

Lecture 1: Introduction

*Lecturer: Matthias Reinsch**January 23**Aditya Sengupta*

Note: *LaTeX* format adapted from template courtesy of UC Berkeley EECS dept.

Note: these lecture notes follow the Griffiths conventions of representing vectors instead of the typical ones, for example, the use of \mathbf{x} instead of \vec{x} for a vector. This is because the homework uses this notation as well as the book.

1.1 Logistics

Pages 1-30 of Griffiths are to be done by Friday (Jan 25), Homework 1 is due two Mondays from now (Feb 4), but every other homework assignment will be due on Friday at 3:15.

Midterm 1 is on March 1, and Midterm 2 is on April 12. Both are during the lecture time. Office hours are Monday 11-12 and Thursday 9-10.

1.2 Cross product and curl

These are fundamental to describing magnetic fields. We will show that if you take a small parcel of fluid, over a short time interval, it will have some angular momentum about its center of mass. Divide that by its moment of inertia and that is its curl (after correcting for a to-be-explained factor of 2).

1.2.1 Notation

We start with the classical notion of a vector field,

$$\mathbf{v} = (v_x, v_y, v_z) = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}} \quad (1.1)$$

where the typical convention of right-handed Cartesian coordinates is followed, and where subscripts are components instead of partial derivatives.

For the cross product, the explicit formula is slightly complicated, so the following abuse of notation is often used:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \quad (1.2)$$

This is a mnemonic. (Much like the word mnemonic, the cross product is not commutative.) This is an abuse of notation because vectors are not supposed to be elements of a determinant. This is interpreted as

the typical determinant algorithm. Essentially, it forms a linear combination of the top-row vectors. This notation makes sense if we assume that the result of the determinant will be dotted with a third vector,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \quad (1.3)$$

This is the scalar triple product of three vectors.

To appreciate the curl, we first consider the del operator:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \quad (1.4)$$

This is an operator, which means we should probably give it something to operate on. Consider the gradient of $T(x, y, z)$, ∇T . This is a vector field. Two new operations arise due to the del operator: divergence and curl. The divergence is denoted by the dot product,

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y + \frac{\partial}{\partial z} v_z \quad (1.5)$$

and the curl by the cross product,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \quad (1.6)$$

1.2.2 Examples

1.2.2.1 Rigid rotation

Define a velocity vector field,

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{\omega} \times \mathbf{r} \quad (1.7)$$

The vector $\boldsymbol{\omega}$ gives us the axis of rotation and the angular velocity. Choose the axis of rotation to be parallel to the z axis, i.e. $\boldsymbol{\omega} = (0, 0, \omega_z)$. Taking the cross product with (x, y, z) , we get $\mathbf{v} = (-y\omega_z, x\omega_z, 0)$. Then, take the curl, and we get

$$\nabla \times \mathbf{v} = 2\boldsymbol{\omega} \quad (1.8)$$

Lecture 2: Math Review

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2.1 The Del Operator

The del operator is defined as

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \quad (2.1)$$

Both of these representations are used in the book.

2.2 Product Rules

Consider the single-variable product rule,

$$\frac{d(f(x)g(x))}{dx} = f(x) \frac{dg(x)}{dx} + g(x) \frac{df(x)}{dx} \quad (2.2)$$

This can be easily extended to multiple dimensions, where we consider functions $f(\mathbf{r})$ and $g(\mathbf{r})$:

$$\nabla(fg) = f\nabla g + g\nabla f \quad (2.3)$$

When we consider cross products, the expression we want to simplify becomes $\nabla \times (\mathbf{A} \times \mathbf{B})$, the curl of the cross-product vector field. The result is

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla) \mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{A}) \quad (2.4)$$

The $(\mathbf{A} \cdot \nabla)$ terms can be considered directional derivatives, where the weights of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are given by the vector \mathbf{A} . The other type of term involves finding the divergence of a vector, and multiplying the resultant scalar by a different vector. Therefore we get a sum of vectors as the result, as we expect for the curl of a vector.

2.3 Application

We know that

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{\omega} \times \mathbf{r} \tag{2.5}$$

Note that $\boldsymbol{\omega}$ does not have to be a constant vector. To make use of the product rule, in order to find the curl of \mathbf{v} , we need to extend this to give some credence to the notion of a derivative of $\boldsymbol{\omega}$. To do this, we make $\boldsymbol{\omega}(\mathbf{r}) = (\omega_x, \omega_y, \omega_z) \forall \mathbf{r} \in \mathbb{R}^3$, changing it slightly from a constant vector to a constant vector *field*.

$$\nabla \times \mathbf{v} = \nabla \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega}(\nabla \cdot \mathbf{r}) - (\boldsymbol{\omega} \cdot \nabla)\mathbf{r} = 3\boldsymbol{\omega} - (\omega_x, \omega_y, \omega_z) = 2\boldsymbol{\omega} \tag{2.6}$$

Note that the two terms involving divergence and curl of $\boldsymbol{\omega}$ vanish because it is a constant vector field. Also note that we used the fact that $\nabla \cdot \mathbf{r} = 3$, which gives us the first term. More explicitly, the second term is given by

$$(\boldsymbol{\omega} \cdot \nabla)\mathbf{r} = \left(\omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z} \right) (x, y, z) = (\omega_x, \omega_y, \omega_z) = \boldsymbol{\omega}$$

2.4 Parameterizing Integrals

The Fundamental Theorem for Gradients helps us to parameterize integrals. Consider an integral

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \tag{2.7}$$

where \mathbf{a} and \mathbf{b} are position vectors identifying points, with some arbitrary path between them. Subdivide this path into a set of small displacements $d\mathbf{l}$. Each of these is a vector that is the same as $d\mathbf{r}$, that is, (dx, dy, dz) .

2.4.1 Parameterized Curves

We select a parameter λ as the input to a function (the *parameterization*) describing the curve, $\mathbf{r}(\lambda)$. This defines a λ axis that is only logically, and not spatially, connected, to the 3D space graph.

As an example, consider the unit circle in the $x - y$ plane, $\mathbf{r}(\lambda) = (\cos \lambda, \sin \lambda, 0)$. For every value of λ , we get a point in space. In this way we can translate a multivariate integral into a single-variable integral along the λ axis,

$$\int_{\lambda_1}^{\lambda_2} (\nabla T) \frac{d\mathbf{r}}{d\lambda} d\lambda \tag{2.8}$$

Lecture 3: Fundamental Theorems of Vector Calculus

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3.1 Fundamental Theorem for Gradients

The Fundamental Theorem for Gradients can be stated as follows:

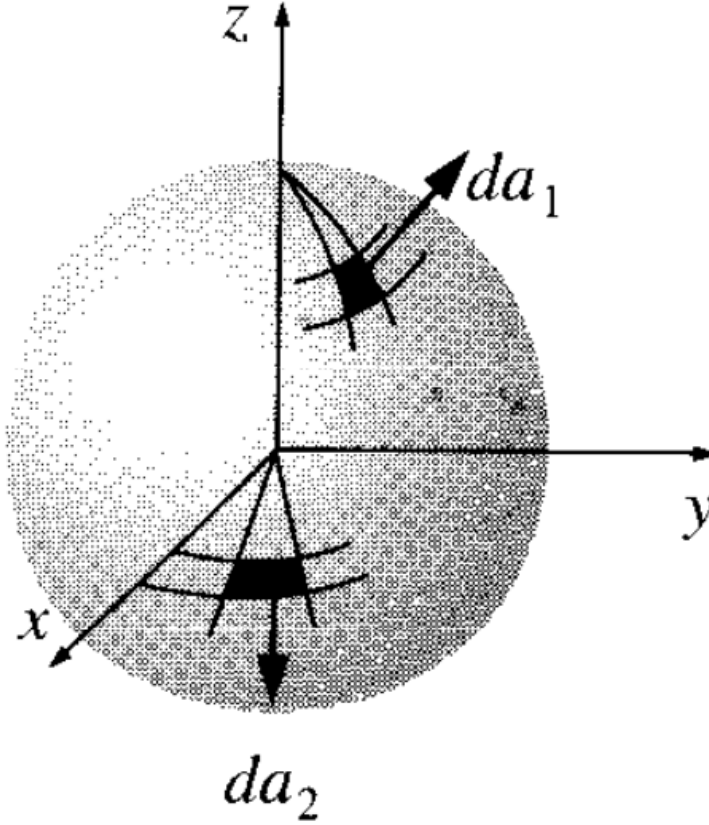
$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \quad (3.1)$$

Consider an arbitrary path through 3D space (that may intersect itself). Integrating a vector field along this path requires a differential element of that vector field, which can be found via a scalar product with a differential element of the path,

$$dT = (\nabla T) \cdot d\mathbf{l} \quad (3.2)$$

This is the first term in a higher-dimensional Taylor series.

Figure 1.39 on page 41 of Griffiths provides an example of a 3D object over which an integral can be done:



Note that the north pole of the sphere, i.e. where the z axis punctures the sphere, is not quite at the same point as the horizon. The equator passes through the points where the x and y axes puncture the sphere, and the meridians go radially down from the north pole to the south pole (the two ends where the z axis punctures the sphere).

