arbitrary exponential solution for both ($\theta_j = A_j e^{i\omega t}$), and we get the following matrix,

$$\begin{bmatrix} \omega_0^2 - \omega^2 & -\mu l \omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 l \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.32}$$

We take the determinant, to find the eigenvalues of ω. We get

$$\omega_\pm^2 = \omega_0^2 \frac{1 + l \pm \sqrt{(1+l)^2 - 4(1-\mu)l}}{2(1-\mu)l} \tag{6.33}$$

and for a general solution based on the eigenvalues, we get

$$\theta_i = C_i^{(+)} A_i^{(+)} e^{i\omega_+ t} + C_i^{(-)} A_i^{(-)} e^{i\omega_- t} \tag{6.34}$$

$i\omega_\pm t$ are the normal modes of the system, and the C_i coefficieents are given by initial conditions.

From the matrix equation above, we get

$$A_1 = \frac{\mu l \omega_+^2}{\omega_0^2 - \omega_+^2} A_2 \tag{6.35}$$

6.4 Circle of Springs

Consider a set of four springs of spring constant k arranged in a circle. There are four masses m evenly spaced between them. The springs can be parameterized by their angles ϕ_i relative to the positive x axis:

$$\mathcal{L} = \frac{1}{2} R^2 \sum_{i=1}^{4} \dot{\varphi}_i^2 - \frac{1}{2} R^2 \frac{k}{m} \left((\varphi_1 - \varphi_2)^2 + (\varphi_1 - \varphi_4)^2 + (\varphi_2 - \varphi_3)^2 + (\varphi_3 - \varphi_4)^2 \right) \tag{6.36}$$

Here we have four possible E-L equations:

$$\ddot{\varphi}_1 = -\omega_0^2 (2\varphi_1 - \varphi_2 - \varphi_4) \tag{6.37}$$
$$\ddot{\varphi}_2 = -\omega_0^2 (2\varphi_2 - \varphi_1 - \varphi_3) \tag{6.38}$$
$$\ddot{\varphi}_3 = -\omega_0^2 (2\varphi_3 - \varphi_2 - \varphi_4) \tag{6.39}$$
$$\ddot{\varphi}_4 = -\omega_0^2 (2\varphi_4 - \varphi_1 - \varphi_3) \tag{6.40}$$

which we can change into a massive matrix problem,

$$\begin{bmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 & 0 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 & -\omega_0^2 & 0 \\ 0 & -\omega_0^2 & 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 0 & -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.41}$$

This has a determinant

$$||D|| = \pm(2\omega_0^2 - \omega^2)^2(4\omega_0^2 - \omega^2)\omega^2 \tag{6.42}$$

When this determinant is zero, $\omega = 0$, or $\omega = 2\omega_0$, or $\omega = \sqrt{2}\omega_0$.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 7: Oscillations

Lecturer: Stuart Bale, Ivan Vasko *12 February* *Aditya Sengupta*

7.1 Linear Oscillations

We can start the study of oscillations using the principle of energy conservation. Consider a one-dimensional case, where there is a force in one direction and an associated potential,

$$F_x = -\frac{\partial V(x)}{\partial x} \tag{7.1}$$

and the energy of the system undergoing this force is

$$E = \frac{1}{2}mv_x^2 + V(x) \tag{7.2}$$

This creates a roughly parabolic graph of potential vs x. There is a well at the bottom, at some $x = x_0$. This is an equilibrium point, where $\frac{dV}{dx} = 0$ and so $F_x = 0$. This is a case of stable equilibrium. If we had a potential well that looked like an inverted parabola, the zero point is also an equilibrium point, but it is unstable; if the particle is perturbed, it does not return to its equilibrium point.

Consider perturbations away from an equilibrium point, $x = x_0 + \eta$. We can Taylor expand about this point,

$$V(x) = V(x_0) + \eta \left.\frac{\partial V}{\partial x}\right|_{x_0} + \frac{1}{2}\eta^2 \left.\frac{\partial^2 V}{\partial x^2}\right|_{x_0} + \dots \tag{7.3}$$

The first-order derivative goes to zero, by definition of the equilibrium point. So the dynamics of a one-dimensional oscillator are entirely determined by the quadratic term. We can drop the constant term because a shift in the potential does not affect the dynamics.

We can write a Lagrangian,

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{\eta}^2 - \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x_0} \eta^2 \tag{7.4}$$

and apply the E-L equation to get the equation of motion,

$$m\ddot{\eta} + \left.\frac{\partial^2 V}{\partial x^2}\right|_{x_0} \eta = 0 \tag{7.5}$$

This is the equation for a simple harmonic oscillator. The solutions to this are of the form of a complex exponential,

$$\eta(t) = Ae^{i(\omega t+\theta)} \tag{7.6}$$

where $\omega = \sqrt{\frac{1}{m} \frac{\partial^2 V}{\partial x^2}\Big|_{x_0}}$ is the angular frequency. The motion is determined by the real part of $\eta(t)$, which is $A\cos(\omega t + \theta)$ for $\omega \in \mathbb{R}$.

The energy is given by

$$E = \frac{1}{2}m\dot{\eta}^2 + \frac{1}{2}k\eta^2 = \text{const} \tag{7.7}$$

We compute the η components in terms of the solution written above,

$$\eta^2 = A^2 \cos^2(\omega t + \theta) \tag{7.8}$$

$$\dot{\eta} = -\omega A \sin(\omega t + \theta) \tag{7.9}$$

$$\dot{\eta}^2 = \omega^2 A^2 \sin^2(\omega t + \theta) \tag{7.10}$$

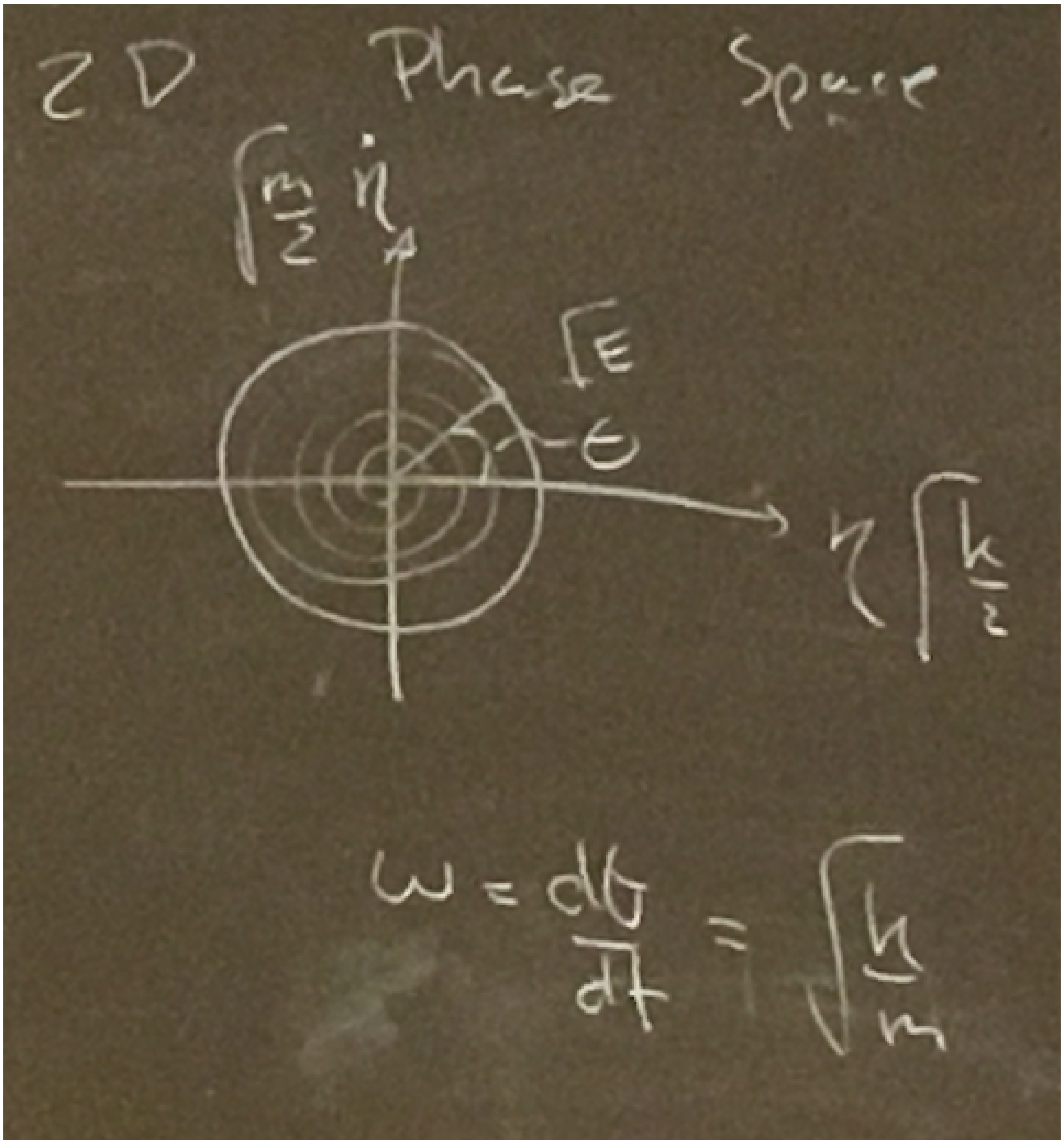
$$E = T + V = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \theta) + \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \theta) = \frac{1}{2}m\omega^2 A^2 \tag{7.11}$$

7.2 Phase space

We can parameterize a system by the values of η and $\dot{\eta}$. The total energy equation can be rewritten in terms of an ellipse,

$$(\sqrt{E})^2 = \left(\sqrt{\frac{m}{2}}\dot{\eta}\right)^2 + \left(\sqrt{\frac{k}{2}}\eta\right)^2 \tag{7.12}$$

and if we rescale the axes, this becomes a circle. The polar representation of a point on the circle is entirely determined by its angle θ, where $\omega = \frac{d\theta}{dt} = \sqrt{\frac{k}{m}}$.



7.3 Linearity

The simple harmonic oscillator is linear, meaning that solutions to its equation of motion have linearity (superposition and homogeneity). This means methods of linear algebra can be used.

SHO has the exponential solution

$$\eta(t) = \eta(0)e^{\alpha t} \tag{7.13}$$

which is invariant under time translation,

$$\eta(t+t_0) = \eta(0)e^{\alpha(t+t_0)} = \left(\eta(0)e^{\alpha t_0}\right)e^{\alpha t} \tag{7.14}$$

7.4 Damped SHO

Consider a simple harmonic oscillator with a mass on a spring, but with frictional forces proportional to the first derivative of the position. The differential equation of motion is

$$m\ddot{z} + b\dot{z} + kz = 0 \tag{7.15}$$

which can be rewritten as

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = 0 \tag{7.16}$$

where $2\beta = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$. To solve this, we assume an exponential of the form $z(t) = e^{pt}$ and take derivatives, which we substitute in to get

$$p^2 + 2\beta p + \omega_0^2 = 0 \tag{7.17}$$

$$p = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \tag{7.18}$$

Based on the sign of the term under the square root, there are three different cases.

7.4.1 $\beta > \omega_0$

In this case, the term is real, and so p is a negative real number. For $z(t)$ we get

$$z(t) = C_1 e^{(-\beta+\sqrt{\beta^2-\omega_0^2})t} + C_2 e^{(-\beta-\sqrt{\beta^2-\omega_0^2})t} \tag{7.19}$$

Here, the $e^{-\beta t}$ term dominates, and so the solution is a decaying exponential. We call the system *overdamped.*

7.4.2 $\beta < \omega_0$

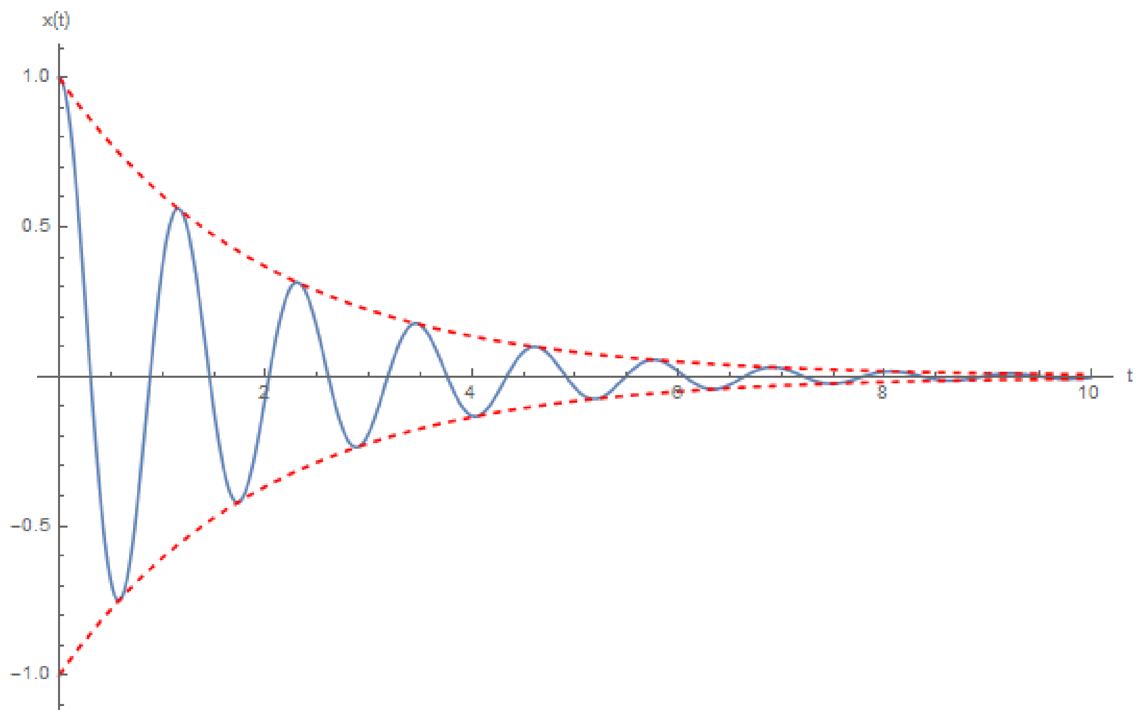
In this case, the term is imaginary, so we write it as $i\sqrt{\omega_0^2 - \beta^2} = i\omega_1$. Using Euler's formula for complex numbers, we find we can rewrite $z(t)$ in this case as

$$z(t) = e^{-\beta t}\left(C_1(\cos\omega_1 t + i\sin\omega_1 t) + C_2(\cos\omega_1 t + i\sin\omega_1 t)\right) \tag{7.20}$$

$$z(t) = e^{-\beta t}(d_1\cos\omega_1 t + d_2\sin\omega_1 t) \tag{7.21}$$

$$z(t) = Ae^{-\beta t}\cos(\omega_1 t + \varphi) \tag{7.22}$$

This solution is oscillating and decaying at the same time.