This surface is orientable, i.e. it has a clear inside and outside, meaning we can parameterize its area for an integral. Here the normal vector to the surface points radially outwards, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. A differential area element $d\mathbf{a}_1$ is a vector with the orientation $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ and with magnitude equal to its area. $d\mathbf{a}_1$ can also be constructed as a cross product of the two vectors (length and width) that specify the differential area element. These two vectors are differential displacements due to θ (from the north pole ($\theta = 0$) to the equator ($\theta = \frac{\pi}{2}$) - it is 90 degrees minus the latitude) and ϕ respectively. Then,

$$d\mathbf{a}_1 = d\mathbf{l}_\theta \times d\mathbf{l}_\phi \quad (3.3)$$

The surface is parameterized by

$$\mathbf{r}(\theta, \phi) = R(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (3.4)$$

“Parameterized” means that for any choice of the two parameterizing variables, the function \mathbf{r} returns a point on the sphere. Explicitly, the vectors representing the differential area element are then

$$d\mathbf{l}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} d\theta \quad (3.5)$$

$$d\mathbf{l}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} d\phi \quad (3.6)$$

3.2 Fundamental Theorem for Divergences

The Fundamental Theorem for Divergences states that

$$\int_{\mathcal{V}} (\nabla \times \mathbf{v}) d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a} \quad (3.7)$$

where $d\tau$ is a differential volume element. This can be stated succinctly as *the boundary of a boundary is zero*. For example, consider a 3D shape whose boundary is a circle. (The fish net example from that one 53 final.) Then, by the Fundamental Theorem of Gradients, the boundary of the boundary is zero because the end points meet up. The volume \mathcal{V} is a three-dimensional region in a three-dimensional space, and the theorem states that this is equivalent to an integral over the 2D *closed* surface S that is the boundary of \mathcal{V} .

The right-hand integral is the flux out of the region.

3.3 Fundamental Theorem for Curls

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l} \quad (3.8)$$

where P is the perimeter of the 2D surface in 3D space S . For the perimeter integral, an orientation consistent with the right-hand rule must be selected.

Note that in physics, the term *sphere* refers to the surface of a solid sphere.

Lecture 4: Vector Calculus Conclusion

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4.1 Fundamental Theorem for Curls

An example of the Fundamental Theorem for Curls can be seen in the definition $\mathbf{v} = \omega \times \mathbf{r}$, with $\omega = (0, 0, \omega)$. Taking the cross product gives us

$$\mathbf{v} = \omega(-y, x, 0) \quad (4.1)$$

This vector field can be drawn, and we find that it is a series of concentric circles in the x-y plane, oriented counterclockwise. We can integrate around the boundary of this circle. Here the normal vector is $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, which defines the area element; $d\mathbf{a} = (da)\hat{\mathbf{n}}$.

We can take the curl to find

$$\nabla \times \mathbf{v} = 2\omega \quad (4.2)$$

and therefore

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 2\omega\pi R^2 \quad (4.3)$$

We can do the integral over the boundary to match. To do this, we parameterize the curve by

$$\mathbf{r}(\lambda) = R(\cos \lambda, \sin \lambda, 0) \quad (4.4)$$

Based on this parameterization, we go around the curve and find that the integral $\oint \mathbf{v} \cdot d\mathbf{l}$ is the same as the surface integral.

4.2 2D parameterization

In 3D space, there are two parameters in a parameterization of a surface. We can use x and y for a surface parameterization $\mathbf{r}(x, y) = (x, y, \sqrt{R^2 - x^2 - y^2})$. Now, we want to find the differential area element, which is not quite as easy in two dimensions. In one dimension, this was simply the derivative of the single parameter, $\frac{d\mathbf{r}}{d\lambda} d\lambda$. This is because the element over which an integral was being defined was a small part of the space, and summing over a lot of them gave us the whole space. The equivalent here is a tiny area

with dimensions dx and dy . Therefore, in one direction, the vector required is $\frac{\partial \mathbf{r}}{\partial x}$ and in the other it is $\frac{\partial \mathbf{r}}{\partial y}$. The combination of these two gives us a tiny parallelogram (the curvature goes to zero), and the sum over a lot of these gives us the area.

The vector representing this tiny area is the cross product of the parallelogram arms:

$$d\mathbf{a} = \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} dx dy \quad (4.5)$$

4.3 Spherical Coordinates

A more natural way to parameterize a sphere is in terms of its r vector and the two angles θ and ϕ , which are respectively due south and due east. We can translate between spherical and Cartesian coordinates by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix} \quad (4.6)$$

The $\hat{\mathbf{r}}$ vector is just these components without the r ,

$$\hat{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (4.7)$$

and the other partial derivatives give us

$$\frac{\partial \mathbf{r}}{\partial \theta} = r (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) \quad (4.8)$$

and similarly for the ϕ derivative. We can get the unit vector by normalizing this derivative,

$$\hat{\theta} = \frac{\frac{\partial \mathbf{r}}{\partial \theta}}{\left| \frac{\partial \mathbf{r}}{\partial \theta} \right|} \quad (4.9)$$

4.4 The Dirac Delta Function

The Dirac delta is a function introduced to formalize the notion of a point object, such as a charge. In one dimension, it is defined as

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \text{“}\infty\text{”} & x = 0 \end{cases} \quad (4.10)$$

and it has the property that

$$\int_{-\infty}^{\infty} \delta(x) dx = 1 \tag{4.11}$$

It helps to think about the Dirac delta as an operator, rather than its own function; it samples another function under an integral, rather than itself being integrated.

Lecture 5: The Dirac Delta

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1 February

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The Dirac delta can be shifted,

$$\delta(x - a) = \begin{cases} 0 & x \neq a \\ \text{"}\infty\text{"} & x = a \end{cases} \quad (5.1)$$

This is in contrast to the Kronecker delta, a discrete function that is well defined,

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad (5.2)$$

5.1 The Dirac delta as the limiting value of a sequence of functions

(that's a mouthful)

Define the following function:

$$R_n(x) = \begin{cases} 0 & |x| > \frac{1}{2n} \\ n & |x| \leq \frac{1}{2n} \end{cases} \quad (5.3)$$

We can see that the area under each of these curves is 1. Then, we take a limit as $n \rightarrow \infty$ to get the Dirac delta.

We can see the effect of the Dirac delta by multiplying it with a different function. Let $f(x)$ be our arbitrary test function, and let it be continuous and generally well-behaved. Then, integrate both of them together:

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(x)R_n(x)dx = \lim_{n \rightarrow \infty} \int_{-1/2n}^{1/2n} f(x)dx = f(0) \quad (5.4)$$

This is called the *sampling property*.

5.2 Dirac Delta Properties

Consider the field

$$\mathbf{A}(r, \theta, \phi) = \frac{1 - \cos \theta}{r \sin \theta} \hat{\phi} \quad (5.5)$$

$$\nabla \times \mathbf{A} = \frac{\hat{\mathbf{r}}}{r^2} \quad (5.6)$$

We take the divergence of this field.

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r}) \quad (5.7)$$

Define $\mathbf{z} = \mathbf{r} - \mathbf{r}'$. We can say the same about this vector,

$$\nabla \cdot \left(\frac{\hat{\mathbf{z}}}{z^2} \right) = 4\pi \delta^3(\mathbf{z}) \quad (5.8)$$

This suggests a three-dimensional delta function. We can define this as follows:

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z) \quad (5.9)$$

Returning to the divergence, we notice the 4π on the expression. We can show this by taking the integral around a spherical surface and letting the radius go to 0.

Theorem 5.1. *In curl-less fields, $\nabla \times \mathbf{F} = 0$ everywhere, $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$ is path-independent, $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any loop, and there exists a $V(\mathbf{r})$ such that $\mathbf{F} = -\nabla V$.*

Lecture 6: Divergence-less Fields, Coulomb's Law

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4 February

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Theorem 6.1. *In divergence-less fields, $\nabla \cdot \mathbf{F} = 0$ everywhere, $\int \mathbf{F} \cdot d\mathbf{a}$ is surface-independent, $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for a closed surface, and there exists an $\mathbf{A}(\mathbf{r})$ such that $\mathbf{F} = \nabla \times \mathbf{A}$.*

An example of a divergence-less field is $\mathbf{A}(r, \theta, \phi) = m_0 \frac{1 - \cos \theta}{r \sin \theta} \hat{\phi}$ defined on $\mathbb{R}^3 - \{0\}$ —negative z axis, and an example of a curl-less field is $\mathbf{B}(r, \theta, \phi) = m_0 \frac{\hat{r}}{r^2}$ defined on $\mathbb{R}^3 - \{0\}$.

All of the statements in the above two theorems are true if any one of them is true. For instance, suppose $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for some \mathbf{F} and some closed path parameterized by $d\mathbf{l}$. Consider any two arbitrary points on the closed curve given by position vectors \mathbf{a} and \mathbf{b} , and set up a path integral between them along one path, then along the other end of the closed curve. We claim that

$$\int_{\mathbf{a}, \text{path 1}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = \int_{\mathbf{a}, \text{path 2}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} \quad (6.1)$$

which we can show by moving the path 2 term to the other side. We flip the minus sign thus generated by switching the start and end points of the path, to get

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} + \int_{\mathbf{b}}^{\mathbf{a}} \mathbf{F} \cdot d\mathbf{l} \quad (6.2)$$

which is zero, because we know that the integral over a closed loop (from \mathbf{a} to \mathbf{a}) is 0. Therefore all integrals of a curl-less field are path independent.

Similarly, we can make statements about a divergence-less field based on any one fact stated above.

Theorem 6.2. *For any \mathbf{F} there exists a scalar potential $V(\mathbf{r})$ and a vector potential $\mathbf{A}(\mathbf{r})$ such that*

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A} \quad (6.3)$$

where the first component has vanishing curl and the second has vanishing divergence.

Note that for any \mathbf{G} such that $\nabla \cdot \mathbf{G} = 0$ and $\nabla \times \mathbf{G} = 0$, then the scalar and vector potentials can be shifted by \mathbf{G} or its gradient because the potential still satisfies the curl-less and divergence-less properties if \mathbf{G} is added to it. For example, the following is still valid:

$$\mathbf{F} = (-\nabla V + G) + (\nabla \times \mathbf{A} - \mathbf{G}) \quad (6.4)$$

Another example of a valid \mathbf{G} is $\mathbf{G} = \nabla(x^2 - y^2)$. This is a nonconstant vector field with zero div and zero curl.

6.1 Coulomb's Law

Note that we will be using MKS units throughout.

Coulomb's law describes a force vector between any two charged species with charge q and Q ,

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{\mathbf{z}} \quad (6.5)$$

where r is defined by $r = \mathbf{r} - \mathbf{r}'$. Component-wise, we can say

$$r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \quad (6.6)$$

Start by thinking of both charges as positive; then, we can draw an arrow between the tips of the two position vectors that give q and Q . The force is directed along this line, and is repulsive in both directions.

6.2 Superposition principle

Consider a system with multiple particles of charge q_i and position vectors \mathbf{r}'_i . Then, the electric field at a certain point is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i^2} \hat{\mathbf{z}}_i \quad (6.7)$$

where the electric field is the force over the charge at the point given by \mathbf{r} . (We assume there is a test charge at \mathbf{r} .)

6.3 Continuous Charge Distributions

We can apply the principle of superposition to specific continuous cases, where we integrate over a lot of infinitesimal charges specifically arranged (i.e. we have some constraint on r):

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r^2} \hat{\mathbf{z}} dq \quad (6.8)$$

Usually, we can parameterize \mathbf{r} by some variable. For example, a linear charge distribution is $dq = \lambda(\mathbf{r}')dl'$, an area distribution is $dq = \sigma(\mathbf{r}')da'$, and a volumetric distribution is $dq = \rho(\mathbf{r}')d\tau'$.

Lecture 7: Parameterizations, Electric Fields

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6 February

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7.1 Parameterizations

7.1.1 1D Parameterizations

We previously saw that we could parameterize a curve in one dimension by $\mathbf{r}(\lambda)$, where the components of \mathbf{r} are all functions of λ , and the line varies along a new λ axis from some λ_1 to some λ_2 . We integrate along λ ,

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_{\lambda_1}^{\lambda_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{d\lambda} d\lambda \quad (7.1)$$

7.1.2 2D Parameterizations

We use the same idea, except with a two-dimensional parameter space. We construct a surface along axes α, β , and parameterize the curve by $\mathbf{r}(\alpha, \beta)$. Then the integral becomes

$$\iint \mathbf{F} \cdot d\mathbf{a} = \iint \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial \alpha} \times \frac{\partial \mathbf{r}}{\partial \beta} \right) d\alpha d\beta \quad (7.2)$$

7.1.3 3D Parameterizations

In three dimensions, consider the example of a tetrahedron with its vertices at $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$. To integrate over its volume, other than slicing along any particular axis and integrating a number of surfaces, e.g. $V = \int \int z dy dx$, we can parameterize the tetrahedron by parameters α, β, γ .

$$\int_0^1 \int_{?}^{?} \int_{?}^{?} \quad (7.3)$$

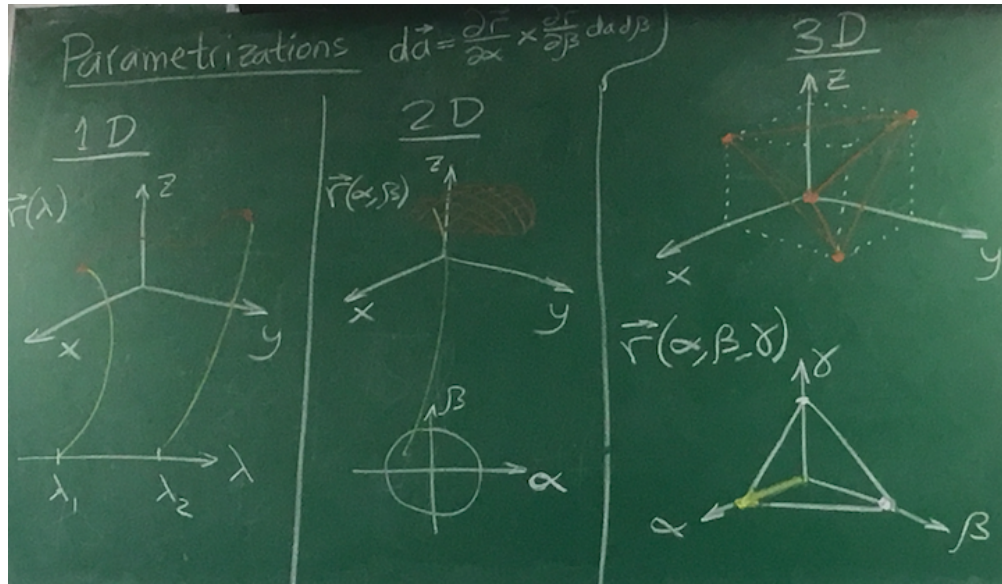
where the question marks can be filled in via geometry. The rest of the integral is yet to be filled in. We can get this by defining the transformation between coordinates and the parameter space, which we do as follows,

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \quad (7.4)$$

and the volume element becomes

$$d\tau = (\det M) d\alpha d\beta d\gamma \quad (7.5)$$

When you define a linear mapping, such as above, the columns of the matrix are the images of the unit vectors. The unit vector along the α axis should be mapped onto one of the given vertices, and the same goes for β and γ .



7.2 Integrating Over Electric Fields

Consider the case from the previous lecture where we wanted to integrate over a charge distribution,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r^2} \hat{\mathbf{r}} dq \quad (7.6)$$

This integral can go over all of space. We can compute the divergence and the curl of this field. Using the previous result on the divergence, namely that

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi\delta^3(\mathbf{r}) \quad (7.7)$$

we show that

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho(\mathbf{r}) \quad (7.8)$$

We can also show that the curl is zero, by $\nabla \times \frac{\hat{z}}{z^2} = 0$.

7.3 Volume/Velocity

Volume can be related to velocity, as a way of visualizing divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{V} \frac{dV}{dt} \quad (7.9)$$

Intuitively, we can think of this as shrinking the volume that encloses a velocity field. We can check dimensions to convince ourselves that this is true.

7.4 Superposition

Consider two charges q on the x -axis. One charge is located at a distance $\frac{3}{4}d$ from the origin on the positive x -axis, and the other is located at $-\frac{4}{3}d$, on the negative x -axis. We can define \mathbf{r}' vectors based on these,

$$\left(-\frac{4}{3}d, 0, 0\right) = \mathbf{r}'_1, \quad \left(\frac{3}{4}d, 0, 0\right) = \mathbf{r}'_2 \quad (7.10)$$

Then, consider a test point at $\mathbf{r} = (0, d, 0)$. We can define \mathbf{r} vectors as the difference between these,

$$\mathbf{r}_1 = \left(\frac{4}{3}d, d, 0\right), \quad \mathbf{r}_2 = \left(-\frac{3}{4}d, d, 0\right) \quad (7.11)$$

Now, the Pythagoreans come in and save the day.

$$r_1^2 = \frac{25}{9}d^2, \quad r_2^2 = \frac{25}{16}d^2 \quad (7.12)$$

Therefore, we get

$$\mathbf{E}_1 = \frac{q}{4\pi\epsilon_0} \frac{(36, 27, 0)}{125d^2} \quad (7.13)$$

7.5 Types of symmetry

Spherical symmetry	Cylindrical symmetry	Planar symmetry
Picture	picture	picture

Lecture 8: Feeling sick today so this won't be detailed enough

Lecturer: Matthias Reinsch

8 February

Aditya Sengupta

8.1 Integrable singularities

Suppose there is a potential function $V(r) \propto \frac{1}{r}$. At the origin, r blows up to infinity, causing a so-called “singularity”. Consider the integral of $\frac{1}{x}$ over the bounds $x = [0, 1]$. It turns out that this has infinite area. However, for the case $\frac{1}{\sqrt{x}}$, the area is actually finite. We can see this via an integral,

$$\int_0^1 \frac{1}{\sqrt{x}} dx = 2 \sqrt{x} \Big|_0^1 = 2 \quad (8.1)$$

8.2 Electric field from charge distribution

There are six arrows on the three-node diagram between ρ , \mathbf{E} , \mathbf{V} . To go from the charge distribution to the electric field, we use

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathbf{z}}}{z^2} \rho d\tau \quad (8.2)$$

Having the squared magnitude of the script-R in the denominator is dangerous. We can define a charge density on a volume such as a sphere of radius R ,

$$\rho(x, y, z) = \begin{cases} 0 & x^2 + y^2 + z^2 > R^2 \\ \frac{Q}{\frac{4}{3}\pi R^3} & x^2 + y^2 + z^2 \leq R^2 \end{cases} \quad (8.3)$$

To go from a charge distribution like this one to a potential, we say

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{z} d\tau' \quad (8.4)$$

For a field point outside the sphere, something happens. When $\rho(\mathbf{r}')$

To get from a field to a potential, we integrate the field along a line:

$$V(\mathbf{r}) = - \int_0^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l} \quad (8.5)$$

8.3 Poisson's Equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad (8.6)$$

This takes us from V to ρ . This is a partial differential equation and it is difficult to solve in general.