7.4.3 $\beta = \omega_0$

In the case $\beta = \omega_0$, we get $z(t) = (C_1 + C_2)e^{-\beta t}$, and we are required to find a second linearly independent solution. To resolve this, we look at the underdamped case; at $t = 0$, $z = z(0)$ and $\dot{z} = \dot{z}(0)$. From above, we know that $z(0) = d_1 = C_1 + C_2$ and $\dot{z}(0) = -\beta d_1 + \omega_1 d_2$. Therefore

$$d_1 = z(0), \;\; d_2 = \frac{\dot{z}(0) + \beta z(0)}{\omega_1} \tag{7.23}$$

We can substitute these into the general underdamped solution,

$$z(t) = e^{-\beta t}\left(z(0)\cos\omega_1 t + \left(\frac{\dot{z}(0) + \beta z(0)}{\omega_1}\sin\omega_1 t\right)\right) \tag{7.24}$$

We let $\omega_1 \to 0$ (because $\omega_0 \approx \beta$) and we get

$$z(t) \approx e^{-\beta t}\left(z(0) + i\left(\dot{z}(0) + \beta z(0)\right)t\right) \tag{7.25}$$

7.5 Analogies to circuits

Consider a circuit with components characterized by R, L, C in series. In a differential equation form, these interact with charge in the system the same way that the mass/damping/spring components interact with the position:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \tag{7.26}$$

Here $L \sim m$, $R \sim \beta$ and $C \sim \frac{1}{k}$.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 8: Forced Oscillators

Lecturer: Stuart Bale, Ivan Vasko *14 February* *Aditya Sengupta*

8.1 Solutions to the forced oscillator

In linear differential equations, we have linearity in the solutions. Consider a damped harmonic oscillator with two solutions z_1, z_2,

$$\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right) z_1(t) = 0 \tag{8.1}$$

$$\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right) z_2(t) = 0 \tag{8.2}$$

Then $z = z_1 + z_2$ is also a solution. Suppose now that the right-hand side of one of the solutions is nonzero, i.e. the solution is not homogeneous:

$$\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right) z_1(t) = \frac{F_1(t)}{m} \tag{8.3}$$

$$\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right) z_0(t) = 0 \tag{8.4}$$

Then $z = z_0 + z_1$ is a solution.

Say $F(t) = C_0 e^{-i\omega t}$, i.e. the force is harmonic. We can try a harmonic solution, $z(t) = \widetilde{A} e^{-i\omega t}$. Then, we can take derivatives and see that $\dot{z} = -i\omega z$, and $\ddot{z} = -\omega^2 z$. Therefore the equation becomes

$$\left(-\omega^2 + 2\beta i\omega + \omega_0^2\right) \widetilde{A} = C_0 \tag{8.5}$$

$$\widetilde{A} = \frac{C_0}{-\omega^2 + 2\beta i\omega + \omega_0^2} \tag{8.6}$$

This is the *particular solution.* This can be split into real and imaginary components,

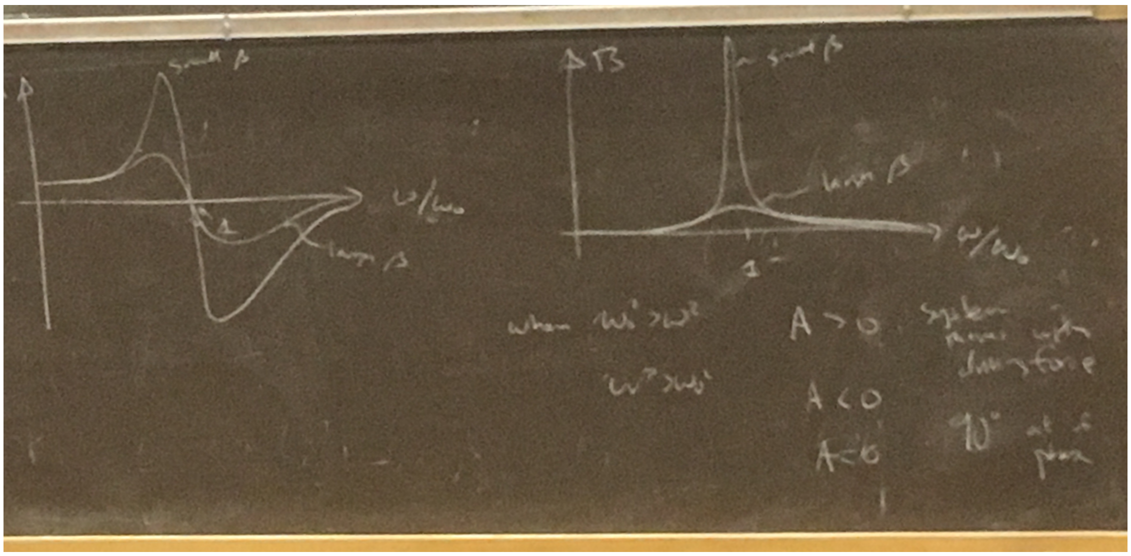
$$\widetilde{A} = \frac{F_0/m\left((\omega_0^2 - \omega^2) + 2\beta i\omega\right)}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} = A + iB \tag{8.7}$$

The *elastic amplitude* $A = \frac{F_0/m(\omega_0^2 - \omega^2)}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}$, and the *absorptive amplitude* $B = \frac{2\beta\omega F_0/m}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$.

We can split the particular solution into real and imaginary components based on these amplitudes,

$$z(t) = \widetilde{A}e^{-i\omega t} = (A + iB)(\cos\omega t - i\sin\omega t) = (A\cos\omega t + B\sin\omega t) + i(B\cos\omega t - A\sin\omega t) \tag{8.8}$$

We can plot A and B against ω/ω_0 to see its behaviour. When $\omega_0^2 > \omega^2$, $A > 0$, the system moves with the driving force; when $\omega^2 > \omega_0^2$, $A < 0$ and the system moves against the driving force; and when $A = 0$ the system is 90 degrees out of phase.



8.2 Power

Consider power, which is work over time, or force times velocity.

$$P(t) = F(t) \cdot \dot{z}(t) \tag{8.9}$$

$$P(t) = F_0 \cos\omega t \cdot \frac{\partial}{\partial t}(A\cos\omega t + B\sin\omega t) \tag{8.10}$$

$$P(t) = -F_0\omega A \sin\omega t \cos\omega t + F_0\omega\beta\cos^2\omega t \tag{8.11}$$

The first term averages to zero in a half cycle.

8.3 Rewriting solutions

We can express the real part of the general solution in the form of an amplitude times a trigonometric phase term,

$$\text{Re}(z) = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}\cos(\omega t - \varphi) \tag{8.12}$$

where $\varphi = \arctan\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)$.

We can also characterize the quality of a system by a parameter Q,

$$Q = \frac{\omega}{2\beta} = \frac{\sqrt{\omega_0^2 - \beta^2}}{2\beta} \tag{8.13}$$

This allows us to rewrite the system simply as

$$\ddot{z} + \frac{1}{Q}\dot{z} + z = 0 \tag{8.14}$$

A low Q factor corresponds to lots of damping, and a high Q factor (on the order of 5000-10000 for radio) corresponds to very little damping.

8.4 Arbitrary Driving Forces

Previously, we looked at harmonic driving forces. If $F(t)$ is not harmonic, this solution changes. Suppose $F(t) = F_0 u(-t+\tau)$, a constant input force that turns off for $t \geq \tau$. For the first part, we get $\ddot{z}+2\beta\dot{z}+\omega_0^2 z = F_0/m$, which has the solution

$$z(t) = e^{-\beta t}(C_1 \cos\omega_1 t + C_2 \sin\omega_1 t) + \frac{F_0}{m\omega_0^2} \tag{8.15}$$

We can Taylor expand to approximately get a solution after $z(t)$ settles to around the average,

$$z(t) = \frac{F_0}{m\omega_0^2}\left(1 - (1 - \beta t + \frac{\beta^2 t^2}{2}) - \frac{\beta t}{\omega_1}(1 - \beta t + \frac{\beta^2 t^2}{2})\omega_1 t\right) \approx \frac{F_0 t^2}{2m\omega_0^2}\left(\omega_1^2 - \beta^2\right) \tag{8.16}$$

After $t = \tau$, we use boundary conditions to ensure the function is valid. The general solution is

$$z_{II}(t) = e^{-\beta(t-\tau)}\left(D_1 \cos\omega_1(t-\tau) + D_2 \sin\omega_1(t-\tau)\right) \tag{8.17}$$

At $t = \tau$, we get $z_{II}(\tau) = D_1$ and $\dot{z}_{II}(\tau) = D_2\omega_1 + \beta D_1$. We can match this up with the first solution,

$$z_I(\tau) = \frac{F_0}{m\omega_0^2}\left((1 - e^{-\beta\tau})\cos\omega_1\tau - \frac{\beta}{\omega_1}e^{-\beta\tau}\sin\omega_1\tau\right) = D_1 \tag{8.18}$$

and the same with the derivative.

8.5 Green Function Solutions

We can also approach this by letting $\tau \to 0$ and $F_0 \to \infty$ such that the area under the force-time curve is constant. This suggests that the displacement can be written as an integral of the impulse,

$$z(t) = \int_{t'=-\infty}^{t'=t} G(t,t')F(t')dt' \tag{8.19}$$

where $G(t)$ is the Green function.

We can individually solve for the z due to each displacement, $\ddot{z}_i + 2\beta\dot{z}_i + \omega_0^2 z_i = F_i(t)$, and we get

$$z_i = \frac{I_1}{m\omega_1} e^{-\beta(t-t_1)} \sin\omega_1(t-t_1) \tag{8.20}$$

After $t > t_2$, this becomes

$$z = \frac{I_1}{m\omega_1} e^{-\beta(t-t_1)} + \frac{I_2}{m\omega_1} e^{-\beta(t-t_1)} \sin\omega_1(t-t_2) \tag{8.21}$$

(I missed a step)

In general, the Green function is

$$G(t,t') = \frac{e^{-\beta(t-t')} \sin\omega_1(t-t')}{m\omega_1} \tag{8.22}$$

We can apply this to a DSHO with $z(0) = \dot{z}(0) = 0$. We try $F(t') = \alpha t'$.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 9: Central Force Motion

Lecturer: Stuart Bale, Ivan Vasko *19 February* *Aditya Sengupta*

9.1 Central Forces

Recall that under Newtonian mechanics,

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt} \tag{9.1}$$

For some direction $\hat{s}$,

$$\vec{F} \cdot \hat{s} = 0 \implies \vec{p} \cdot \hat{s} = \text{const.} \tag{9.2}$$

That is, momentum is conserved along $\hat{s}$. Analogous to this, we can make a similar statement about torque,

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{9.3}$$

$$\vec{L} = \vec{r} \times \vec{p} \tag{9.4}$$

$$\vec{\tau} \cdot \hat{s} = 0 \implies \vec{L} \cdot \hat{s} = \text{const} \tag{9.5}$$

Suppose we wanted to set up a system in which angular momentum were conserved in any direction. This leads naturally to the notion of a *central force*, one in which a force depends on the position vector between objects,

$$\vec{F}(\vec{r}) = f(r)\hat{r} \tag{9.6}$$

The torque due to a central force is zero,

$$\tau = \vec{r} \times \vec{F} = \vec{r} \times \hat{r} f(r) = 0 \tag{9.7}$$

Therefore there is zero torque and angular momentum is conserved for any direction.

9.2 Two-Body Problem

Let there be masses m_1 and m_2 in the $x - y$ plane, with position vectors $\vec{r}_1$ and $\vec{r}_2$ from the origin of the inertial coordinate system to m_1 and to m_2 respectively. Define a position vector $\vec{r}$ from m_1 to m_2. By vector addition, we can say

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \tag{9.8}$$

Now, define a vector $\vec{R}$ that points from the origin to the CM. We have not yet found the CM, but the point defined by $\vec{R}$ will be lie somewhere on the line defined by $\vec{r}$.

We can write a Lagrangian,

$$\mathcal{L} = T(\dot{\vec{R}}, \dot{\vec{r}}) - U(\vec{r}, \dot{\vec{r}}) \tag{9.9}$$

This is not yet very specific. We can add terms to the coordinate system to make easy expressions for the energy. Call the vectors from the CM to m_1 and m_2 respectively $\vec{r}_1'$ and $\vec{r}_2'$. Then, we can say

$$\vec{R} + \vec{r}_1' = \vec{r}_1 \tag{9.10}$$
$$\vec{R} + \vec{r}_2' + \vec{r}_2 \tag{9.11}$$

or, differentiating,

$$\dot{\vec{r}}_1 + \dot{\vec{R}} + \dot{\vec{r}}_1' \tag{9.12}$$
$$\dot{\vec{r}}_2 + \dot{\vec{R}} + \dot{\vec{r}}_2' \tag{9.13}$$

So, the kinetic energy is

$$T = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 = \frac{1}{2}m_1\left(\dot{\vec{R}} + \dot{\vec{r}}_1'\right)^2 + \frac{1}{2}m_2(\dot{\vec{R}} + \dot{\vec{r}}_2')^2 \tag{9.14}$$
$$T = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}m_1\dot{r}_1'^2 + m_1\dot{R}\dot{r}_1' + \frac{1}{2}m_2\dot{r}_2'^2 + m_2\dot{R}\dot{r}_2' \tag{9.15}$$

This is a confusing expression, but the correct definition of the CM will save it.

$$\sum_i m_i\vec{r}_i = \vec{R}\sum_i m_i \tag{9.16}$$

This is essentially a weighted average. Using the expression we derived before, we can say

$$\sum_i m_i\vec{r}_i = \sum_i m_i\vec{R} + \sum_i m_i\vec{r}_i' \tag{9.17}$$

In our case, this simplifies (how?) to

$$m_1 r_1' + m_2 r_2' = 0 \tag{9.18}$$

Therefore the cross terms $m_1 \dot{R} \dot{r}_1'$ and $m_2 \dot{R} \dot{r}_2'$ cancel. We simplify T to

$$T = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2} m_1 \dot{r}_1'^2 + \frac{1}{2} m_2 \dot{r}_2'^2 \tag{9.19}$$

We want to convert this to an expression in r_1 and r_2 so that it matches up with the definition of the potential energy. We do this by the definition of the center of mass,

$$\vec{r}_2' = \frac{-m_1}{m_1 + m_2}\vec{r}\ \vec{r}_1' = \frac{m_2}{m_1 + m_2}\vec{r} \tag{9.20}$$

Therefore we can do algebra to simplify the noncentral terms in the kinetic energy,

$$\frac{1}{2} m_1 \dot{r}_1'^2 + \frac{1}{2} m_2 \dot{r}_2'^2 = \frac{1}{2}\left(\frac{m_1 m_2^2}{(m_1 + m_2)^2} + \frac{m_1^2 m_2}{(m_1 + m_2)^2}\right)\dot{r}^2 = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{r}^2 \tag{9.21}$$

This is equivalent to the kinetic energy of a single particle at the CM with the reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

$$T = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2 \tag{9.22}$$

The Lagrangian is

$$\mathcal{L} = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2 - U(r) \tag{9.23}$$

We can set up E-L equations in both coordinates. First we do this in R,

$$\frac{\partial \mathcal{L}}{\partial R} = 0 \tag{9.24}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{R}} = M \dot{R} \tag{9.25}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{R}} = \frac{\partial \mathcal{L}}{\partial R} = 0 \tag{9.26}$$

$$M \dot{R} = \text{const} \tag{9.27}$$

This is the statement of conservation of linear momentum of the center of mass.