Lecture 9: Boundary Conditions, Energy

*Lecturer: Matthias Reinsch**11 February**Aditya Sengupta*

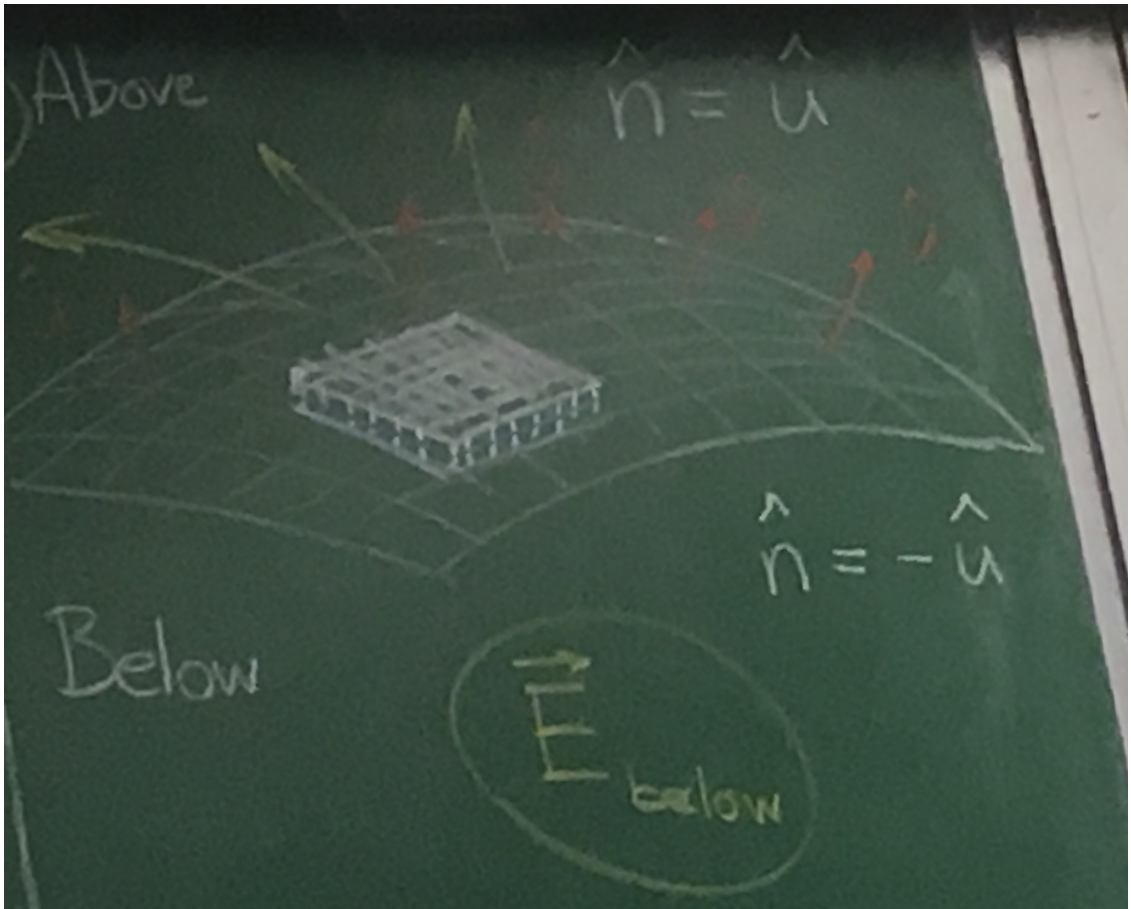
9.1 Boundary Conditions

When there are two different solutions to a differential equation in different regions, we need boundary conditions to connect them. Consider a two-dimensional surface with some curvature in a three-dimensional space. There are regions above and below this surface. At every point on the surface, there is a normal vector, and this is oppositely oriented depending on the position (above or below); say above, $\hat{\mathbf{n}} = \hat{\mathbf{u}}$ and below $\hat{\mathbf{n}} = -\hat{\mathbf{u}}$. We can analyze the perpendicular component of the electric field on this surface,

$$E_{above}^\perp = \hat{\mathbf{u}} \cdot \mathbf{E}_{above} \quad (9.1)$$

$$E_{above}^\perp - E_{below}^\perp = \frac{\sigma}{\epsilon_0} \quad (9.2)$$

Above and below the surface, there are vector fields representing the electric field. We define $\hat{\mathbf{u}}$ to be the same above and below the surface. Immediately below the surface, we would use the same normal vector as just above, when using $\hat{\mathbf{u}}$. $\hat{\mathbf{n}}$ is the outward pointing normal vector.



9.2 Gaussian Pillbox

Consider a “pillbox” with very thin edges, whose upper surface is above the plane and lower surface is below. We can invoke Gauss’s law on this box.

We can draw a regular outward-pointing normal for the Gaussian pillbox, with the upper surface having \hat{n} straight upwards, the lower surface having \hat{n} downwards, and the normal vector along the sides pointing out of those sides.

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0} \quad (9.3)$$

The differential area element is

$$d\mathbf{a} = (da)\hat{n} \quad (9.4)$$

Now, we let the sides of the pillbox go to zero, so that the upper surface represents E_{above}^\perp and the lower surface represents E_{below}^\perp . We use Gauss’s law; $dq = \sigma da$ is the differential charge enclosed by the pillbox.

Therefore the integral over $\mathbf{E} \cdot d\mathbf{a}$ becomes just the infinitesimal $\mathbf{E} \cdot \hat{\mathbf{n}}da$, so dividing by da and employing Gauss's law, we can get the above result.

9.3 Tangential Components

The above analysis only deals with one part of the electric field, that perpendicular to the surface. To deal with the parallel component, we write

$$\mathbf{E}_{above}^{\parallel} = \mathbf{E}_{below}^{\parallel} \quad (9.5)$$

To get this, we consider a family of closed loops in the plane, from \mathbf{a} to \mathbf{b} . Consider a structure similar to the Gaussian pillbox with curves in space, where two possible paths from \mathbf{a} to \mathbf{b} are either above or below the surface. We can add these together to make a closed loop,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad (9.6)$$

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}_{above} \cdot d\mathbf{l} + \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}_{below} \cdot (-d\mathbf{l}) = 0 \quad (9.7)$$

9.4 Energy in Electric Charge Configurations

This argument will have several stages. We will start with a point charge at rest, and we will bring a second point charge in from an infinite distance to into close proximity with the first charge.

Eventually, we will find that the work required to assemble a configuration of charges is

$$W = \frac{1}{2} \sum_{i=1}^n q_i V(\mathbf{r}_i) \quad (9.8)$$

where $V(\mathbf{r}_i)$ describes the potential at \mathbf{r}_i due to the other charges (those with index not equal to i .)

Let q_2 be infinitely far away from q_1 . Then, we bring it to some location close to q_1 , along an arbitrary path. By Coulomb's law, we know that the work done (and therefore the energy required) is

$$\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}} \quad (9.9)$$

We can see this arise from Coulomb's law by realizing that work is the path integral of force,

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} \quad (9.10)$$

There is no reference to the time, velocity, or kinetic energy of the particle, so in theory this could be done infinitely slowly with a vanishingly small kinetic energy.

Now, we can bring in a third particle q_3 . Then, the work to be done is, by superposition,

$$\frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) \quad (9.11)$$

Lecture 10: Work and Energy in Charge Distributions

Lecturer: Matthias Reinsch

13 February

Aditya Sengupta

10.1 Work to assemble a configuration of point charges

We know that the work required to move a charge q_2 near q_1 is

$$W = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r} = q_1 V(\mathbf{r}_1) \quad (10.1)$$

where V is the potential at \mathbf{r}_1 due to q_2 . By superposition, we can say that

$$W = \frac{1}{2} \sum_{i=1}^n q_i V(\mathbf{r}_i) \quad (10.2)$$

The factor of $\frac{1}{2}$ comes from removing double-counting. For example, if we explicitly write out the summation for the case of three charges, we have

$$\frac{1}{2} \left(\frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_1}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_3 q_1}{r_{13}} + \frac{q_3 q_2}{r_{23}} \right) \right) \quad (10.3)$$

We see that every term appears twice, so we have to divide by 2 to compensate.

10.2 Continuous Charge Distributions

We can naturally generalize this summation to an integral,

$$W = \frac{1}{2} \int \rho V d\tau \quad (10.4)$$

We can integrate this by parts and rewrite the charge density as $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$, to get

$$W = \frac{\epsilon_0}{2} \int_{all\ space} E^2 d\tau \quad (10.5)$$

This integral suggests we can define the energy density of an electric field as $\frac{\epsilon_0}{2} E^2$.

10.3 Conductors

Conductors have the property that $\mathbf{E} = 0$ inside them; they allow charge to flow and so no field “accumulates”. We can investigate the electric field around a conductor. We do this by employing boundary conditions. Immediately outside the conductor, the electric field is given by the surface charge,

$$\mathbf{E}_{outside} = \frac{\sigma}{\varepsilon_0} \hat{\mathbf{n}} \quad (10.6)$$

Lecture 11: Conductors, Potentials

Lecturer: Matthias Reinsch

15 February

Aditya Sengupta

11.1 Capacitors

Consider a configuration of conductors with charges Q_i on them. We can find the potential difference between any of them by computing a line integral of the electric field between them.

The special case we are interested in is a parallel-plate capacitor, which consists of two conductors parallel to one another. Suppose they have an area A , and a distance between them d . We get a simple path integral between these two, which gives us a ratio between charge and potential difference $C = \frac{Q}{V}$,

$$C = \epsilon_0 \frac{A}{d} \quad (11.1)$$

11.2 Concentric Spherical Shells

We know that for concentric spherical shells, the electric field goes as $\frac{1}{r^2}$ and the potential goes as $\frac{1}{r}$. In the special case where the inner radius a and the outer radius b are very close, we will end up reproducing the result for parallel plates:

$$a \approx b \implies C = 4\pi\epsilon_0 \frac{ab}{b-a} \approx \epsilon_0 \frac{A}{d} \quad (11.2)$$

where the area is approximately $A = 4\pi a^2$ and the distance is $b - a$. This approximation holds if $A \gg d^2$.

11.3 Potentials

Consider the general scalar field $V(\mathbf{r})$, which assigns a scalar to every point in space. We can connect this to a charge distribution by solving Poisson's equation,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad (11.3)$$

In one dimension, consider the case of a parallel-plate capacitor with $\rho = 0$ between the plates. We get

$$\frac{d^2 V}{dx^2} = 0 \quad (11.4)$$

$$V(x) = mx + b \implies V(x) = -Ex \quad (11.5)$$

In two dimensions, this becomes more complicated. The PDE to be solved is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad (11.6)$$

which gives us a solution that is given by a path integral,

$$V(x, y) = \frac{1}{2\pi R} \oint V dl \quad (11.7)$$

We can parameterize the path around this circle by $(x, y) = (x_0, y_0) + R(\cos \theta, \sin \theta)$. We get $dl = |d\mathbf{l}| = \frac{d\mathbf{r}}{d\theta} d\theta$ and can set up the integral.

Lecture 12: Laplace's Equation and Uniqueness

*Lecturer: Matthias Reinsch**20 February**Aditya Sengupta*

Consider a sphere centered around the origin, and a charge q at a point $(0, 0, z)$. We want to find the average value of the potential due to this charge over the sphere. To do this, it is useful to shift the solution.

12.1 Shifting a solution

Consider a point charge at $(0, 0, 0)$. The potential drops off with the inverse distance from the origin,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2 + y^2 + z^2}} \quad (12.1)$$

We can shift this to being centered around $(0, 0, z_0)$ just by shifting the z coordinate,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2 + y^2 + (z - z_0)^2}} \quad (12.2)$$

To average over the sphere, we can compute a surface integral,

$$V_{ave} = \frac{q}{4\pi\epsilon_0} \oint \frac{1}{r} da \quad (12.3)$$

Here, the sphere is just a surface for convenience. Suppose the situation were reversed; the sphere now carries a surface charge, and we want to find the potential at $(0, 0, z)$ due to this sphere. The integral becomes

$$V((0, 0, z)) = \frac{q}{4\pi\epsilon_0} \oint \frac{1}{r} da \quad (12.4)$$

12.2 Laplace's Equation

In two dimensions, Laplace's equation states

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad (12.5)$$

A surface that satisfies this equation is a circle, and we can average the potential over this circle,

$$V(x_0, y_0) = \frac{1}{2\pi R} \oint V dl \quad (12.6)$$

In three dimensions, Laplace's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad (12.7)$$

and the equivalent averaged potential is

$$V(x_0, y_0, z_0) = \frac{1}{4\pi R^2} \oint_{sphere} V da \quad (12.8)$$

12.3 Uniqueness Theorems

The first uniqueness theorem states that $V(\mathbf{r})$ is unique in a volume \mathcal{V} if V is specified on the boundary \mathcal{S} . This can be proved using the principle of superposition. Assume the existence of $V_1(\mathbf{r})$ and $V_2(\mathbf{r})$, two solutions that are not the same. Define $V_3 = V_1 - V_2$. Since V_1 and V_2 both satisfy the boundary conditions, V_3 must be zero on the boundary. Since V at any point in space is proportional to its surface integral, which is zero, we conclude that V_3 is identically zero, $V_3(\mathbf{r}) \equiv 0$. Therefore $V_1 = V_2$.

The second uniqueness theorem involves conductors. Consider a boundary of conductors \mathcal{S} , surrounding a volume \mathcal{V} with a specified ρ . The Q on each conductor is also specified. The second uniqueness theorem claims that these are enough to specify $V(\mathbf{r})$.

12.4 Image Charges

Consider a charge q on the z axis, above the $x - y$ plane. We are interested in the volume $\mathcal{V}, z \geq 0$. This is equivalent to the problem that has a charge $-q$ on the other end of the z axis.

Lecture 13: Method of Images

Lecturer: Matthias Reinsch

22 February

Aditya Sengupta

13.1 Image Charges

Suppose there is a charge q at a position $(0, 0, d)$. We want to solve Laplace's equation over the half-space $z \geq 0$ (say there is a conductor for all space $z \leq 0$ - the potential over space is affected by the surface charge on this conductor). We can construct a system in which there is a charge $-q$ at $(0, 0, -d)$ replacing the conductor. It turns out that these two systems have the same potential, so we get

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{z_+} - \frac{q}{z_-} \right) \quad (13.1)$$

We can make the distance vectors more explicit by performing a vector subtraction,

$$V(x, y, z) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} \right) \quad (13.2)$$

This is defined over all space except for the points where the two charges are located, which are singularities.

If we set $z = 0$, we get $V = 0$, so the $x - y$ plane is an equipotential.

Physically, we expect some distribution of negative charge over the conductor at $z = 0$. We can find this distribution by

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \quad (13.3)$$

In this case, this becomes

$$\sigma(x, y) = -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0} = \frac{-qd}{2\pi(x^2 + y^2 + d^2)^{3/2}} \quad (13.4)$$

If we integrate this over all space, we get $-q$, as we expect.

13.2 Solving PDEs

13.2.1 Separation of Variables

Suppose there is a three-dimensional problem defined as follows: we have a region defined by three planar conducting surfaces: the $x - z$ plane, a wall on the $y - z$ plane, and an upper surface parallel to the $x - z$

plane. This extends to infinity for positive x , and for positive and negative z . We want to solve Laplace's equation in this region. Since this is completely z symmetric, we can draw a trace in the $x - y$ plane to simplify. We can use the two-dimensional Laplacian,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad (13.5)$$

We use boundary conditions to find a specific solution. We know that the potential vanishes on all boundaries, so $V(x, 0, z) = 0$, i.e. the potential vanishes when $y = 0$. Additionally, $V(x, a, z) = 0$, describing the upper surface. The third surface boundary condition can be compactly described by $V(0, y, z) = V_0(y)$. Finally, we can say that $\lim_{x \rightarrow \infty} V = 0$.

Now, we can introduce the method of separation of variables. Consider the following form for the solution:

$$V(x, y) = X(x)Y(y) \quad (13.6)$$

Note that many functions cannot be written in this form, for example $x + y$.

We take a second-order derivative in x and another in y , and we get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \quad (13.7)$$

We conclude that the two terms must be equal and opposite constants,

$$\frac{X''}{X} = C, \frac{Y''}{Y} = -C \quad (13.8)$$

Lecture 15: Solving Laplace's Equation

Lecturer: Matthias Reinsch

27 February

Aditya Sengupta

15.1 In Spherical Coordinates

For problems with azimuthal symmetry, we can set up the general variable-separable form of the potential,

$$V(r, \theta) = R(r)\Theta(\theta) \quad (15.1)$$

This is an *ansatz*, an assumption about the form a solution will take which will be verified later. We apply Laplace's equation to this general form to get

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = 0 \quad (15.2)$$

The derivatives in r and θ can be expanded out using the product rule, but this form is sufficient. The general solution in R is

$$R(r) = Ar^l + \frac{B}{r^{l+1}} \quad (15.3)$$

We get this by assuming that the first term in Laplace's equation is $l(l+1)$, for some constant l . In Θ , we can assume the term is equal to $-l(l+1)$, which gives us

$$\Theta(\theta) = P_l \cos \theta \quad (15.4)$$

where P_l is the l th Legendre polynomial.

15.2 Legendre Polynomials

The Legendre polynomials are defined as follows,

$$P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2 - 1)^l. \quad (15.5)$$

For example, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{3x^2 - 1}{2}$. Note that $r^l P_l(\cos \theta)$ is a polynomial in x, y, z .

The general solution to Laplace's equation in spherical coordinates for problems with azimuthal symmetry can be written as

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l \cos \theta \quad (15.6)$$

If we plug in $r = R$ in the asymptotic case where $V \rightarrow E_0 z$, we get

$$V(r, \theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos \theta \quad (15.7)$$

Lecture 16: Spelio Is Here

Lecturer: Matthias Reinsch

4 March

Aditya Sengupta

16.1 Dipole expansion

With a multipole expansion, you get a much better qualitative understanding of the field due to a system of multiple charges. To motivate this, consider a dipole aligned along the y axis, with its positive charge at $\frac{1}{2}\mathbf{d}$ and its negative at $-\frac{1}{2}\mathbf{d}$. The position vector of the test point is \mathbf{r} relative to the origin, and we can define position vectors \mathbf{r}_+ and \mathbf{r}_- which are defined as we would expect (the position vectors from the positive and negative charges.)

The potential is

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_+} - \frac{1}{r_-} \right) \quad (16.1)$$

And we can explicitly write out the position vectors,

$$\mathbf{r}_\pm = \mathbf{r} \mp \frac{1}{2}\mathbf{d} \quad (16.2)$$

Now, we need the length:

$$r_\pm = \sqrt{\left(\mathbf{r} \mp \frac{1}{2}\mathbf{d}\right) \cdot \left(\mathbf{r} \mp \frac{1}{2}\mathbf{d}\right)} = \sqrt{r^2 \mp \mathbf{d} \cdot \mathbf{r} + \frac{d^2}{4}} \quad (16.3)$$

$$r_\pm = \left(r^2 \mp dr \cos \theta + \frac{d^2}{4}\right)^{1/2} \quad (16.4)$$

Therefore, we substitute this back into the potential expression to get

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{\left(r^2 - dr \cos \theta + \frac{d^2}{4}\right)^{1/2}} - \frac{1}{\left(r^2 + dr \cos \theta + \frac{d^2}{4}\right)^{1/2}} \right) \quad (16.5)$$

This expression is exact, but this does not help us understand the physics at all. One of the most important dipoles in nature is that in a water molecule, which has a d of a few angstroms. Just about always, our length scales are going to be bigger than that. So, at least in this case, the approximation $r \gg d$ can be considered to be valid.