Then, we do this in r. We can drop the $\frac{1}{2} M \dot{R}^2$ term for this. Here, the potential energy term matters, so to simplify we first take conservative forces that depend only on r.

$$\vec{F}(\vec{r}) = f(r)\hat{r} \tag{9.28}$$
$$\vec{F} = -\vec{\nabla}V(r) = f(r)\vec{r} \tag{9.29}$$
$$V(r) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}')d\vec{r}' \tag{9.30}$$

$\vec{F}$ being a central force implies conservation of angular momentum. This suggests that $\vec{r}$ and $\vec{p}$ are always in a plane normal to $\vec{L}$ (the plane is defined by the cross product of the two, i.e. the normal vector, being a constant). We describe this using polar coordinates in the plane, and the new Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{9.31}$$

We take derivatives,

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{9.32}$$
$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{9.33}$$
$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{9.34}$$

Therefore the angular momentum $mr^2\dot{\theta}$ is a constant. This can also be interpreted as an area swept out in polar coordinates, $\frac{1}{2}r(rd\theta) = dA$. We therefore find that assuming constant mass, $\frac{dA}{dt} = 0$. This gives us Kepler's second law, that the radius vector sweeps out equal areas in equal times.

In this, we have not specified the form of the potential, so this applies to all central forces.

We can find the equation of motion by taking derivatives in r,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0 \tag{9.35}$$
$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{9.36}$$
$$m\ddot{r} - mr\dot{\theta}^2 = f(r) \tag{9.37}$$

We can replace $\dot{\theta}$ by $\frac{l}{mr^2}$, to get a one-dimensional equation of motion,

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r) \tag{9.38}$$

There is no time dependence here, so energy is conserved as well. This comes from the fact that $\frac{\partial \mathcal{L}}{\partial t} = \frac{dH}{dt} = 0$. We can write an expression for the energy.

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V(r) = \text{const} \tag{9.39}$$

To solve for $r(t)$, we can try to integrate the equation of motion.

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{l}{mr^2} \tag{9.40}$$

$$\theta = \int d\theta = \int \frac{l}{mr^2}dt \tag{9.41}$$

$$\theta - \theta_0 = l\int_0^t \frac{dt'}{mr^2(t')} \tag{9.42}$$

Here, we need to find r to get θ. Using the energy equation, we can isolate $\dot{r}$ and get

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)} \tag{9.43}$$

which we can integrate for time,

$$t = \int dt = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}} \tag{9.44}$$

This allows us to define an "effective" potential,

$$V'(r) = V(r) + \frac{l^2}{2mr^2} \tag{9.45}$$

which allows us to define energy simply as $E = \frac{1}{2}m\dot{r}^2 + V'(r)$.

Now, we can start looking at what happens when we specify a form of the potential. Let $V = -\frac{k}{r}$, which implies $f = -\frac{k}{r^2}$. Then the effective potential is

$$V'(r) = -\frac{k}{r} + \frac{l^2}{2mr^2} \tag{9.46}$$

This allows us to graph the kinetic and potential energies. With a positive total energy, the potential energy becomes hyperbolic. With zero total energy, it is a parabola. With a negative total energy just above the bottom point of the potential well, the potential energy looks like an ellipse. At the bottom of the potential well, we have $\dot{r} = 0$ and a constant $r = r_1$. We get $f = -mr_1\dot{\theta}^2$ and a circular path.

Qualitatively, the potential dominates the centrifugal force at large r, which is equivalent to saying that V falls off slower than $\frac{1}{r^2}$. At small r, the centrifugal term dominates.

We can change any time dependence terms to θ dependence, and solve for $r(\theta)$ or $\theta(r)$ using the integral expressions in terms of t for both of these,

$$\int_{\theta_0}^{\theta} d\theta = \int_{r_0}^{r} \frac{l dr}{mr^2\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}} \tag{9.47}$$

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{k}{r^2}}} \tag{9.48}$$

In principle, we can now solve for $r(\theta)$.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 10: Central Force Motion

Lecturer: Stuart Bale, Ivan Vasko *28 February* *Aditya Sengupta*

10.1 Recap

In general, with a central force, we have

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r), \tag{10.1}$$

which has associated energy

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) \equiv \text{const.} \tag{10.2}$$

Based on this, we can integrate for θ in terms of r.

$$d\theta = \frac{ldr}{mr^2\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}} \tag{10.3}$$

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{dr}{mr^2\sqrt{\frac{2m(E-V)}{l^2} - \frac{1}{r^2}}} \tag{10.4}$$

Let $u = \frac{1}{r}$. Then the integral becomes

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2m(E-V)}{l^2} - u^2}} \tag{10.5}$$

If $f \sim \frac{1}{r^2}$, then $V = -\frac{k}{r} = -ku$. Using trig substitution, we can solve this:

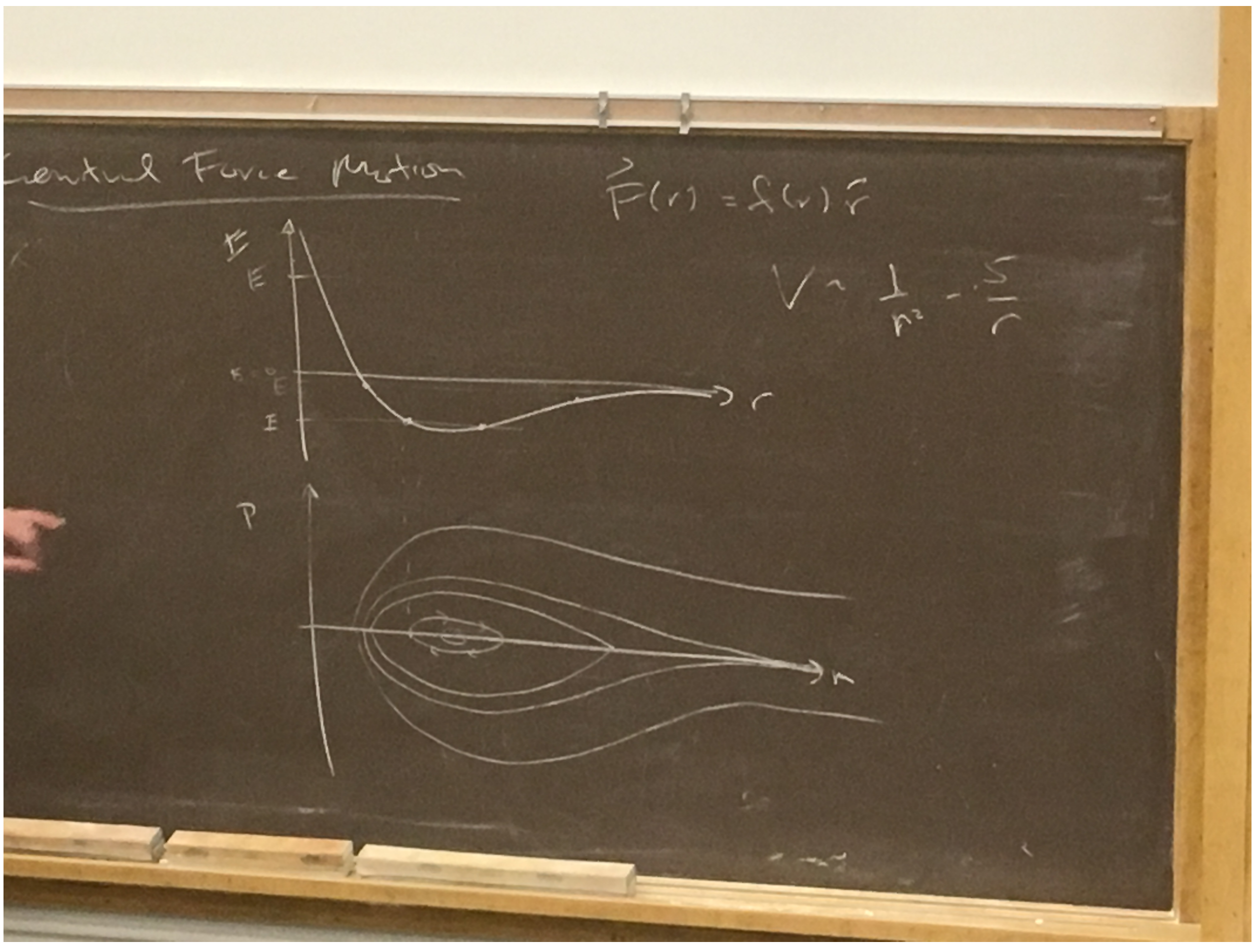
$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{mk^2}}\cos(\theta - \theta_0)\right) \tag{10.6}$$

10.2 Equation of Conic Sections

The above equation can be written simply as

$$\frac{1}{r} = C(1 + \epsilon \cos(\theta - \theta_0)), \tag{10.7}$$

where $\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$ is the eccentricity. We can construct a phase-space diagram with axes $\frac{p}{\sqrt{2m}}$ and $\sqrt{\frac{k}{2}}x$. Using this relationship as well as the energy/position dependence, we can construct diagrams representing the system. These can help us visualize the phase-space trajectories of the particle.



10.3 Limits of the Orbit

Consider the minimum value of r. This is achieved at the maximum value of the right hand side, at $\cos\theta = 1$. We get

$$\frac{1}{r_{min}} = \frac{mk}{l^2}(1 + \epsilon). \tag{10.8}$$

To find the maximum value of r, we break this up into two cases, depending on whether $\epsilon > 1$ or $\epsilon < 1$. We get

$$r_{max} = \begin{cases} \frac{l^2}{mk(1-\epsilon)} & \epsilon < 1 \\ \infty & \epsilon > 1 \end{cases} \tag{10.9}$$

We get ∞ in the $\epsilon > 1$ case because the right hand side of the equation can be zero.

In the case of a circle, we get simply

$$r = \frac{l^2}{mk}. \tag{10.10}$$

The fact that the radius is constant with θ reflects the fact that this is the equation of a circle. If we let $\alpha = \frac{l^2}{mk}$, then we can express the equation of the orbit in terms of the equivalent circular orbit,

$$\frac{1}{r} = \frac{1}{\alpha}(1 + \epsilon\cos\theta). \tag{10.11}$$

$$\alpha = r + \epsilon x \tag{10.12}$$

$$r^2 = \alpha^2 - 2\alpha\epsilon x + \epsilon^2 x^2 = x^2 + y^2 \tag{10.13}$$

We complete the square on this equation.

$$\frac{1}{a^2}\left(x + \frac{\alpha\epsilon}{1-\epsilon^2}\right)^2 + \frac{1}{b^2}y^2 = 1 \tag{10.14}$$

where

$$a = \frac{\alpha}{1-\epsilon^2}, b = \frac{\alpha}{\sqrt{1-\epsilon^2}}. \tag{10.15}$$

a and b define the semimajor and semiminor axes, respectively. This is the equation of an ellipse centered at $x_0 = -\frac{\alpha\epsilon}{1-\epsilon^2}$.

Suppose $\epsilon = 1$. We get

$$y^2 = \alpha^2 - 2\alpha x, \tag{10.16}$$

which is the equation of a parabola whose vertex is at $\left(\frac{\alpha}{2}, 0\right)$.

In the case $\epsilon > 1$, we get a hyperbola, through similar reasoning.

10.4 Kepler's Laws

We can connect the behaviour of central forces to Kepler's laws in orbital mechanics, specifically the second and third ones. The statement of equal areas being swept out in equal times can be expressed mathematically as

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2m}, \tag{10.17}$$

which is essentially a different statement of the conservation of angular momentum. Similarly, the third law states that for a gravitational force,

$$T^2 = \frac{4\pi^2 a^3}{GM_o}. \tag{10.18}$$

We start from the second law, and get

$$\frac{dA}{dt} = \frac{l}{2m} \implies A = \frac{l}{2m}T = \pi ab \tag{10.19}$$

Substituting in the relations between the axes and the eccentricity,

$$ab = a^2\sqrt{1-\epsilon^2} \tag{10.20}$$

$$\pi a^4 = \frac{l^2}{m(1-\epsilon^2)}\frac{T^2}{4m} \tag{10.21}$$

with $k = GMm$.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 11: Central Forces, Scattering, Hamiltonian Mechanics

Lecturer: Stuart Bale, Ivan Vasko *5 March* *Aditya Sengupta*

11.1 Higher-Order Forces

Previously, we found the orbits for a potential with $V \sim \frac{1}{r}$. Now, consider a force of the form $F(r) = \frac{k}{r^n}$, so that the potential is

$$V(r) = \frac{k}{n-1} r^{-(n-1)} \tag{11.1}$$

So the effective potential in a circular orbit becomes

$$V_{eff}(r) = \frac{l^2}{2mr^2} - \frac{k}{n-1}\frac{1}{r^{n-1}} \tag{11.2}$$

We take an r derivative and set it to zero, and we get

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{k}{r^n} - \frac{l^2}{mr^3} = 0 \tag{11.3}$$

$$r_0^{n-3} = \frac{mk}{l^2} \tag{11.4}$$

For stability, we look at the second derivative,

$$\frac{\partial^2 V}{\partial r^2} = -\frac{nk}{r_0^{n+3}} + \frac{3l^2}{mr_0^4} > 0 \tag{11.5}$$

$$(3-n)\frac{l^2}{m} > 0 \tag{11.6}$$

Therefore, for $n < 3$, we have stability in a circular orbit. We do not get this for $n \geq 3$.

The equation of motion is

$$m\ddot{r} - \frac{l^2}{mr^3} = -f(r) = -\frac{\partial \mathcal{L}}{\partial r} \tag{11.7}$$

and we can find the frequency of small oscillations around r_0 by setting $r = r_0 + \eta$ and $\ddot{r} = \ddot{\eta}$. We get

$$\ddot{\eta} - \frac{l^2}{mr_0^3\left(1+\frac{\eta}{r_0}\right)^3} = -g(r_0+\eta) \tag{11.8}$$

$$\ddot{\eta} - \frac{l^2}{m^2r_0^3}\left(1-\frac{3\eta}{r_0}\right) = -g(r_0) - \eta\frac{\partial g}{\partial \eta}\bigg|_{r_0} \tag{11.9}$$

where in general $g = g(r)$, and for the case of $\frac{1}{r^2}$ dependence g happens to be constant.

11.2 Scattering

Consider Rutherford's alpha particle scattering experiment, in which He^{2+} particles were fired through a lead "stop" into gold foil at a range of angles θ, then onto a zinc sulphide screen to see the effects of scattering. This can be formulated classically as a two-body problem. Let Θ be the scattering angle for a given particle, and let ψ be the angle between the path of the α particle and the path off the target of a reflected particle. $\psi = \frac{\pi}{2} - \frac{\Theta}{2}$.