What sorts of approximations might be valid on z_{\pm} with this assumption? We look at z_{\pm}^{-1} :

$$z_{\pm}^{-1} = \left(r^2 \mp dr \cos \theta + \frac{d^2}{4} \right)^{-1/2} = r^{-1} \left(1 \mp \frac{d}{r} \cos \theta + \frac{d^2}{4r^2} \right)^{-1/2} \quad (16.6)$$

The d^2 term is tiny here, so we drop it.

$$z_{\pm}^{-1} = r^{-1} \left(1 \mp \frac{d}{r} \cos \theta \right) \quad (16.7)$$

Here, we can use a binomial expansion to first-order: $f(x) = (1+x)^p \approx 1+px$. If we drop the px term in addition to dropping terms in x^2 and higher, we just end up with $f(x) = 1$, so the potential is zero. We've moved so far away that we can't say anything about the physics. With this approximation, we get

$$z_{\pm}^{-1} = r^{-1} \left(1 + \frac{1}{2} \frac{d}{r} \cos \theta \right) \quad (16.8)$$

So, the potential is

$$V(\mathbf{r}) \approx \frac{q}{4\pi\epsilon_0} \frac{1}{r} \left(\left(1 + \frac{d}{2r} \cos \theta \right) - \left(1 - \frac{d}{2r} \cos \theta \right) \right) + O\left(\frac{d^2}{r^2}\right) \approx \frac{qd \cos \theta}{4\pi\epsilon_0 r^2} + O\left(\frac{d^2}{r^2}\right) \quad (16.9)$$

We can write the numerator in terms of the dipole moment,

$$V(\mathbf{r}) \approx \frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2} + O\left(\frac{d^2}{r^2}\right) \quad (16.10)$$

16.2 Arbitrary distribution of charge

Consider a body with a completely arbitrary distribution of charge $\rho(\mathbf{r})$. From Gauss's law, we know that the potential is

$$V(\mathbf{r}) = \int \frac{1}{4\pi\epsilon_0} \frac{\rho(\mathbf{r}')}{z} d\tau' \quad (16.11)$$

This is an exact formula, based on breaking up the body into lots of little charge elements dq and summing up the potential due to each one. It is also useless; we don't know what the potential looks like, and differentiating it does not give us the electric field. So we apply the same method, a multipole expansion. This allows us to talk about the overall behaviour of a system, and deviations from this overall behaviour.

The charge distribution is finite, and so it has some notion of a length scale. \mathbf{r}' , the distance from the center to each individual dq , will be less than this length scale. This is not an exact definition, but an exact

definition is not necessary. We consider the case $r \gg D$, cases where the charge distribution is reasonably far away from the test point. This implies that $r \gg r'$, because for every possible r' , $D > r'$ by definition.

$$\mathbf{z} = \mathbf{r} - \mathbf{r}' \quad (16.12)$$

$$z^{-1} = (r^2 - 2rr' \cos(\alpha(\mathbf{r}')) + r'^2)^{-1/2} \approx r^2 \left(1 - 2\frac{r'}{r} \cos \alpha + \left(\frac{r'}{r}\right)^2 \right)^{-1/2} \quad (16.13)$$

Let $\epsilon = \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r} \cos \alpha$. Then we use a binomial approximation to get

$$z^{-1} \approx \frac{1}{r} \left(1 - \frac{\epsilon}{2} + \frac{3}{8}\epsilon^2 + O(\epsilon^3) \right) \quad (16.14)$$

$$z^{-1} = \frac{1}{r} \left(1 - \frac{1}{2} \left(-2\frac{r'}{r} \cos \alpha + \left(\frac{r'}{r}\right)^2 \right) + \frac{3}{8} \left(-2\frac{r'}{r} \cos \alpha + \left(\frac{r'}{r}\right)^2 \right)^2 + O(\epsilon^3) \right) \quad (16.15)$$

This is annoying, but we can expand it systematically. We get

$$z^{-1} = \frac{1}{r} \left(1 + \frac{r'}{r} \cos \alpha - \frac{1}{2} (3 \cos^2 \alpha - 1) \left(\frac{r'}{r}\right)^2 + O \left[\left(\frac{r'}{r}\right)^3 \right] \right) \quad (16.16)$$

and we can realize that these terms are Legendre polynomials:

$$(1 + \epsilon)^{-1/2} = \sum_i P_i(\cos \alpha) \left(\frac{r'}{r}\right)^i \quad (16.17)$$

This is because $\frac{1}{r}$ is the solution to Laplace's equation for a delta function (read up more), and all solutions to Laplace's equation can be written in terms of the Legendre polynomials.

Then, we can write the potential,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos \alpha) \rho(\mathbf{r}') d\tau' \quad (16.18)$$

and integrate term by term,

$$V(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{1}{4\pi\epsilon_0} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos \alpha) \rho(\mathbf{r}') d\tau' \quad (16.19)$$

This is the *multipole expansion*. We can isolate the integral and write it as m_n , which is nice because the integral is independent of r , and so it is entirely determined by the system's geometry. We get

$$V(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{1}{4\pi\epsilon_0} \frac{m_n}{r^{n+1}} \quad (16.20)$$

We can find the first few values of m_n .

$$m_0 = \int \rho(\mathbf{r}') d\tau = Q \quad (16.21)$$

$$m_1 = \int r' \cos \alpha \rho(\mathbf{r}') d\tau \quad (16.22)$$

$$m_2 = \int (r')^2 P_2(\cos \alpha) \rho(\mathbf{r}') d\tau' \quad (16.23)$$

m_0 is the total charge, m_1 is the dipole moment, m_2 is the quadrupole moment, and so on.

Lecture 17: Solutions to Laplace's Equation contd.

*Lecturer: Matthias Reinsch**6 March**Aditya Sengupta*

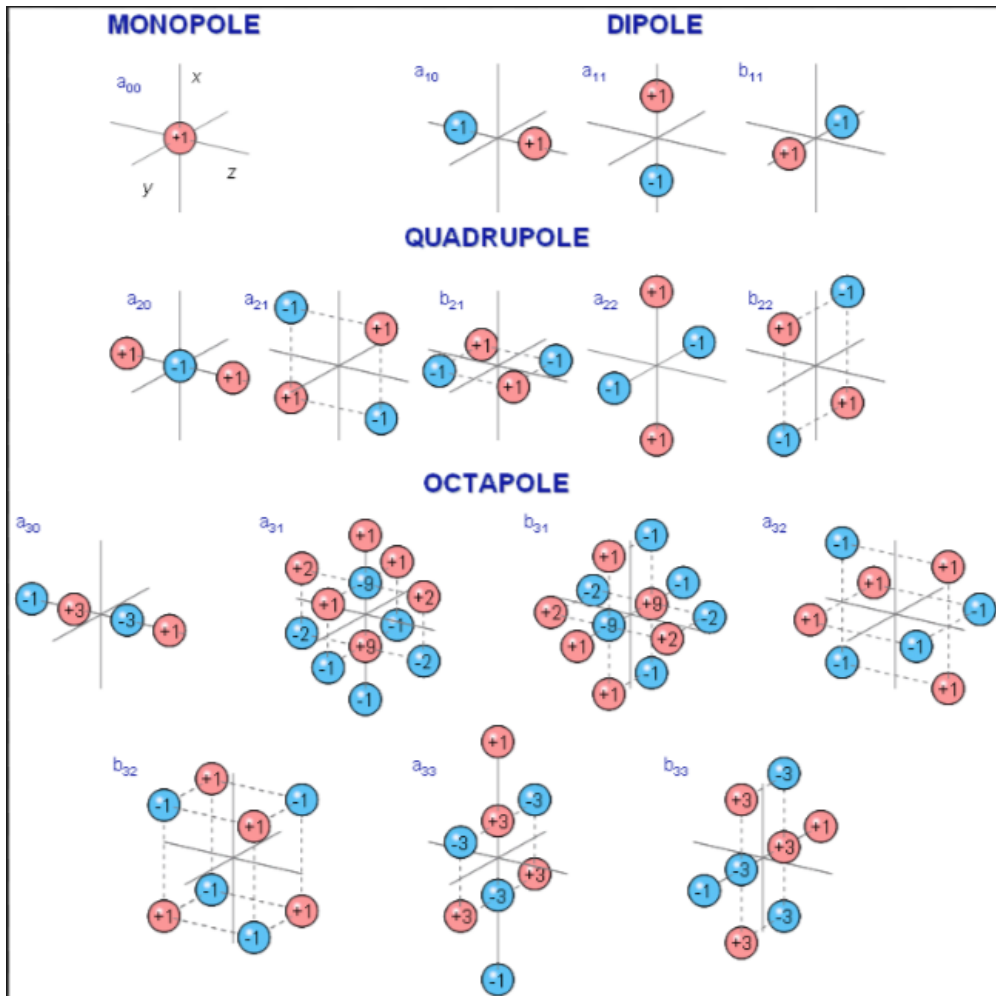
Previously, we found

$$V(r, \theta) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-l-1}) P_l(\cos \theta) \quad (17.1)$$

based on which we can find the inverse of the script-R vector. Let there be an angle α between vectors \mathbf{r} and \mathbf{r}' , with the script-R vector between them. Then

$$\frac{1}{z} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos \alpha) \quad (17.2)$$

The first few terms of the expansion define the monopole, dipole, quadrupole, octopole, and hexadecapole. These have $l = 0$ through $l = 4$ and have radial dependence r^{-1} through r^{-5} . The first few poles look like this:



Although these are the components of a solution that has azimuthal symmetry, they do not in themselves have that symmetry. A superposition of them has that symmetry, which we can find through Fourier series methods as in the homework.

17.1 Matter

Matter can be broadly categorized into conductors and insulators. Conductors have 1 or 2 electrons per atom, which are free to move. Insulators (dielectrics) have all of their electrons bound. These can have induced dipoles due to an applied electric field, which we denote by \mathbf{p} . This points in the same direction as the applied electric field. If $Q = 0$ then \mathbf{p} does not depend on your choice of origin.

Lecture 18: Superposition and Dipole Moments

Lecturer: Matthias Reinsch

8 March

Aditya Sengupta

18.1 Superposition in Solutions to Laplace's Equation

Consider a sphere on which there is some surface charge density, in a region where the potential is known. Say $V_{far} = -E_0 \mathbf{r} \cdot \hat{\mathbf{z}}$. Under rotation, $V_{far} = -E_0 \mathbf{r} \cdot \hat{\mathbf{n}}$. These are the $l = 1$ Legendre polynomial terms, that define the boundary conditions for far fields (far away from the sphere with charge, we expect the potential to go to its known value).

18.2 Induced Dipoles

Consider a charge within a conducting metal sphere. Under no applied field, the field due to the charge is uniform in the sphere. When there is an applied electric field, the charge moves over to a new equilibrium point. The electric field thus induced is given by

$$E_e = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3} \quad (18.1)$$

where d is the distance from the center and a is the radius. The dipole moment is given by

$$qd = 4\pi\epsilon_0 a^3 E \quad (18.2)$$

The dipole moment is therefore proportional to the electric field, with a constant of proportionality $\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 V$. This is a vector relationship, so if the dipole moment and electric field have components p_i and E_i respectively, then $p_i = \alpha E_i$. In some materials, this has the structure of a matrix multiplication,

$$p_i = \alpha_{ij} E_j \quad (18.3)$$

Define \mathbf{P} to be the dipole moment per unit volume.

18.3 Superposition contd.

Recall that for one dipole, the potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{z}}}{r^2} \quad (18.4)$$

With a superposition of dipoles, we get

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\mathbf{z}}}{z^2} d\tau' \quad (18.5)$$

We use integration by parts. Recall that $\nabla \left(\frac{1}{z} \right) = \frac{\hat{\mathbf{z}}}{z^2}$, therefore

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \mathbf{P} \cdot \nabla' \left(\frac{1}{z} \right) d\tau' \\ &= \frac{1}{4\pi\epsilon_0} \oint_S \frac{1}{z} \mathbf{P} \cdot d\mathbf{a}' - \int_{\mathcal{V}} \frac{1}{z} (\nabla' \cdot \mathbf{P}) d\tau' \end{aligned} \quad (18.6)$$

This allows us to define the surface charge density as $\mathbf{P} \cdot \hat{\mathbf{n}}$, and the bulk charge density as $\rho_b = -\nabla \cdot \mathbf{P}$.

Lecture 19: Forces and torques on dipoles

Lecturer: Matthias Reinsch

11 March

Aditya Sengupta

19.1 Torque on a dipole

The torque on a dipole can be found by integrating differential force elements crossed with the position vector,

$$\mathbf{N} = \int \mathbf{r}' \times d\mathbf{F}' = \int \mathbf{r}' \times \mathbf{E} dq' \quad (19.1)$$

We can exchange the order of the differential dq and \mathbf{r}' , and for a constant electric field, we can pull it out. We get

$$\mathbf{N} = \left(\int \mathbf{r}' dq' \right) \times \mathbf{E} \quad (19.2)$$

Then, we can recognize the remaining factor as the dipole moment. Note that this is dependent on the origin; if the total charge is zero, then this dependence does not matter.

19.2 Potential energy

We can extract this information about torques from a scalar, namely the *potential energy*,

$$U(\mathbf{r}, \hat{\mathbf{n}}) = -p\hat{\mathbf{n}} \cdot \mathbf{E}(\mathbf{r}) \quad (19.3)$$

The inputs to the potential energy can be represented in configuration space, $\mathbb{R}^3 \times S_2$. The \mathbb{R}^3 is the three-dimensional space we live in, and S_2 is the surface of a solid sphere.

If we know the angle between the dipole and electric field vectors, we can simplify the scalar equation to

$$U = -\mathbf{p} \cdot \mathbf{E} = -pE \cos \theta \quad (19.4)$$

Recall for one-dimensional linear motion, we have $dW = Fdx$. Similarly, we get $dW = |\mathbf{N}|d\theta$.

We can differentiate the potential energy to get a relationship looking like this,

$$\frac{dU}{d\theta} = pE \sin \theta = |\mathbf{p} \times \mathbf{E}|. \quad (19.5)$$

19.3 Force on dipole

To first order, the change in electric field in the x direction due to a displacement \mathbf{d} is

$$\begin{aligned}\Delta E_x &= (\nabla E_x) \cdot \mathbf{d} \\ &= \left(d_x \frac{\partial}{\partial x} + d_y \frac{\partial}{\partial y} + d_z \frac{\partial}{\partial z} \right) E_x = (\mathbf{d} \cdot \nabla) E_x\end{aligned}\tag{19.6}$$

In general, we get

$$\Delta \varepsilon_0 c E = (\mathbf{d} \cdot \nabla) \mathbf{E}\tag{19.7}$$

and so

$$\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E}\tag{19.8}$$

19.4 Polarization

Let \mathbf{P} be the dipole moment per unit volume of a charge configuration. Then, by superposition, we can find the potential,

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\mathbf{z}}}{r'^2} d\tau'\tag{19.9}$$

and by analogy, we can define surface and bulk charge densities,

$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}\tag{19.10}$$

$$\rho_b \equiv -\nabla \cdot \mathbf{P}\tag{19.11}$$

19.5 Electric Displacement

The electric displacement is defined as follows,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}\tag{19.12}$$

This accounts for the effects of both bound and free charge, $\rho = \rho_b + \rho_f$. We take the divergence of \mathbf{E} , and by Gauss's law we get

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \rho = -\nabla \cdot \mathbf{P} + \rho_f \quad (19.13)$$

$$\nabla \cdot \mathbf{D} = \rho_f \quad (19.14)$$

This appears to resemble Gauss's law, but here, the curl of \mathbf{D} could be nonzero.

Lecture 20: Linear Dielectrics

Lecturer: Matthias Reinsch

15 March

Aditya Sengupta

20.1 Boundary Value Problems with Linear Dielectrics

Recall that the following equations govern linear dielectrics:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \quad (20.1)$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 (1 + \chi_e) \mathbf{E} \quad (20.2)$$

$$\mathbf{P} = \frac{\chi_e}{1 + \chi_e} \mathbf{D} \quad (20.3)$$

$$\rho_b = -\nabla \cdot \mathbf{P} = -\frac{\chi_e}{1 + \chi_e} \rho_f \quad (20.4)$$

We can use this to solve boundary value problems using linear dielectrics. Consider a linear dielectric spanning the extent of the xy plane, with a point charge q on the z axis at some height d . This setup allows us to say $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ for the dielectric. Therefore

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P_z \quad (20.5)$$

Now we invoke the fundamental property of the linear dielectric: that P is proportional to E .

$$\rho_b = P_z = \varepsilon_0 \chi_e E_z|_{z=0^-} \quad (20.6)$$

Recall that in a small patch of charge, meaning that its length scale is less than $\frac{\sigma}{\partial\sigma/\partial x}$, the electric field is $\frac{\sigma}{2\varepsilon_0}$. Therefore we can say $E_z|_{z=0^-}$ is the sum of this surface charge contribution and a point charge contribution from Coulomb's law. We get

$$\sigma_b = -\frac{1}{2\pi} \left(\frac{\chi_e}{\chi_e + 2} \right) \frac{qd}{(r^2 + d^2)^{3/2}} \quad (20.7)$$

and the total bound surface charge comes out to

$$q_b = -\frac{\chi_e}{\chi_e + 2} q \quad (20.8)$$

We can also solve this as an image charge problem. Place a charge q_b at $-d$ on the z axis; then the potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \begin{cases} \frac{q}{z_+} + \frac{q_b}{z_-} & z > 0 \\ \frac{q+q_b}{z_+} & z < 0 \end{cases} \quad (20.9)$$

Lecture 21: Dielectrics

Lecturer: Matthias Reinsch

18 March

Aditya Sengupta

21.1 Linear Dielectrics

By definition, $\mathbf{D} = \epsilon\mathbf{E}$. Consider the quantity $\frac{1}{2}\Delta(\mathbf{D} \cdot \mathbf{E})$, which simplifies to $\frac{1}{2}\epsilon\Delta E^2$. (With a susceptibility tensor, in general \mathbf{D} is not parallel to \mathbf{E} . The relationship is still linear.)

$$\frac{1}{2}\Delta(\mathbf{D} \cdot \mathbf{E}) = \frac{1}{2}\Delta(\epsilon E^2) = \frac{1}{2}\epsilon(2\mathbf{E} \cdot \Delta\mathbf{E} + \Delta E^2) \quad (21.1)$$

In the approximation where the change is small, we get

$$\frac{1}{2}\Delta(\mathbf{D} \cdot \mathbf{E}) \approx \epsilon(\Delta\mathbf{E}) \cdot \mathbf{E} \quad (21.2)$$

21.2 Bring in free charge

We are interested in the amount of work ΔW necessary to bring in a small amount of free charge.

$$\Delta W = \int (\Delta\rho_f)V d\tau \quad (21.3)$$

We know that $\rho_f = \nabla \cdot \mathbf{D}$, so $\Delta\rho_f = \nabla \cdot (\Delta\mathbf{D})$. Therefore the work is

$$\Delta W = \int (\nabla \cdot (\Delta\mathbf{D}))V d\tau \quad (21.4)$$

It is a great day, because we can use Product Rule Five:

$$\nabla \cdot ((\Delta\mathbf{D})V) = (\nabla \cdot (\Delta\mathbf{D}))V + (\Delta\mathbf{D}) \cdot \nabla V \quad (21.5)$$

Therefore

$$\Delta W = \int_{\mathbb{V}} \nabla \cdot ((\Delta\mathbf{D})V) d\tau + \int_{\mathbb{V}} (\Delta\mathbf{D}) \cdot \mathbf{E} d\tau \quad (21.6)$$

Skip to the grand conclusion.

$$W = \frac{1}{2} \int_{all\ space} \mathbf{D} \cdot \mathbf{E} d\tau \quad (21.7)$$

21.3 Capacitor

Consider a parallel plate capacitor with a dielectric slab partially inserted. There is an electric field going from one plate to the other, with fringing fields coming out the sides making loops. The capacitance we get is

$$C = \frac{\epsilon_0 W}{d} (\epsilon_r l - \chi_e x) \quad (21.8)$$

Lecture 22: Magnetism

Lecturer: Matthias Reinsch

1 April

Aditya Sengupta

We know that the equation for the field due to an electric dipole is

$$\mathbf{E}_{dip}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) \quad (22.1)$$

and the equivalent magnetic dipole equation is

$$\mathbf{B}_{dip}(r, \theta) = \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) \quad (22.2)$$

Berkeley is at 37 degrees north, so the magnetic field is approximately proportional to $\hat{\mathbf{r}} + \frac{4}{5}\hat{\theta}$.