Define the cross-section $d\sigma = \frac{d\sigma}{d\Omega}d\Omega$, the amount of flux per unit solid angle, where $d\Omega$ is the differential cross-section. Then σ can be found via an integral,

$$\sigma = \int \frac{d\sigma}{d\Omega}d\Omega = \int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi\frac{d\sigma}{d\Omega}(\theta,\varphi) \tag{11.10}$$

The flux density is defined as the number of particles per unit area per unit time; call this I. Over a differential element $d\Theta$, the number of particles scattered is

$$2\pi Is|ds| = 2\pi\sigma(\Theta)I\sin\Theta d\Theta \tag{11.11}$$

and so we get

$$\sigma(\Theta) = \frac{s}{\sin|\Theta|}\left|\frac{ds}{d\Theta}\right| \tag{11.12}$$

We can also invoke conservation of angular momentum, $l = mv_0s = s\sqrt{2mE}$. From the results in the previous section, we know that

$$\psi = \int_{r_1}^{\infty}\frac{dr}{r^2\left(\frac{2m(E-V(r)))}{l^2} - \frac{1}{r^2}\right)^{1/2}} \tag{11.13}$$

We know that this yields a hyperbolic orbit, and $k = -z_1z_2e^2$. Therefore

$$\frac{1}{r} = -\frac{mz_1z_2e^2}{l^2}(1 + \epsilon \cos \psi) \tag{11.14}$$

In the limit $r \to \infty$, we get $\epsilon = -\csc \frac{\theta}{2}$. Using the previous relationships we can get $s(\Theta)$,

$$\cot^2 \frac{\theta}{2} = \epsilon^2 - 1 = \frac{2Es}{z_1z_2e^2} \tag{11.15}$$

$$s = \frac{z_1z_2e^2}{2E} \cot \frac{\theta}{2} \tag{11.16}$$

$$\sigma(\Theta) = \frac{1}{4}\left(\frac{z_1z_2e^2}{2E}\right) \csc^4 \frac{\theta}{2} \tag{11.17}$$

11.3 Hamiltonian mechanics

Recall that in Lagrangian mechanics, a Lagrangian in each coordinate is a function $\mathcal{L}(q, \dot{q}, t) = T - V$, and the Euler-Lagrange equation allows us to find equations of motion based on this. Consider the *generalized momenta* $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$. Then, we define the Hamiltonian as a function of these generalized momenta,

$$H = \sum_i p_i \dot{q}_i - \mathcal{L}(q, \dot{q}(q, p), t) = H(q, p, t) \tag{11.18}$$

This exists in $2n$-dimensional *phase space*. We can derive an equation of motion from the Hamiltonian by taking a p derivative,

$$\frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\left(\sum p\dot{q} - L(q, \dot{q}, t)\right) = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p} = \dot{q} \tag{11.19}$$

and by taking a q derivative,

$$\frac{\partial H}{\partial q} = \frac{\partial}{\partial q}\left(\sum p\dot{q} - L(q, \dot{q}, t)\right) = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q} = p \tag{11.20}$$

Therefore

$$\frac{\partial H}{\partial q} = -\frac{\partial \mathcal{L}}{\partial q} = -\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = -\frac{d}{dt}p \tag{11.21}$$

This gives us *Hamilton's equations of motion*,

$$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}, \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{11.22}$$

This has a couple of advantages over the Lagrangian method. It is a system of first-order equations rather than second-order, and any time dependence is explicit.

11.4 Harmonic Oscillator

Consider a simple harmonic oscillator, with Lagrangian $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$. We first write down p,

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \implies \dot{x} = \frac{p}{m} \tag{11.23}$$

so the Hamiltonian is

$$H = p\dot{x} - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{p^2}{m} - \left(\frac{p^2}{2m} - \frac{1}{2}kx^2\right) = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{11.24}$$

Then, we use the Hamiltonian equations, we get

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \dot{p} = -\frac{\partial H}{\partial x} = kx, \frac{\partial H}{\partial t} = 0 \tag{11.25}$$

The third equation is essentially the statement of conservation of energy.

11.5 Particle in EM Field

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - e\varphi(r,t) + \frac{e}{2}\vec{\dot{r}} \cdot \vec{A}(\dot{r},t) \tag{11.26}$$

Using $\vec{B} = \vec{\nabla} \times \vec{A}$, we get $\vec{E} = -\vec{\nabla}\varphi - \frac{1}{2}\frac{\partial \vec{A}}{\partial t}$. Then, we take derivatives in x, to get

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + \frac{e}{2}A_x = p_x \tag{11.27}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -e\frac{\partial \varphi}{\partial x} + \frac{e}{2}\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right) \tag{11.28}$$

Bashing out algebra (come back here!), we get

$$H = \frac{1}{2m}\left(p - \frac{e}{2}\vec{A}\right) + e\phi(r,t) \tag{11.29}$$

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 12: The Hamiltonian

Lecturer: Stuart Bale, Ivan Vasko *7 March* *Aditya Sengupta*

12.1 Deriving the Hamiltonian equations

Recall that the Hamiltonian represents a system in terms of its generalized momenta,

$$H = p_i \dot{q}_i - L(q_i, \dot{q}_i(q_i, p_i), t) = H(q_i, p_i, t) \tag{12.1}$$

where $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is the canonical momentum. Then we get the equation of motion and conserved quantities from the Hamiltonian equations,

$$\dot{q} = \frac{\partial H}{\partial p} \tag{12.2}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{12.3}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{12.4}$$

We can derive these in the same way as for the Lagrangian, by requiring a stationary value of the action (which here becomes the phase(?)):

$$S = \int_{t_1}^{t_2} \mathcal{L} dt = \int_{t_1} t_2 \left(p\dot{q} - H\right) dt \tag{12.5}$$

Perturb $q \to q + \epsilon\eta$, and $p \to p + \epsilon x$. Then we get

$$S = \int_{t_1}^{t_2} \left((p + \epsilon x)(\dot{q} + \epsilon\dot{\eta}) - H(q + \epsilon\eta, p + \epsilon x, t)\right) dt \tag{12.6}$$

Keeping the terms that are linear in ϵ, we get the deviation,

$$\delta S = \epsilon \int \left(p_i \dot{\eta} + x\dot{q} - \frac{\partial H}{\partial q}\eta - \frac{\partial H}{\partial p} x\right) dt \tag{12.7}$$

We require that this is zero, meaning that the coefficients on η and x be zero. We apply integration by parts to shift the derivative in the first term from η to p_i,

$$p_i \dot{\eta} = \frac{d}{dt}(p\eta) - \eta\dot{p} \tag{12.8}$$

and so the deviation in S is

$$\delta S = \epsilon \int_{t_1}^{t_2} dt \left(\left(\dot{q} - \frac{\partial H}{\partial p} \right) x - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \eta \right) \tag{12.9}$$

Therefore, $\dot{q} - \frac{\partial H}{\partial p} = 0$ and $\dot{p} + \frac{\partial H}{\partial q} = 0$.

12.2 Connection to Quantum Mechanics

Intuitively, having a stationary phase corresponds to the path with the greatest probability due to no destructive addition of phases.

We can apply the Hamiltonian method to a wave packet described by $p = \hbar k$ and $E = \hbar\omega$. We use the Hamiltonian equations,

$$\dot{q} = \frac{\partial H}{\partial p} \implies v = \frac{dx}{dt} = \frac{\partial}{\partial p}(\hbar\omega) = \frac{\partial \omega}{\partial k} \tag{12.10}$$

$$\dot{p} = -\frac{\partial H}{\partial x} \implies \frac{dk}{dx} = -\frac{\partial H}{\partial x} \tag{12.11}$$

The first equation is the definition of the group velocity, and the second describes the time evolution of the wavepacket. In general, these are $\vec{v}_g = \vec{\nabla}_k \omega$ and $\frac{dk}{dt} = -\vec{\nabla}\omega$.

12.3 Legendre Transformation

A Legendre transformation allows us to describe a (necessarily convex) curve in terms of an intercept as a function of a slope.Consider the curve $y(x) = e^x$. At some point x, the slope is $\frac{\partial y}{\partial x} = e^x = m \implies x = \ln m$. At that point, $b = y - mx = y - m \ln m$. Therefore we get $b(m) = m - m \ln m$ as the Legendre representation.

Now, consider a function of two variables, and let the two parameterizing variables in the Legendre representation be the function's partial derivatives.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = u dx + v dy \tag{12.12}$$

Let $g = f - ux$. Then

$$dg = df - u dx - x du = (u dx + v dy) - y dx - x du = v dy - x du = \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial v} du \tag{12.13}$$

Then, define $v = \frac{\partial g}{\partial y}$ and $x = -\frac{\partial g}{\partial u}$. This is akin to a transformation $(q, \dot{q}, t) \rightarrow (q, p, t)$. From this perspective, a two-variable Legendre transformation is the same as the transformation from generalized positions to generalized momenta. We can do this explicitly in the Lagrangian,

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial q} dq + \frac{\partial \mathcal{L}}{\partial \dot{q}} d\dot{q} + \frac{\partial \mathcal{L}}{\partial t} dt \tag{12.14}$$

Set $\frac{\partial \mathcal{L}}{\partial q} = \dot{p}$ and $\frac{\partial \mathcal{L}}{\partial \dot{q}} = p$. Then, we get

$$d(\mathcal{L} - p\dot{q}) = \frac{\partial \mathcal{L}}{\partial q} dq + \frac{\partial \mathcal{L}}{\partial t} dt - \dot{q} dp \tag{12.15}$$

This suggests a definition for the Hamiltonian,

$$dH = \dot{q} dp - \dot{p} dq - \frac{\partial \mathcal{L}}{\partial t} dt = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial t} dt \tag{12.16}$$

Equating the coefficients on each differential element gives us the Hamiltonian equations of motion.

12.4 Solving a problem with Hamiltonian dynamics

1. Choose coordinates q_i and construct a Lagrangian.
2. Set $p = \frac{\partial \mathcal{L}}{\partial q_i}$.
3. $H(q, \dot{q}, p, t) = \dot{q}p - \mathcal{L}(q, \dot{q}, t)$.
4. Invert $p = \frac{\partial \mathcal{L}}{\partial q}$ to get $\dot{q}(q, p, t)$.
5. Eliminate $\dot{q}$ from H to get $H(q, p, t)$.
6. Use the Hamiltonian equations of motion to solve.

For example, consider the case of a particle moving in gravity.

$$\mathcal{L} = T - V = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - mgz \tag{12.17}$$

By taking partial derivatives, we find that $p_x = m\dot{x}$ and the same for the other two coordinates.

$$H = p\dot{q} - \mathcal{L} = \frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \left(\frac{1}{2} m \left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2} + \frac{p_z^2}{m^2} \right) - mgz \right) \tag{12.18}$$

which can be simplified to

$$H(q,p) = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + mgz \tag{12.19}$$

The Hamiltonian equations give us $\dot{x} = \frac{p_x}{m}$ and the same in y, and $F_z = \dot{p}_z = -\frac{\partial H}{\partial q} = -mg$.

12.5 Hamiltonian Phase Space

Consider a space parameterized by independent coordinate q and dependent coordinate p, and look at a box between (p,q) and $(p+\Delta p, q+\Delta q)$. The phase space density ρ is the density of points in this region. This allows us to find a number of particles crossing in a specific region in phase space. Consider the left and right faces, respectively going from (p,q) to $(p+\Delta p, q)$ and from $(p, q+\Delta q)$ to $(p+\Delta p, q+\Delta q)$. The number of particles crossing is $dN = \rho\Delta q\Delta p$, where $\Delta q = \frac{dq}{dt}dt = \dot{q}dt$. At faces 1 and 2, we can subtract this quantity, to get

$$\frac{\partial}{\partial q}(\rho\dot{q}) = \frac{\rho\dot{q}|_{q+\Delta q} - \rho\dot{q}|_q}{\Delta q} \tag{12.20}$$

or

$$dN_{12} = -\frac{\partial}{\partial q}(p\dot{q})\Delta q\Delta p dt \tag{12.21}$$

We can translate this to a continuity equation,

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial q}(p\dot{q}) + \frac{\partial}{\partial p}(\rho\dot{p}) = 0 \tag{12.22}$$

This is the Liouville theorem

In phase space, the continuity equation becomes

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) = 0 \tag{12.23}$$

$$\frac{\partial\rho}{\partial t} + \vec{\boldsymbol{\nabla}}\cdot(\rho\vec{v}) = 0 \tag{12.24}$$

which, applying the Hamiltonian equations, becomes

$$\frac{\partial\rho}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial\rho}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial\rho}{\partial p} = 0 \tag{12.25}$$

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 13: Applying Hamilton's Principle, Rigid Body Motion

Lecturer: Stuart Bale, Ivan Vasko *12 March* *Aditya Sengupta*

13.1 Coordinate Transforms

We are searching for coordinates that lead to a cyclic (i.e. conserved) coordinate. More concretely, we want to find $Q_i(p_i, q_i, t)$ and $P_i(q_i, p_i, t)$ that make H cyclic in Q, so that $H = H(P)$. The pair (q, p) is called canonically conjugate if Hamilton's equations hold and $(q, p) \iff (Q, P)$ is the canonical transform.

Hamilton's Principle states that $\delta \int \mathcal{L}(q, \dot{q}, t)dt = \delta \int L(Q, \dot{Q}, t)$, therefore

$$\delta \int (L - \mathcal{L})dt = 0 \tag{13.1}$$

Therefore L and $\mathcal{L}$ differ by some differential in time. Call this $\frac{dF}{dt}$, where $F = F(q_i, p_i, Q_i, P_i, t)$ is the generating function. Then, using the Fundamental Theorem of Calculus, we get

$$\delta \int_{t_0}^{t_1} \frac{dF}{dt} dt = \delta(F(t_1) - F(t_0)) = 0 \tag{13.2}$$

There are four kinds of generating functions, by taking pairwise each of p_i, q_i, P_i, Q_i and specifying the others as functions of these:

$$\begin{aligned} F_1 &= F_1(q_i, Q_i, t) \\ F_2 &= F_2(q_i, P_i, t) \\ F_3 &= F_3(p_i, Q_i, t) \\ F_4 &= F_4(p_i, P_i, t) \end{aligned}$$

For a type 1 function, for example,

$$L = \mathcal{L} + \frac{dP}{dt} = \sum p\dot{q} - H = \sum P\dot{Q} - \mathbb{H} + \frac{dF}{dt} \tag{13.3}$$

Based on the form of F, we can expand the derivative,

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t} \tag{13.4}$$

Therefore we get

$$\sum p\dot{q} - \sum P\dot{Q} - H + \mathbb{H} = \sum \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t} \tag{13.5}$$

By matching terms, we get

$$p_i = \frac{\partial F}{\partial q_i} \tag{13.6}$$

$$P_i = -\frac{\partial F}{\partial Q} \tag{13.7}$$

$$\mathbb{H} = H + \frac{\partial F}{\partial t} \tag{13.8}$$

13.2 Simple Harmonic Oscillator

We can apply this transformation to the case of an SHO. The Lagrangian is

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \tag{13.9}$$

Using the Hamiltonian equations, we get $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$. Therefore we can write down the Hamiltonian,

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \tag{13.10}$$

where $\omega^2 = \frac{k}{m}$. Now, we want to apply a coordinate transformation. We know that the solution is oscillating, so we try $p = f(P)\cos Q$ and $q = \frac{f(P)}{m\omega}\sin Q$. We get

$$p^2 + m^2\omega^2 q^2 = f(P)^2\cos^2 Q + f(P)^2\sin^2 Q = f(P)^2 \tag{13.11}$$

So

$$\mathbb{H} = \frac{f(P)^2}{2m} \tag{13.12}$$

to which we apply the Hamiltonian equations again to get $p = \frac{\partial F}{\partial q} = m\omega q \cot Q$ Therefore

$$F = \frac{m\omega q^2}{2} \cot Q \tag{13.13}$$

$$P = -\frac{\partial F}{\partial Q} = \frac{m\omega^2 q^2}{2} \frac{1}{\sin^2 Q} \tag{13.14}$$

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \tag{13.15}$$

$$p = m\omega \sqrt{\frac{2P}{m\omega}} \cos Q \tag{13.16}$$

Finally, we can write down the Hamiltonian in the new coordinates,

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = \frac{1}{2m}\left(2Pm\omega\cos^2 Q + m\omega \cdot 2P \sin^2 Q\right) \tag{13.17}$$

The total energy is then $E = \mathbb{H} = \omega P$. Therefore $P = \frac{E}{\omega}$, and the Hamiltonian equations give us the time-evolution of Q simply as

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega \implies Q = \omega t + Q_0 \tag{13.18}$$

Therefore, substituting in known values, we finally get

$$p = f(P)\cos Q = \sqrt{2mE}\cos(\omega t + Q_0) \tag{13.19}$$

$$\sqrt{\frac{2E}{m\omega}} \sin(\omega t + Q_0) \tag{13.20}$$

13.3 Rigid Body Motion

A rigid body is a collection of mass elements fixed with respect to each other, i.e. for all elements i, j, it is given that $\vec{r}_i - \vec{r}_j$ is constant.

Consider a rigid body rotating about some axis with angular velocity ω. Pick a point P on the axis. Then, the motion of the body can be described by the translation of P combined with a rotation about $\vec{\omega}$ through P.

The main relationship describing rigid body motion is $\vec{v} = \vec{\omega} \times \vec{r}$. This describes the tangential velocity of P. Each mass element δm has an angular momentum $\vec{L} = \vec{r} \times \vec{p}$. Summing these up, we get

$$\vec{L} = \int dm(\vec{r} \times (\vec{\omega} \times \vec{r})) \tag{13.21}$$

or in a system of discrete masses, we get

$$\vec{L} = \sum_i m_i \vec{r}_i \times (\omega \times \vec{r}_i) \tag{13.22}$$

Explicitly, the cross product being summed or integrated is

$$\begin{aligned} \vec{r} \times (\vec{\omega} \times \vec{r}) = (\omega_1(y^2+z^2) - \omega_2 xy - \omega_3 xz)\hat{x} \\ + (\omega_2(x^2+z^2) - \omega_3 yz - \omega_1 xy)\hat{y} \\ + (\omega_3(x^2+y^2) - \omega_1 zx - \omega_2 yz)\hat{z} \end{aligned} \tag{13.23}$$

which can be translated into a matrix relationship,

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} \int y^2+z^2 dm & -\int xydm & -\int zxdm \\ -\int xydm & \int z^2+x^2 dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int x^2+y^2 dm \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{13.24}$$

Compactly, we write this as $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$, where $L_i = I_{ij}\omega_j$. We note that the inertia tensor $\overleftrightarrow{I}$ is symmetric and positive definite. It depends on geometry and not on $\int \omega$. It is only specified after choosing an origin and coordinate system.

For example, a point mass in a plane has $\vec{\omega} = (0, 0, \omega_3)$, and $z = 0$. Since the first two components of ω are zero, only the rightmost column matters, of which two of the components are zero. Therefore the relationship reduces to

$$\vec{L} = \begin{bmatrix} 0 \\ 0 \\ \omega_3 \int r^2 dm \end{bmatrix} \tag{13.25}$$

which is the familiar resut $\vec{L} = mr^2\omega_3\hat{z} = mvr\hat{z}$.

The kinetic energy of a rigid body is given by $T = \frac{1}{2}\omega^T \overleftrightarrow{I} \omega = \frac{1}{2}\omega \cdot \vec{L}$ by definition of the angular momentum vector.

We can write everything in terms of center of mass coordinates, $\vec{r} = \vec{R} + \vec{r}'$. We get

$$\vec{L} = \int dm(\vec{r} \times \vec{v}) = \int (\vec{R} + \vec{r}') \times (\vec{v} + (\vec{\omega}' \times \vec{r}'))dm \tag{13.26}$$

which simplifies to

$$\vec{L} = \int (\vec{R} \times \vec{V})dm + \int \vec{r}' \times (\vec{\omega} \times \vec{r}')dm = M\vec{R} \times \vec{V} + \vec{L}_{CM} \tag{13.27}$$

Therefore the kinetic energy is

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\vec{\omega}' \cdot \vec{L}_{CM} \tag{13.28}$$

We want to find special basis vectors that diagonalize the inertia tensor. These are referred to as the principal axes, and the diagonal elements are called the principal moments. We do this by diagonalization.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 14: π Day

Lecturer: Stuart Bale, Ivan Vasko *14 March* *Aditya Sengupta*

14.1 Rigid Body Motion

We can write the kinetic energy of a rotating rigid body by the sum of its CM kinetic energy and that due to rotation,

$$T = \frac{1}{2}MV_C^2 + \frac{1}{2}\sum_{\alpha,\beta} I_{\alpha\beta}\omega_\alpha\omega_\beta = \frac{1}{2}MV_C^2 + \frac{1}{2}\vec{\omega}\hat{I}\vec{\omega} \tag{14.1}$$

where the components of the inertia tensor are given by

$$I_{\alpha,\beta} = \iiint \rho(\vec{r})(r^2\delta_{\alpha\beta} - x_\alpha x_\beta)d^3\vec{r} \tag{14.2}$$

We can write the Lagrangian for this system,

$$\mathcal{L} = T - V(\vec{r}_{CM}, \theta, \varphi, \psi) \tag{14.3}$$

We can use conservation of angular momentum to gain more information about this system.

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i \implies \dot{\vec{L}} = \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i = \vec{N} \tag{14.4}$$

We can write $\vec{F}_i = m_i \times (\vec{\omega} \times \vec{r}_i)$, so the angular momentum is

$$\vec{L} = \sum_i m_i \left(\vec{\omega}\vec{r}_i^2 - \vec{r}_i(\vec{\omega}\vec{r}_i)\right) \tag{14.5}$$

Comparing this to the definition $L_\alpha = I_{\alpha\beta}\omega_\beta$, we see that the above is the definition of the components of the inertia tensor.

14.2 Calculating the Inertia Tensor and Rotational Dynamics

To translate the inertia tensor from being centered around a point O to being centered around a point C, with position vector $\vec{R}$ between them, the formula is

$$I_{ij}^{(O)} = I_{ij}^{(C)} + M(R^2\delta_{ij} - X_iX_j) \tag{14.6}$$

Consider a rectangle in the $x-y$ plane with width a and height b, rotating about its positive-slope diagonal. In this case, we can compute the components of the inertia tensor,

$$I_{xy} = I_{yx} = \int dxdy\sigma_0(r^2\delta_{xy} - xy) = 0 \tag{14.7}$$

$$I_{xz} = I_{zx} = \int dxdz\sigma_0(r^2\delta_{xz} - xz) = 0 \tag{14.8}$$

$$I_{xx} = \int dxdy\sigma_0(r^2 - x^2) = \sigma_0 \int dxdyy^2 = \frac{1}{12}Mb^2 \tag{14.9}$$

$$I_{yy} = \frac{1}{12}Ma^2 \tag{14.10}$$

$$I_{zz} = \int \sigma_0(x^2 + y^2)dxdy = I_{xx} + I_{yy} = \frac{1}{12}M(a^2 + b^2) \tag{14.11}$$

In general, for any rigid body with principal axes along x, y, z, if a body rotates about any of these principal axes (say x) then its inertia tensor is diagonal, so $L_x = I_{xx}\omega_x$ and if there is rotation only along x, then $L_y = L_z = 0$.

Returning to the case of the rectangle, we write

$$L_x = I_{xx}\omega_x = I_{xx}\omega\cos\theta = \frac{1}{12}Mb^2\omega\cos\theta \tag{14.12}$$

$$L_y = I_{yy}\omega_y = I_{yy}\omega\sin\theta = \frac{1}{12}Ma^2\omega\sin\theta \tag{14.13}$$

$$L_z = 0 \tag{14.14}$$

We can use this definition and $\vec{\omega} = (\omega\cos\theta, \omega\sin\theta, 0)$ to write $\vec{N} = \frac{d\vec{L}}{dt}$ as

$$\vec{N} = (I_{xx} - I_{yy})\omega_x\omega_y\hat{z}(t) = \frac{1}{12}M(b^2 - a^2)\omega^2\frac{ab}{a^2 + b^2} \tag{14.15}$$

Consider a cube of side a with the origin at a corner. The components of the inertia tensor can be calculated by shifting relative to the center, $\vec{R} = \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$. (Get details later)

Cone:

$$\vec{N} = \left(\frac{d\vec{L}}{dt}\right)_L = \left(\frac{d\vec{L}}{dt}\right)_B + \vec{\omega} \times \vec{L} \tag{14.16}$$

From this, we find the components of $\vec{N}$,

$$N_1 = I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\omega_2\omega_3 \tag{14.17}$$

$$N_2 = I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\omega_3\omega_1 \tag{14.18}$$

$$N_3 = I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\omega_1\omega_2 \tag{14.19}$$

These equations describe the rotational dynamics of the system. We can solve these as a coupled system of differential equations to get

$$\vec{\omega} = (A\cos(\Omega t + \varphi), A\sin(\Omega t + \varphi), \omega_3) \tag{14.20}$$

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 15: Euler Equations, Rotation

Lecturer: Stuart Bale, Ivan Vasko *19 March* *Aditya Sengupta*

$$\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \tag{15.1}$$
$$\tau_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \tag{15.2}$$
$$\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \tag{15.3}$$

For small perturbations around the I_1 axis, we say ω_2 and ω_3 are small, so $\omega_2\omega_3 \approx 0$. From the Euler relations, we get $\dot{\omega}_1 \approx 0$, so ω_1 is approximately a constant. This gives us a frequency of small oscillations relationship in the ωs.

$$\ddot{\omega}_2 + \left(\frac{I_1 - I_3}{I_2}\right)\omega_1\dot{\omega}_3 = 0 \tag{15.4}$$
$$\ddot{\omega}_2 + \left(\frac{I_1 - I_3}{I_2}\right)\left(\frac{I_1 - I_2}{I_3}\right)\omega_1\omega_2 = 0 \tag{15.5}$$

Therefore we can define

$$\Omega_1 = \omega_1\sqrt{\left(\frac{I_1 - I_3}{I_2}\right)\left(\frac{I_1 - I_2}{I_3}\right)} \tag{15.6}$$

which admits the solution

$$\omega_2 = Ae^{i\Omega_1 t} + Be^{-i\Omega_1 t} \tag{15.7}$$

We can define rotation in terms of space coordinates $\vec{r}'$ and body coordinates $\vec{r}$ that rotate with the top. We define angles of rotation θ, φ, ψ to describe this. To change the coordinate system, we first rotate about the $\hat{z}$ axis by some angle φ, then around the α axis (the rotated x) by some angle θ, then finally around the γ' axis (the doubly rotated z) by some angle ψ.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{15.8}$$

The overall transformation matrix is

$$\begin{bmatrix} \cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta & \cos\psi\sin\varphi + \sin\psi\cos\varphi\cos\theta & \sin\theta\sin\psi \\ -\sin\psi\cos\varphi - \cos\psi\sin\varphi\cos\theta & -\sin\varphi\sin\psi + \cos\psi\cos\varphi\cos\theta & \sin\theta\cos\phi \\ \sin\theta\sin\phi & -\cos\varphi\sin\theta & \cos\theta \end{bmatrix} \tag{15.9}$$

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 16: Gravitational Torques

Lecturer: Stuart Bale, Ivan Vasko *2 April* *Aditya Sengupta*

Recall that we previously had two sets of coordinates: space coordinates $\vec{r}'$ which are fixed, and body coordinates $\vec{r}$ that rotate with the body. We call the transformation between them U, so that $\vec{r}' = U\vec{r}$ or $\vec{r} = U^{-1}\vec{r}' = \widetilde{U}\vec{r}'$.

By the product rule, the velocity in space coordinates is

$$\frac{d\vec{r}'}{dt} = \frac{dU}{dt}\vec{r} + U\frac{d\vec{r}}{dt} \tag{16.1}$$

i.e. the frame rotation plus the body velocity in space coordinates. This can be rewritten as

$$\vec{v}'_{space} = \vec{v}'_{body} + \dot{U}\vec{r} = \vec{v}'_{body} + \dot{U}\widetilde{U}U\vec{r} \tag{16.2}$$

Let $A = \dot{U}\widetilde{U}$. We know that $U\widetilde{U} = I$, so

$$\dot{U} + \widetilde{U} + U\dot{\widetilde{U}} = 0 \tag{16.3}$$

which tells us that A is antisymmetric. In terms of A, the space rotation velocity is

$$\vec{v}'_{space} = \vec{v}'_{body} + A\vec{r}' \tag{16.4}$$

where the $A\vec{r}'$ term is $\vec{\omega} \times \vec{r}$. Using the fact that it is antisymmetric and the definition of the cross product, we can write

$$A = \begin{bmatrix} 0 & -\omega'_3 & \omega'_2 \\ \omega'_2 & 0 & -\omega'_1 \\ -\omega'_2 & \omega'_1 & 0 \end{bmatrix} \tag{16.5}$$

In the Euler coordinates, with $\omega_{space} = (\omega'_1, \omega'_2, \omega'_3) = (A'_{32}, A'_{13}, A'_{21})$ and $\omega_{body} = (A_{32}, A_{13}, A_{21})$, we get

$$\vec{\omega}_{body} = \begin{bmatrix} \dot{\omega}\cos\phi + \dot{\varphi}\sin\psi\sin\theta \\ -\dot{\omega}\sin\psi + \dot{\varphi}\cos\psi\sin\theta \\ \dot{\psi} + \dot{\varphi}\cos\theta \end{bmatrix} \tag{16.6}$$

$$\vec{\omega}_{space} = \begin{bmatrix} \dot{\theta}\cos\varphi + \dot{\psi}\sin\varphi\sin\theta \\ \dot{\theta}\sin\varphi - \dot{\psi}\cos\varphi\sin\theta \\ \dot{\varphi} + \dot{\psi}\cos\theta \end{bmatrix} \tag{16.7}$$

Now, we can write down the kinetic energy $T = \frac{1}{2}\vec{\omega} \cdot (I \cdot \vec{\omega})$. Consider a symmetric top with $I_1 = I_2 = I$ and a distinct I_3, so that we get

$$T = \frac{1}{2} I(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 \tag{16.8}$$

Substituting in the ω_i values from above and simplifying, we get

$$T = \frac{I}{2}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\varphi}\cos\theta)^2 \tag{16.9}$$

Also, assume we have gravity, so that $V = mgl\cos\theta$. This gives us a Lagrangian. We calculate the generalized momenta in terms of the coordinates θ, φ, ψ. Since the potential energy only has θ dependence, φ and ψ are ignorable so their momenta are conserved.

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\varphi}\cos\theta) = I_3\omega_3 \tag{16.10}$$

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = I\dot{\varphi}\sin^2\theta + I_3(\dot{\psi} + \dot{\varphi}\cos\theta)\cos\theta \tag{16.11}$$

Define $a = \frac{I_3\omega_3}{I}$ and $b = \frac{p_\varphi}{I} = \dot{\varphi}\sin^2\theta + a\cos\theta$. Both of these are conserved.

The total energy is

$$E = T + V = \frac{I}{2}(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{I_3\omega_3^2}{2} + mgl\cos\theta \tag{16.12}$$

which we can express in terms of the parameters we introduced by substituting in $\dot{\varphi} = \frac{b - a\cos\theta}{\sin^2\theta}$, and by dropping the $I_3\omega_3^2/2$ term (because it is constant, equal to angular momentum). We get

$$E' = \frac{I}{2}\dot{\theta}^2 + \frac{I}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + mgl\cos\theta \tag{16.13}$$

Call the second and third terms combined $V_{eff}(\theta)$. We can solve for θ in the same way as the central force, but it may be easier to introduce a quantity $u = \cos\theta \implies \dot{u} = -\sin\theta\dot{\theta}$. The energy equation then gives us

$$\alpha(1 - u^2) = \dot{u}^2 + (b - au)^2 + \beta u(1 - u^2) \tag{16.14}$$

where $\alpha = \frac{2E'}{I}$ and $\beta = \frac{2mgl}{I}$.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 17: Small Oscillations Return

Lecturer: Stuart Bale, Ivan Vasko *9 April* *Aditya Sengupta*

Consider the simple case of a Lagrangian with one coordinate,

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x) \tag{17.1}$$

Solving the Euler-Lagrange equations, we get (by now predictably)

$$m\ddot{x} = -\frac{\partial V}{\partial x} \tag{17.2}$$

Consider a stationary point x_0. A stationary point is one at which if $\dot{x}_0 = 0$ then $\dot{x} = 0$. At x_0, $\frac{\partial V}{\partial x} = 0$.

We can also consider stability around a stationary point. Consider a point x_1 in a potential well, and x_2 at a potential peak. We can analyze stability by perturbing a coordinate and plugging it into the Lagrangian,

$$x = x_i + \delta x \tag{17.3}$$

$$\mathcal{L} = \frac{1}{2}m\delta\dot{x}^2 - V(x_i + \delta x) \tag{17.4}$$

Expanding, we get

$$\mathcal{L} = \frac{1}{2}m\delta\dot{x}^2 - V(x_i) - \left(\frac{\partial V}{\partial x}\right)_{x=x_i}\delta x - \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)_{x=x_i}\delta x^2 \tag{17.5}$$

We drop the V term as it is a constant, and we drop the first derivative in V because it is 0 by definition.

$$\mathcal{L} = \frac{1}{2}m\delta\dot{x}^2 - \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)_{x=x_2}\delta x^2 \tag{17.6}$$

This is the form of a harmonic oscillator, with a solution $\delta x = Ae^{-i\omega t}$ where $\omega^2 = \frac{1}{m}\left(\frac{d^2V}{dx^2}\right)_{x=x_i}$.

We can consider a more complicated case, with generalized coordinates $q = \{q_i\}_{i=1,\ldots,N}$. The kinetic energy is

$$T = \frac{1}{2}\sum T_{ik}\dot{q}_i\dot{q}_k \tag{17.7}$$

This can be derived from the known expression for kinetic energy,

$$T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i(q_k) \tag{17.8}$$

$$T = \frac{1}{2}\sum_{i,\alpha,\beta} m_i \frac{\partial \vec{r}_i}{\partial q_\alpha} \frac{\partial \vec{r}_i}{\partial q_\beta \dot{q}_\alpha \dot{q}_\beta} \tag{17.9}$$

$$T = \frac{1}{2}\sum_{\alpha,\beta}\left(\sum_i m_i \frac{\partial \vec{r}_i}{\partial q_\alpha}\frac{\partial \vec{r}_i}{\partial q_\beta}\dot{q}_\alpha \dot{q}_\beta\right) \tag{17.10}$$

From the kinetic energy, we subtract a generic $V(q)$ to make a Lagrangian. Solving the E-L equations in each q_α, we get

$$\sum_k T_{\alpha k}\ddot{q}_k = \frac{\partial \mathcal{L}}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha} \tag{17.11}$$

For a point to be stationary, you need $\frac{\partial V}{\partial q_\alpha} = 0$ for all α. (Notes: $T_{ik} = T_{ki}$, and $T_{ik} = T_{ik}(q)$). The above expression assumed T_{ik} constant, so the expression changes slightly with the coordinate dependence,

$$\frac{d}{dt}\left[\sum_k T_{\alpha k}\dot{q}_k\right] = \frac{1}{2}\sum_{i,k}\frac{\partial T_{ik}}{\partial q_\alpha}\dot{q}_i\dot{q}_k - \frac{\partial V}{\partial q_\alpha} \tag{17.12}$$

We can expand the left side as $\sum_{k,p}\frac{\partial T_{\alpha,k}}{\partial q_p}\dot{q}_p\dot{q}_k + \sum_k T_{\alpha k}\ddot{q}_k$, and cause a first-order perturbation in q_i to get

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} m_{ik}\dot{q}_i\dot{q}_k - \frac{1}{2}\sum_{ik} V_{ik} q_i q_k \tag{17.13}$$

m_{ik}, V_{ik} are constants. The equations of motion that we get out of this Lagrangian are $\sum_k m_{\alpha k}\ddot{q}_k = -\sum_k V_{\alpha k} q_k$. This is essentially a generalization of $m\ddot{x} = -kx$.

Using this, we can solve systems like the double pendulum. Consider a double pendulum with masses m on both arms and a length l of both. We describe the positions of both masses and the kinetic energy from that,

$$\vec{r}_1 = \langle l\sin\varphi_1, -l\cos\varphi_1\rangle \tag{17.14}$$

$$T_1 = \frac{1}{2}m\dot{x}_1^2 = \frac{1}{2}ml^2\dot{\varphi}_1^2 \tag{17.15}$$

$$\vec{r}_2 = \langle l\sin\varphi_1 + l\sin(\varphi_1+\varphi_2), -l\cos\varphi_1 - l\cos(\varphi_1+\varphi_2)\rangle \tag{17.16}$$

$$T_2 = \frac{1}{2}m\left(l^2\dot{\varphi}_1^2 + l^2(\dot{\varphi}_1+\dot{\varphi}_2)^2 + 2l^2\dot{\varphi}_1(\dot{\varphi}_1+\dot{\varphi}_2)\cos\varphi_2\right) \tag{17.17}$$

and we also write the gravitational potential energy,

$$V = -mgl(2\cos\varphi_1 + \cos(\varphi_1 + \varphi_2)) \tag{17.18}$$

The only stable stationary points, from the equation $\frac{\partial V}{\partial \varphi_i} = 0$, are $\varphi_1 = \varphi_2 = 0$. We can see this mathematically by expanding the expression for kinetic energy to get linear combinations of pairs of coordinates. We get

$$T = \frac{1}{2}ml^2\left((3 + 2\cos\varphi_2)\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1\dot{\varphi}_2(1 + \cos\varphi_2)\right) \tag{17.19}$$

We see that $T_{11} = (3 + 2\cos\varphi_2)ml^2, T_{12} = T_{21} = (1 + \cos\varphi_2)ml^2, T_{22} = ml^2$. Similarly we write out the potential energy in terms of paired coordinates, $V = \frac{1}{2}mgl(3\varphi_1^2 + 2\varphi_1\varphi_2 + \varphi_2^2)$. At the stationary point, we can simplify the kinetic energy to $T = \frac{1}{2}ml^2\left(5\dot{\varphi}_1^2 + 4\dot{\varphi}_1\dot{\varphi}_2 + \dot{\varphi}_2^2\right)$. Finally, this allows us to write out a Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(5\dot{\varphi}_1^2 + 4\dot{\varphi}_1\dot{\varphi}_2 + \dot{\varphi}_2^2\right) - \frac{1}{2}\frac{g}{l}(3\varphi_1^2 + 2\varphi_1\varphi_2 + \varphi_2^2) \tag{17.20}$$

Using this, we can write out matrices specifying the system's mass and potential constants,

$$m_ik = m\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \tag{17.21}$$

$$V_{ik} = mgl\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = ml^2\omega_0^2\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \tag{17.22}$$

We can solve this as a system of differential equations. Recall that the solutions for stable coordinates were completely oscillatory, $q_k = A_k e^{-i\omega t}$. From the equation we derived for the generic case previously, we get $(-\omega^2 m_{\alpha k} + V_{\alpha k})A_k = 0$. In a matrix form this is $||\hat{V} - \omega^2\hat{m}|| = 0$. This has N solutions for ω^2, which we will call ω_s. We get

$$\sum_k(-\omega_s^2 m_{ik} + V_{ik})A_k^{(s)} = 0 \tag{17.23}$$

$$\omega_s^2 = \frac{V_{ik}A_i^{(s)}}{m_ikA_i^{(s)}A_k^{(s)}} \tag{17.24}$$

Having found these eigenvalues, we make a generic linear combination to get the coordinates,

$$q_k = \sum_{s=1}^{N} C_s A_k^{(s)} e^{-i(\omega_s t + \varphi_s)} \tag{17.25}$$

From the earlier determinant relation, we get

$$\begin{vmatrix} 3\omega_0^2 - 5\omega^2 & \omega_0^2 - 2\omega^2 \\ \omega_0^2 - 2\omega^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0 \tag{17.26}$$

We get $\omega_1^2 = (2+\sqrt{2})\omega_0^2$ and $\omega_2^2 = (2-\sqrt{2})\omega_0^2$. These are eigenvalues, for which we can find the corresponding eigenvectors. For ω_1, this is

$$\begin{bmatrix} A_1^{(1)} \\ A_2^{(1)} \end{bmatrix} = \begin{bmatrix} 1 - \sqrt{2} \\ 1 \end{bmatrix} \tag{17.27}$$

and for ω_2, this is

$$\begin{bmatrix} A_1^{(2)} \\ A_2^{(2)} \end{bmatrix} = \begin{bmatrix} 1 + \sqrt{2} \\ 1 \end{bmatrix} \tag{17.28}$$

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 18: Generalized Lagrangians and Stability Analysis

Lecturer: Stuart Bale, Ivan Vasko *11 April* *Aditya Sengupta*

Recall that last time, we expressed the Lagrangian of any system in terms of generalized coordinates and wrote a generic component-wise expression for it in terms of kinetic and potential energy components:

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} m_{ik}\dot{q}_i\dot{q}_k - \frac{1}{2}\sum_{i,k} V_{ik}q_iq_k \tag{18.1}$$

We can write down the Hamiltonian,

$$\mathcal{H} = \sum p_j\dot{q}_j - \mathcal{L} = \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}q_j - \mathcal{L} \tag{18.2}$$

which requires that we write down the generalized momenta,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\sum_{i,k} T_{ik}\dot{q}_i\dot{q}_k\right) = \frac{1}{2}\sum T_{ik}\delta_{ij}\dot{q}_k + \frac{1}{2}\sum T_{ik}\dot{q}_i\delta kj \tag{18.3}$$

This can be simplified to

$$p_j = \frac{1}{2}\sum T_{ji}\dot{q}_i + \frac{1}{2}\sum T_{ij}\dot{q}_i = \sum_i T_{ij}\dot{q}_i \tag{18.4}$$

Then the Hamiltonian is

$$\mathcal{H} = \sum T_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum T_{ij}\dot{q}_i\dot{q}_j + V(q) = \frac{1}{2}\sum T_{ik}(q)\dot{q}_i\dot{q}_k + V(q) \tag{18.5}$$

Then the Hamiltonian is just the sum of kinetic and potential energies, $\mathcal{H} = T + V = \frac{1}{2}\sum m_{ik}\dot{q}_i\dot{q}_k + \frac{1}{2}\sum V_{ik}q_iq_k$.

To examine stability of a start point $q^{(0)}$, perturb each individual coordinate by some η_i. The condition for stability, i.e. for oscillation or decay of the perturbation, turns out to be $\sum V_{ik}q_iq_k > 0$.

By solving the previous eigenvalue equation, i.e. $(\hat{V} - \omega^2\hat{m})\vec{A} = 0$, we get the normal-mode frequencies,

$$\omega_s^2 = \frac{\sum V_{ik}A_i^{*(s)}A_k^{(s)}}{\sum m_{ik}A_i^{*(s)}A_k^{(s)}} \tag{18.6}$$

Again we cause a perturbation in each coordinate. If $V_{ik}\eta_i\eta_k > 0 \forall \eta_i$ where $||\eta|| \neq 0$ then the stability condition is met. (why?)

For any $\omega_s^2 \neq \omega_\alpha^2$, we should get $\sum_{i,k} m_{ik} A_i^{(\alpha)} A_k^{(s)} = 0$.

Having found the normal-mode frequencies, we can write down the normal modes,

$$q_k = \sum_s C_s A_k^{(s)} e^{-i\omega_s t} \tag{18.7}$$

where C_s may be complex, $C_s = |C_s|e^{i\varphi_s}$. We can apply this normal-mode construction to physical systems. Consider two springs of spring constant K, with masses m at the extreme ends and a mass M connecting the two in the middle. The Lagrangian for this system is

$$\mathcal{L} = \frac{1}{2}(m\dot{x}_1^2 + M\dot{x}_2^2 + m\dot{x}_3^2) - \frac{1}{2}\left(K(x_2 - x_1)^2 + K(x_3 - x_2)^2\right) \tag{18.8}$$

which allows us to write the m and V matrices,

$$m_{ik} = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix} \tag{18.9}$$

$$V_{ik} = K \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{18.10}$$

The determinant equation is then

$$\begin{vmatrix} K - \omega^2 m & -K & 0 \\ -K & 2K - \omega^2 M & -K \\ 0 & -K & K - \omega^2 m \end{vmatrix} = 0 \tag{18.11}$$

which admits the solutions $\omega = 0, \omega^2 = \frac{K}{m}, \omega^2 = \frac{K}{m}\left(1 - 2\frac{m}{M}\right)$. We can substitute these back in to find the corresponding eigenvectors (but they're on the far board and I'm too lazy to do it myself.) Eventually we get the amplitudes $A_1 = A_3$ and $A_2 = -2\frac{m}{M}A_1$.

Consider N masses in a line separated by equilibrium distance d and with tension τ pulling each mass up or down by an amount y_i. The force in x is $F_x = \tau\cos\alpha - \tau\cos\beta \approx \tau\left(\frac{\alpha^2}{2} - \frac{\beta^2}{2}\right)$, and that in y is $F_y = \tau\sin\alpha + \tau\sin\beta$. The angles can be found geometrically by

$$\sin\alpha_{k-1} = \frac{y_{k-1} - y_k}{\sqrt{d^2 + (y_k - y_{k-1})^2}} \approx \frac{y_k - y_{k-1}}{d} \tag{18.12}$$

$$\sin\beta_{k-1} \approx \frac{y_{k-1} - y_{k-2}}{d} \tag{18.13}$$

The final Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2} m \sum_{k} \dot{y}_k^2 - \sum_{k=0}^{N} \frac{\tau}{2d} \left(y_k - y_{k+1} \right)^2 \tag{18.14}$$

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 19: Coupled Oscillators contd.

Lecturer: Stuart Bale, Ivan Vasko *16 April* *Aditya Sengupta*

Consider the Lagrangian from last time,

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} m_{ik}\dot{y}_k\dot{y}_i - \frac{1}{2}\sum_{i,k} V_{ik}y_iy_k \tag{19.1}$$

where $m_{ik} = m\delta_{ik}$, i.e. the identity matrix, and $V_{ik} = \frac{\tau}{d}\begin{bmatrix} -2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix} k$.

We get this matrix by taking derivatives on the k-th part of the potential,

$$V = \frac{1}{2}\left(\frac{\tau}{d}(y_k - y_{k+1})^2 + \frac{\tau}{d}(y_k - y_{k-1})^2\right) \tag{19.2}$$

$$V = \frac{1}{2}\frac{\tau}{d}(2y_k^2 - 2y_ky_{k+1} - 2y_ky_{k-1} + y_{k+1}^2 + y_{k-1}^2) \tag{19.3}$$

$$\frac{\partial V}{\partial y_k} = \frac{1}{2}\frac{\tau}{d}(4y_k - 2y_{k+1} - 2y_{k-1}) \tag{19.4}$$

$$\frac{\partial^2 V}{\partial y_k^2} = 2\frac{\tau}{d} \tag{19.5}$$

$$\frac{\partial^2 V}{\partial y_k \partial y_{k+1}} = -\frac{\tau}{d} \tag{19.6}$$

Then, we want to solve the eigenvalue equation $||\hat{V} - \omega^2\hat{m}|| = 0$. We end up getting $2 - \frac{\omega^2}{\omega_0^2}$ all along the diagonals, -1 in the positions one place off from the diagonal, and 0 everywhere else.

For the case $N = 2$ this ends up giving us $\omega = \omega_0$ and $\omega = \sqrt{3}\omega_0$. For the case $N = 3$ this gives us $\omega = \omega_0\sqrt{2}$ and $\omega = \omega_0\sqrt{2 \pm \sqrt{2}}$. Here $\omega_0^2 = \frac{\tau}{md}$.

In general, for the Nth-order eigenfrequencies, we can compute the determinant Λ_n denoting the determinant $||\hat{V} - \omega^2\hat{m}||$. We get $\Lambda_1 = x$ (where we take $x = 2 - \frac{\omega^2}{\omega_0^2}$) and $\Lambda_N = x\Lambda_{N-1} - \Lambda_{N-2}$.

In general we can solve the equation of motion for the k-th mass with the assumption $y_k = A_k\cos(\omega t + \varphi)$ where we assume $A_k = e^{i(k\gamma+\delta)}$ where γ is a constant. Then we get

$$m\ddot{y}_k = \frac{\partial\mathcal{L}}{\partial y_k} = -\frac{\tau}{d}(y_k - y_{k+1}) - \frac{\tau}{d}(y_k - y_{k-1}) \tag{19.7}$$

$$\omega^2 A_k = \omega_0^2(A_k - A_{k+1}) + \omega_0^2(A_k - A_{k-1}) \tag{19.8}$$

$$\omega^2 = \omega_0^2(1 - e^{i\gamma}) + \omega_0^2(1 - e^{-i\gamma}) \tag{19.9}$$

$$\omega^2 = 4\omega_0^2\sin^2\frac{\gamma}{2} \tag{19.10}$$

We further constrain this at the endpoints, $y_0 = y_{N+1} = 0$. To fit both of these we get $\gamma = \frac{\pi n}{N+1}$ where $n = 1, \ldots, N$, and $\delta = \frac{\pi}{2}, \frac{3\pi}{2}$. Then we get $\omega_n = 2\omega_0 \left|\sin\left(\frac{\pi n}{2(N+1)}\right)\right|$.

In the limit where there is an infinite number of particles, $N \to \infty$ and $d, m \to 0$. Here the eqation for the k-th mass becomes a continuous equation $y(x,t)$. In the limit, we can rewrite the difference $y_k - y_{k+1} \approx y_{k+\frac{1}{2}} d$; we get

$$\frac{\partial^2 y}{\partial t^2} = \omega_0^2 \cdot \frac{\partial y_{k+1/2}}{\partial x} d - \omega_0^2 \frac{\partial y_{k-1/2}}{\partial x} d \tag{19.11}$$

$$\frac{\partial^2 y}{\partial t^2} = \omega_0^2 d^2 \frac{\partial^2 y}{\partial x^2} \tag{19.12}$$

This is the wave equation with $c_s^2 = \frac{\tau}{m/d}$.

To solve this, we assume separable solutions $y = A(x)\cos(\omega t + \varphi)$. Substituting into the equation, we get $A''(x) + \frac{\omega^2}{c_s^2} A(x) = 0$. With the boundary conditions $y(x=0) = y(x=L) = 0$, we get a sinusoidal form for A, namely $A = asin\left(\frac{\omega}{c_s} x\right)$ where $\omega = \frac{n\pi}{L} c_s$.

These are standing wave solutions to the wave equation. It is also possible to have traveling-wave solutions of the form $y = Ae^{i(\omega t - kx)}$.

It is also possible to write down a Lagrangian in this continuous case,

$$L[y] = \frac{1}{2}\int_0^L \rho(x) \left(\frac{\partial y}{\partial t}\right)^2 dx - \frac{1}{2}\int dx \tau(x) \left(\frac{\partial y}{\partial x}\right)^2 \tag{19.13}$$

or, as a function of the discrete case,

$$L[y] = \int dx \mathcal{L}(y, \partial_x y, \partial_t y, x, t) \tag{19.14}$$

$$\mathcal{L} = \frac{1}{2}\rho(x)\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{\partial y}{\partial x}\right)^2 \tag{19.15}$$

We refer to $\mathcal{L}$ here as the Lagrange density. From the principle of least action we know that $\delta \int_{t_1}^{t_2} L dt = 0$, from which we can say that

$$\delta \int_{t_1}^{t_2} L[y] dt = 0 \tag{19.16}$$

$$\delta \int_{t_1}^{t_2} \int_D \mathcal{L}(x, t, y, \partial_x y, \partial_t y) = 0 \tag{19.17}$$

$$\int_{t_1}^{t_2} \int_D dx dt \left[\mathcal{L}(x, t, y + \delta y, \partial_x y + \partial x \delta y, \partial_t y + \partial_t \delta y) - \mathcal{L}(x, t, y, \partial_x y, \partial_t y)\right] = 0 \tag{19.18}$$

Taking partial derivatives, we get

$$\iint dxdt\frac{\partial\mathcal{L}}{\partial y}\delta y - \iint dxdt\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\partial_x y}\right)\delta y - \iint dxdt\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\partial_t y}\right)\delta y = 0 \tag{19.19}$$

Using integration by parts, we can simplify this case (somehow) to

$$\rho(x)\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(\tau(x)\frac{\partial y}{\partial x}\right) \tag{19.20}$$

Similarly to the Lagrangian density, we can define a Hamiltonian density. Recall that $\mathcal{H} = p\partial_t y - \mathcal{L} = \frac{\partial\mathcal{L}}{\partial\partial_t y}\partial_t y - \mathcal{L}$. We differentiate this in time to get

$$\frac{\partial\mathcal{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\partial_t y}\right)\partial_t y + \frac{\partial\mathcal{L}}{\partial\partial_t y}\frac{\partial^2 y}{\partial t^2} - \frac{\partial\mathcal{L}}{\partial t} - \frac{\partial\mathcal{L}}{\partial y}\frac{\partial y}{\partial t} - \frac{\partial\mathcal{L}}{\partial\partial_x y}\partial_t\left(\frac{\partial y}{\partial x}\right) - \frac{\partial\mathcal{L}}{\partial\partial_t y}\frac{\partial^2 y}{\partial t^2} \tag{19.21}$$

Simplifying significantly, we get a density in time,

$$\frac{\partial\mathcal{H}}{\partial t} = -\frac{\partial\mathcal{L}}{\partial t} - \frac{\partial}{\partial x}\left[\frac{\partial\mathcal{L}}{\partial\partial_x y}\frac{\partial y}{\partial t}\right] \tag{19.22}$$

In the case with no time dependence, we get $\frac{\partial\mathcal{H}}{\partial t} = -\frac{\partial S}{\partial x}$ where $S = \frac{\partial\mathcal{L}}{\partial\partial_x y}\frac{\partial y}{\partial t} = \rho(x)\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}$ is the Poynting flux.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 20: Perturbation of Normal Modes

Lecturer: Stuart Bale, Ivan Vasko *18 April* *Aditya Sengupta*

Let $\zeta = x - c_s t$ and let $\eta = x + c_s t$ represent the perturbations to $y(x,t) \to y(\zeta, \eta)$. Taking derivatives, we get that

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial \zeta^2} + 2\frac{\partial^2 y}{\partial \zeta \partial \eta} + \frac{\partial^2 y}{\partial \eta^2} \tag{20.1}$$

and

$$\frac{\partial^2 y}{\partial t^2} = c_s^2 \frac{\partial^2 y}{\partial \zeta^2} - 2c_s \frac{\partial^2 y}{\partial \zeta \partial \eta} + c_s^2 \frac{\partial^2 y}{\partial \eta^2} \tag{20.2}$$

Therefore, substituting into the wave equation, we see that $\frac{\partial^2 y}{\partial \zeta \partial \eta} = 0$, i.e. $y = \frac{\int f(\eta) d\eta}{f(\eta)} + g(\zeta)$. We get a solution that is of the form of a traveling wave,

$$y(x,t) = f(x - c_s t) + g(x + c_s t) \tag{20.3}$$

These are arbitrary functions determined by intial conditions. At $t = 0$, $y_0(x) = f(x) + g(x)$ and $\frac{dg}{dx} - \frac{df}{dx} = \frac{1}{c_s}\dot{y}_0(x)$. These give us

$$g(x) - f(x) = \frac{1}{c_s}\int_{x_0}^{x} y_0(x')dx' + g(x_0) - f(x_0) \tag{20.4}$$

which allows us to isolate for both perturbations individually,

$$f(x) = \frac{1}{2}y_0(x) + \frac{1}{2c_s}\int_{x_0}^{x} y_0(x')dx' - A_0 \tag{20.5}$$

$$g(x) = \frac{1}{2}y_0(x) + \frac{1}{2c_s}\int_{x_0}^{x} y_0(x')dx' \tag{20.6}$$

So the general solution ends up being

$$y(x,t) = \frac{1}{2}\left[y_0(x - c_s t) + y_0(x + c_s t)\right] + \frac{1}{2c_s}\int_{x - c_s t}^{x + c_s t} \dot{y}_0(x')dx' \tag{20.7}$$

In general, for an arbitrary number of dimensions, the wave equation generalizes to the Helmholtz equation,

$$\vec{\nabla}^2 A(\vec{r}) + \frac{\omega^2}{c_s^2} A(\vec{r}) = 0 \tag{20.8}$$

for a standing wave of the form $\psi(\vec{r}, t) = A(\vec{r}) e^{-i\omega t}$. The wave can be expanded into a Fourier series; we find the cosine and sine coefficients are

$$a_n = \frac{2}{L} \int_0^L \psi(x, 0) \sin\left(\frac{\pi n}{L} x\right) dx \tag{20.9}$$

$$b_n = \frac{2}{L\omega_n} \left(\frac{\partial \psi}{\partial t}\right)_{t=0} \sin\left(\frac{\pi n}{L} x\right) dx \tag{20.10}$$

For the magnitude, assume it is separable into $A(x, y) = X(x)Y(y)$, and taking derivatives gives us

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{\omega^2}{c_s^2} = 0 \tag{20.11}$$

Let's say $X''(x) = -\lambda X(x)$ and $Y''(y) = -\mu Y(y)$. This gives us $\frac{\omega^2}{c_s^2} = \lambda + \mu$, so $\omega_{nm} = c_s \frac{\pi}{L} \sqrt{n^2 + m^2}$ where $n, m = 1, \ldots, \infty$.

In two-dimensional polar coordinates, it turns out we only have radial dependence and we can explicitly solve for the φ dependence:

$$A(r, \varphi) = R(r) \times e^{-im\varphi} \tag{20.12}$$

Substituting into the Helmholtz equation and using the del operator for polar coordinates, we get

$$R''(r) + \frac{1}{r} R'(r) + \left(\frac{\omega^2}{c_s^2} - \frac{m^2}{r^2}\right) R = 0 \tag{20.13}$$

This is the Bessel equation. Its solutions are represented as $R = J_m\left(\frac{\omega}{c_s} r\right)$. Therefore we get

$$\psi_{nm} = J_m\left(\frac{\omega_{nm}}{c_s} r\right) \times \begin{bmatrix} \cos(m\varphi) \\ \sin(m\varphi) \end{bmatrix} \times e^{-i\omega_{nm} t} \tag{20.14}$$

(the above is not a matrix, it means that either one can be selected. It's weird notation.)

We generalize this to having r and z dependence; we get

$$R'' + \frac{1}{r} R' - \frac{m^2}{r^2} R + \frac{\omega^2}{c_s^2} R - \left(\frac{\pi n}{L}\right)^2 R = 0 \tag{20.15}$$

where we found using the assumption of a constant Z''/Z that $Z = \sin\left(\frac{\pi n}{L} z\right)$.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 21: Nonlinear Mechanics

Lecturer: Stuart Bale, Ivan Vasko *23 April* *Aditya Sengupta*

Consider a system of two springs, one attached at a wall at $y = -l$ and one attached at a wall at $y = l$, and both attached to a mass on the x axis. The springs' relaxed length is l_0, so we can write down a potential.

$$U = \frac{1}{2}k\left(\sqrt{x^2 + l^2} - l_0\right)^2 \times 2 \tag{21.1}$$

We Taylor expand this to fourth order,

$$U(x) = U(0) + \left.\frac{\partial U}{\partial x}\right|_{x=0} x + \frac{1}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=0} x^2 + \frac{1}{6}\left.\frac{\partial^3 U}{\partial x^3}\right|_{x=0} x^3 + \frac{1}{24}\left.\frac{\partial^4 U}{\partial x^4}\right|_{x=0} x^4 \tag{21.2}$$

Computing each term, we get

$$U(0) = k(l - l_0)^2 \tag{21.3}$$

$$\frac{\partial U}{\partial x} = 2k\left(x - \frac{xl_0}{\sqrt{l^2 + x^2}}\right) \tag{21.4}$$

$$\frac{\partial^2 U}{\partial x^2} = 2k\left(1 - \frac{l_0 l^2}{(l^2 + x^2)^{3/2}}\right) \tag{21.5}$$

$$\frac{\partial^3 U}{\partial x^3} = 6kl_0 l^2 (x^2 + l^2)^{-3/2} x \tag{21.6}$$

$$\frac{\partial^4 U}{\partial x^4} = 6kl_0 l^2 \left(-\frac{5}{2}(x^2 + l^2)^{3/2}(2xk) + (x^2 + l^2)^{-5/2}\right) \tag{21.7}$$

Therefore, substituting in $x = 0$ to each of these, we get

$$U(x) \approx U(0) + k\left(1 - \frac{l_0}{l}\right)x^2 + \frac{1}{4}\frac{kl_0}{l^3}x^4 + \dots \tag{21.8}$$

and the force is

$$F = -\frac{\partial U}{\partial x} = -2k\left(1 - \frac{l_0}{l}\right)x - \frac{kl_0}{l^3}x^3 + \dots \tag{21.9}$$

The equation of motion is therefore

$$m\ddot{x} + 2\beta m\dot{x} + 2k\left(1 - \frac{l_0}{l}\right)x + \frac{kl_0}{l^3}x^3 = f(t) \tag{21.10}$$

This allows us to construct phase-space plots of the energy $E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4$. For a hard spring, with $k_{eff} = 2k\left(1 - \frac{l_0}{l}\right) > 0$, the phase-space energy plots of $\dot{x}$ against x with contours as the fixed energy states, we approximately get rounded squares. For a soft spring, we try and analytically find equilibrium states; we get

$$x_0 = \pm\sqrt{2}\frac{l^{3/2}}{l_0}\left(1 - \frac{l_0}{l}\right)^{1/2} \tag{21.11}$$

Therefore we get phase-space plots of roughly intersecting circles centered around these two points on the x axis with $\dot{x} = 0$, eventually creating figure-eight type constant-energy contours.

We can try a Fourier expansion of the differential equation, with quality and nonlinearity constants replacing the physical constants in the problem. We look for solutions to $\ddot{x} + \frac{\dot{x}}{Q} + x + \epsilon x^3 = f(\cos\omega t)$, in the limit $Q \to \infty$ so that there is no damping, of the form $x(t) = \sum_n A_n(\omega)\cos n\omega t$.

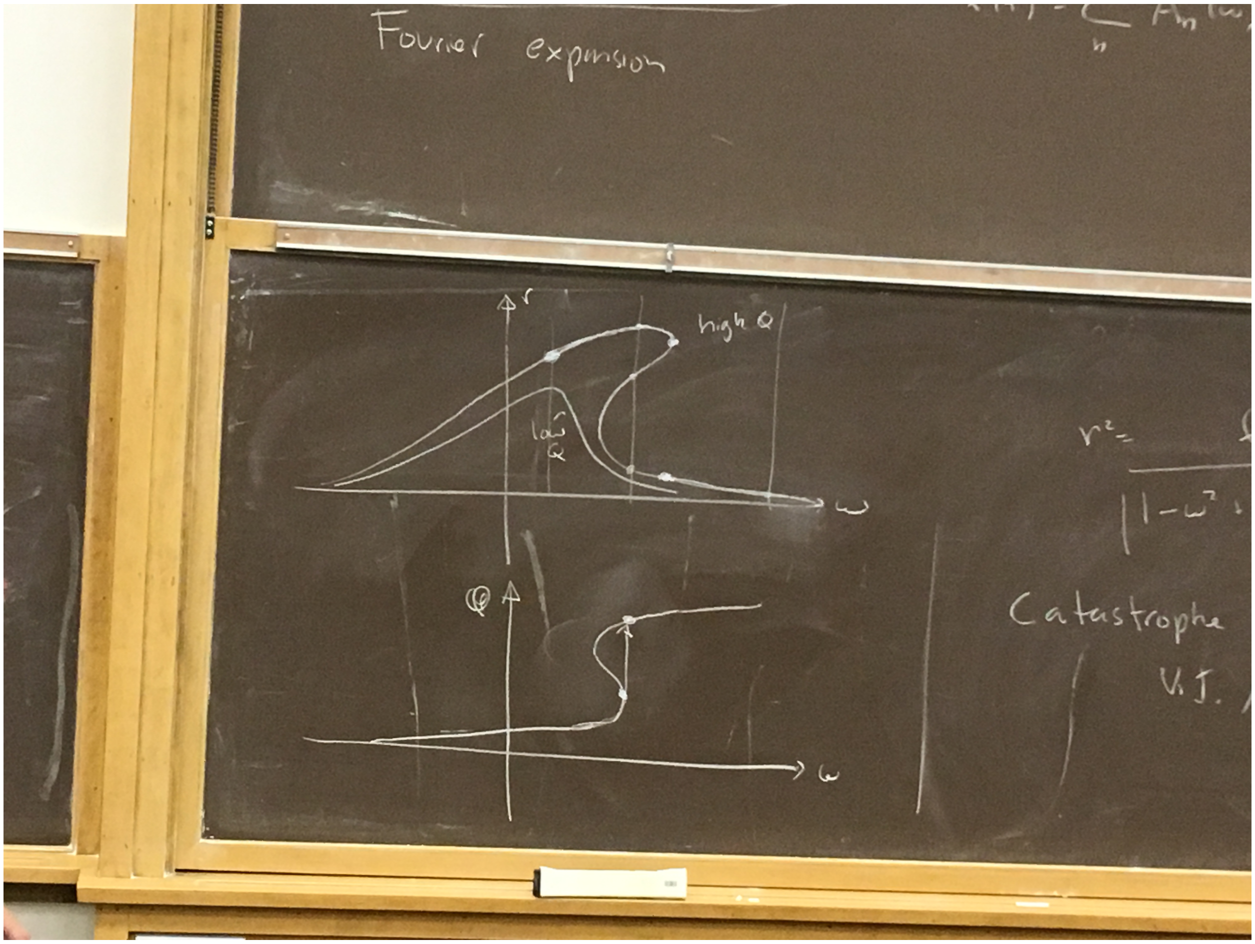
Solving for the Fourier coefficients, we get $(1-\omega^2)A_1 + \epsilon\frac{3}{4}A_1^3 = f$ and $(1-9\omega^2)A_3 + \epsilon\frac{1}{4}A_1^3 = 0$. This allows us to define a phase-space plot for A_1 against ω.

We do the same thing for the case of finite Q, which gives us

$$a\left(1 - \omega^2 + \frac{3\epsilon}{4}(a^2+b^2)\right) + \frac{b\omega}{Q} = f \tag{21.12}$$

$$b\left(1 - \omega^2 + \frac{3\epsilon}{4}(a^2+b^2)\right) - \frac{a\omega}{Q} = 0 \tag{21.13}$$

In both $r \equiv \sqrt{a^2+b^2}$ against ω and A_1 against ω, there is a region with one solution per ω and there is a region with three solutions per ω at higher frequencies.


Fourier expansion
high Q
low Q
r
ω
Q
ω
Catastrophe
V. I.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 22: Nonlinear Dynamical Systems

Lecturer: Stuart Bale, Ivan Vasko *25 April* *Aditya Sengupta*

22.1 Extending an Oscillator Model

Consider a pendulum with arbitrarily distributed mass under gravity, with its CM at a length d and angle θ relative to the y axis (straight down). The torque is $\tau = mgd\sin\theta$. The position vector to any point on the distributed mass is

$$\vec{r} = r(\hat{y}\cos(\theta+\varphi) + \hat{x}\sin(\theta+\varphi)) \tag{22.1}$$

The velocity is

$$\vec{v} = \frac{d\theta}{dt}\frac{dr}{dt} = \dot{\theta}r(-\hat{y}\sin(\theta+\varphi) + \hat{x}\cos(\theta+\varphi)) \tag{22.2}$$

The angular momentum is

$$\vec{L} = -\dot{\theta}\hat{z}\int r^2 dm \tag{22.3}$$

Therefore, we can write the equation of motion as

$$\ddot{\theta} + \frac{mgd}{I}\sin\theta = 0 \tag{22.4}$$

We can define the radius of gyration $r_0 = \sqrt{\frac{I}{m}}$ to reduce this to something like the usual form of a pendulum,

$$\ddot{\theta} + \frac{gd}{r_0^2}\sin\theta = 0 \tag{22.5}$$

Under the small-angle approximation, this is just the form of a simple harmonic oscillator. Consider the third-order term, so that the equation of motion is

$$\ddot{\theta} + \frac{gd}{r_0^2}\left(\theta - \frac{1}{6}\theta^3\right) = 0 \tag{22.6}$$

We can write down the energy of the system,

$$E = \frac{1}{2} I\dot{\theta}^2 + mgd(1 - \cos\theta) \tag{22.7}$$

and for a fixed value of energy we can write down $\dot{\theta}$,

$$\dot{\theta} = \pm\sqrt{\frac{2}{I}\left(E - mgd(1 - \cos\theta)\right)} \tag{22.8}$$

At the peak, the system energy is entirely potential, so we can substitute in for E and use trig identities to get

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{4}{I} mgd\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)} \tag{22.9}$$

This allows us to solve for $\theta(t)$,

$$\int dt = \int \frac{d\theta}{\dot{\theta}} \tag{22.10}$$

$$t = \frac{2r_0}{\sqrt{gd}} \int_0^{\theta_0} \frac{d\theta}{\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)^{1/2}} \tag{22.11}$$

This is an elliptic integral, which we try solving by substituting $k = \sin\frac{\theta_0}{2}$ and $kz = \sin\frac{\theta}{2}$. Then expanding out $(1 - k^2z^2)^{-1/2}$ we get

$$t = \frac{4r_0}{\sqrt{gd}} \int_0^1 \frac{dz}{\sqrt{1 - z^2}} + \frac{1}{2}k^2 \int_0^1 \frac{z^2 dz}{\sqrt{1 - z^2}} + \ldots \tag{22.12}$$

We get

$$t = \frac{2\pi r_0}{\sqrt{gd}}\left(1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \ldots\right) \tag{22.13}$$

22.2 Perturbation Theory

Consider a force of the form

$$F = -kx + \lambda m x^2 \tag{22.14}$$

where λ is a small perturbation. We want an $x(\lambda, t)$ that satisfies $\ddot{x} + \omega_0^2 x - \lambda x^2 = 0$. We say $x(\lambda, t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t)$.

We get

$$\dot{x} = \dot{x}_0 + \lambda \dot{x}_1 + \dots \tag{22.15}$$
$$\ddot{x} = \ddot{x}_0 + \lambda \ddot{x}_1 + \dots \tag{22.16}$$
$$\ddot{x}_0 + \lambda_1 \ddot{x}_1 + \omega_0^2(x_0 + \lambda x_1) - \lambda(x_0 + \lambda x_1)^2 = 0 \tag{22.17}$$
$$\ddot{x}_0 + \omega_0^2 x_0 + \lambda \ddot{x}_1 + \omega_0^2 \lambda x_1 - \lambda x_0^2 - 2\lambda^2 x_0 x_1 - \lambda^3 x_1^2 = 0 \tag{22.18}$$

We split this up and eventually get

$$x(t) = A \cos \omega_0 t - \frac{\lambda A^2}{6\omega_0^2} \left(\cos(2\omega_0 t) - 3\right) \tag{22.19}$$

22.3 Fluid Turbulence

To first order, the eigenvalues of a fluid mechanical system can be approximated by a system of oscillators, but this does not take into account the different length scales of fluid motion. We can plot power against k, the spatial frequency, and use the fact that the energy transfer ratio is constant in the inertial regime to define $\epsilon = \frac{\delta v^2}{\tau}$. Based on further conservation laws, we get $\omega_1 = \omega_2 + \omega_3$ and $\vec{k}_1 = \vec{k}_2 + \vec{k}_3$.

Physics 105: Analytic Mechanics **Spring 2019**

Lecture 23: Infinitesimal Canonical Transformations

Lecturer: Stuart Bale, Ivan Vasko *2 May* *Aditya Sengupta*

Consider a transformed Hamiltonian $\widetilde{\mathcal{H}} = \mathcal{H} + \frac{\partial F}{\partial t}$. Then we define momenta $p_i = \frac{\partial F}{\partial q_i}$ and $P_i = -\frac{\partial F}{\partial Q_i}$. Let F be the generating function for this transformation. For example, say $S = qP$ is the generating function. Then $p = \frac{\partial S}{\partial q} = P$ and $Q = \frac{\partial S}{\partial P} = q$. This is the identity transformation. Perturb this to create an infinitesimal transformation,

$$S = qP + \epsilon G(q, P, t) \tag{23.1}$$
$$\widetilde{\mathcal{H}} = \mathcal{H} + \frac{\partial S}{\partial t} \tag{23.2}$$

Then the momenta are

$$p = \frac{\partial S}{\partial q} \tag{23.3}$$
$$Q = \frac{\partial S}{\partial P} \tag{23.4}$$

We want to find $P(\epsilon) \approx P(0) + \epsilon \left.\frac{\partial P}{\partial \epsilon}\right|_{\epsilon=0}$ and $Q(\epsilon) = Q(0) + \epsilon \left.\frac{\partial Q}{\partial \epsilon}\right|_{\epsilon=0}$.

$$p = \frac{\partial S}{\partial q} = P + \epsilon \frac{\partial G}{\partial q} \tag{23.5}$$
$$\frac{dp}{d\epsilon} \approx \frac{dP}{d\epsilon} + \frac{\partial G}{\partial q} + \epsilon \frac{d}{d\epsilon}\left(\frac{\partial G}{\partial q}\right) \tag{23.6}$$

At $\epsilon = 0$, this gives us

$$\left.\frac{dP}{d\epsilon}\right|_{\epsilon=0} = -\frac{\partial G}{\partial q} \tag{23.7}$$

This suggests that we let our generating function $G(q, p, t)$ be the Hamiltonian. Then we get

$$P \approx p - \epsilon \frac{\partial \mathcal{H}}{\partial q} \tag{23.8}$$
$$Q \approx q + \epsilon \frac{\partial \mathcal{H}}{\partial p} \tag{23.9}$$

This ends up propagating the system through time:

$$P\epsilon p + \epsilon\dot{p} = p + \frac{dp}{dt}dt = p(t + dt) \tag{23.10}$$

$$Q \approx q + \epsilon\dot{q} \approx q + \frac{dq}{dt}dt \approx q(t + dt) \tag{23.11}$$

Consider a function $u(Q, P, t)$ under infinitesimal canonical transformations.

$$\left.\frac{du}{d\epsilon}\right|_{\epsilon=0} = \left(\frac{\partial u}{\partial Q}\frac{\partial Q}{\partial \epsilon} + \frac{\partial u}{\partial P}\frac{\partial P}{\partial \epsilon}\right)_{\epsilon=0} \tag{23.12}$$

Therefore, for a first-order Taylor expansion of u, we get

$$u(t) = u(q, p, t) + \epsilon\left(\frac{\partial u}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial G}{\partial q}\right) \tag{23.13}$$

The bracketed term is called the Poisson bracket $[u, G]$. We get this by replacing $\dot{q}$ and $\dot{p}$ in the chain rule expansion of $\frac{du}{dt}$ by derivatives of the Hamiltonian, using Hamilton's equations of motion. In a simplified form, we get

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} \tag{23.14}$$

Poisson brackets have the properties that $[u, v] = -[v, u]$, that $[u, u] = 0$, and that $[u, c] = 0$ for any constant c. Poisson brackets are linear:

$$[(u_1 + u_2), v] = [u_1, v] + [u_2, v] \tag{23.15}$$

$$[u_1 u_2, v] = u_1[u_2, v] + [u_1, v]u_2 \tag{23.16}$$

There is also the Jacobi identity on Poisson brackets:

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 \tag{23.17}$$

We can use Poisson brackets to show that something is a conserved quantity: if $\frac{du}{dt} = [u, H] = 0$ for some quantity u, then u is conserved over time. Suppose there are two such conserved quantities u, v. Then

$$\frac{d}{dt}([u, v]) = [[u, v], H] = -[[v, H], u] - [[H, u], v] = 0 \tag{23.18}$$

so $[u, v]$ is another conserved quantity!