22.1 Surface Current Density

Consider a surface of charge moving with a velocity v . Over a time dt , the total charge that exists in the slice that is moved through is $dq = \sigma(vdt)L_{\perp}$, so $I = \frac{dq}{dt} = \sigma v L_{\perp}$ and the current density $K = \frac{I}{L_{\perp}} = \sigma v$.

The vector version of this can generalize the one-dimensional motion to any direction, $\mathbf{K} = \sigma \mathbf{v}$.

22.2 Volume Current Density

By similar reasoning, we get

$$dq = \rho A_{\perp} (vdt) \quad (22.3)$$

$$J = \rho v \quad (22.4)$$

and more generally $\mathbf{J} = \rho \mathbf{v}$. In general ρ and J are independent quantities.

22.3 Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{z}}}{r^2} d\tau' \quad (22.5)$$

We take the divergence of this under the integral sign,

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{z}}}{r^2} \right) d\tau \quad (22.6)$$

Then we use one of our favourite product rules, $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = -\mathbf{A} \cdot (\nabla \times \mathbf{B})$ for a constant \mathbf{A} . I don't see where we went with that, but sure.

Lecture 24: Multipole expansion of \mathbf{A}

Lecturer: Matthias Reinsch

8 April

Aditya Sengupta

The vector potential \mathbf{A} is defined as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{z} d\mathbf{l}' \quad (24.1)$$

which can be expanded,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \left[\frac{1}{r} \oint d\mathbf{l}' + \frac{1}{r^2} \oint r' \cos \alpha d\mathbf{l}' + \dots \right] \quad (24.2)$$

plus some higher-order terms. The first term goes away because the loop is closed, and the second term can be computed. To do this, we compute the following,

$$\mathbf{a} = \int_S d\mathbf{a} = \frac{1}{2} \oint \mathbf{r} \times d\mathbf{l} \quad (24.3)$$

Then, define $\mathbf{V}(\mathbf{r}) = \mathbf{c}(\mathbf{e} \cdot \mathbf{r})$ where \mathbf{c} and \mathbf{e} are constant vectors. Using product rule 7, we get $\nabla \times \mathbf{V} = \mathbf{e} \times \mathbf{c}$. Then, we use Stokes' Theorem to get a nice result from this. If we integrate the curl of \mathbf{V} over an arbitrary surface, we get

$$\int_S (\mathbf{e} \times \mathbf{c}) \cdot d\mathbf{a} = \oint (\mathbf{e} \cdot \mathbf{r})(\mathbf{c} \cdot d\mathbf{l}) \quad (24.4)$$

We can consider boundary conditions, taking a Gaussian pillbox which allows us to find $\mathbf{B}_{above} - \mathbf{B}_{below} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}$. We use $\mathbf{B} = \nabla \times \mathbf{A}$ to relate these two,

$$\mathbf{B}_{above} - \mathbf{B}_{below} = \left(-\frac{\partial A_y^{(above)}}{\partial z} + \frac{\partial A_y^{(below)}}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial A_x^{(above)}}{\partial z} - \frac{\partial A_x^{(below)}}{\partial z} \right) \hat{\mathbf{y}} \quad (24.5)$$

Using the multipole expansion, we can write out the \mathbf{A} vector potential for a dipole,

$$\frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') d\mathbf{l}' = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \quad (24.6)$$

Lecture 25: Torque on magnetic dipole

Lecturer: Matthias Reinsch

10 April

Aditya Sengupta

Consider a rectangular loop inclined at an azimuthal angle θ , with normal magnetic dipole vector \mathbf{m} and current flowing counterclockwise. This exists in a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The torque on this dipole, in general, is given by $\mathbf{N} = \mathbf{m} \times \mathbf{B}$, the counterpart in magnetism to $\mathbf{p} \times \mathbf{E}$.

In general, we can solve for the force by taking the cross product, in terms of the Levi-Civita tensor,

$$C_i = \sum_{j,k=1}^3 \epsilon_{ijk} A_j B_k \quad (25.1)$$

We calculate the force by Taylor expanding for the magnetic field,

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{0}) + \overleftrightarrow{C} \mathbf{r} + \dots + \mathbf{B}(\mathbf{0}) + \left[C_{ik} = \frac{\partial B_i}{\partial r_k} \right] \mathbf{r} + \dots \quad (25.2)$$

Note that because the divergence of \mathbf{B} is 0, the matrix has a zero trace. We set up the integral for force,

$$\mathbf{F} = I \oint d\mathbf{l} \times (\overleftrightarrow{C} \mathbf{r}) \quad (25.3)$$

where we drop the constant term because it goes to 0 when being integrated.

$$F_i = I \sum_{j,k} \epsilon_{ijk} \oint dl_j (\overleftrightarrow{C} \mathbf{r})_k \quad (25.4)$$

$$I \sum_{j,k,m} \epsilon_{ijk} \oint dl_j C_{km} r_m \quad (25.5)$$

Using a convenient product rule (BAC-CAB), we simplify this to just $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$.

$$\Delta \mathbf{m} = -\frac{e^2 R^2}{4m_e} \mathbf{B} \quad (25.6)$$

Lecture 26: Magnetic field lines

*Lecturer: Matthias Reinsch**15 April**Aditya Sengupta*

Magnetic field lines cannot terminate on any kind of magnetic charge because those do not exist.

In general, the magnetization of something is a function of position. We define it completely in analogy with polarization of an electric field. This suggests that there is an analogy to bound charges.

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{z}}}{r^2} d\tau' = \frac{\mu_0}{4\pi} \int \frac{1}{r} [\nabla' \times \mathbf{M}(\mathbf{r}')] d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{r} \mathbf{M}(\mathbf{r}) \times d\mathbf{a}' \quad (26.1)$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_b(\mathbf{r}')}{r} d\tau' + \frac{\mu_0}{4\pi} \oint \frac{K_b(\mathbf{r}')}{r} da' \quad (26.2)$$

This gives us a bound volume current density $\mathbf{J}_b = \nabla \times \mathbf{M}$, and a bound surface current density $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$.

Consider a spinning spherical shell with a surface charge density σ . We write $\mathbf{K} = \sigma \mathbf{v} = \sigma \omega R \sin \theta \hat{\phi}$. We use the bound current density formula.

Lecture 27: Boundary value problems

*Lecturer: Matthias Reinsch**22 April**Aditya Sengupta*

Consider a cylinder with conductivity σ and with its axis aligned to the z -axis, with $V = V_0$ at $z = -L$ and $V = 0$ at $z = 0$. Inside, the potential varies linearly, by $V(x, y, z) = -V_0 \frac{z}{L}$.

This gets more complicated for more boundaries or where the boundaries are not just surfaces of constant potential. For example, consider a cylinder of conductivity σ and radius a , around which is a perfect conductor ranging from radial distances a to b . By solving the boundary value problem, with the same conditions on endpoints as before(?), we get

$$V = -\frac{Iz}{\pi a^2 \sigma} \frac{\ln \frac{s}{b}}{\ln \frac{b}{a}} \quad (27.1)$$

where $a \leq s \leq b$.

27.1 Electrodynamics

The electromotive force (emf), which is not a force, is defined as

$$\varepsilon \equiv \oint \mathbf{f} \cdot d\mathbf{l} \quad (27.2)$$

This is not the Lorentz force law. Here, f is the force per unit charge, such as \mathbf{E} . We see that emf has units of volts. emf is related to magnetic flux by

$$\varepsilon = -\frac{d\Phi}{dt} \quad (27.3)$$

We can compute the emf of an electrodynamic setup by computing a loop integral. For an arbitrary loop in a magnetic field, that may be varying with time, consider a total velocity $\mathbf{w} = \mathbf{v} + \mathbf{u}$, the sum of the velocity of the loop and the velocity of a charge relative to the loop.

Lecture 28: Electric and Magnetic Flux

*Lecturer: Matthias Reinsch**29 April**Aditya Sengupta*

$$\phi = \oint \mathbf{A} \cdot d\mathbf{l} \quad (28.1)$$

We can write down all of Maxwell's equations!

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \quad (28.2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (28.3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (28.4)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (28.5)$$

and we take the divergence of the magnetic curl equation to get a continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad (28.6)$$

Lecture 29: Flux,

Lecturer: Matthias Reinsch

3 May

Aditya Sengupta

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l} \quad (29.1)$$

Dirac's formula for the vector potential in spherical coordinates:

$$\mathbf{A}_{\pm}(r, \theta, \phi) = m_0 \frac{\pm 1 - \cos \theta}{r \sin \theta} \hat{\phi} \quad (29.2)$$

We compute the curl of this,

$$\nabla \times \mathbf{A}_{\pm} = m_0 \frac{\hat{\mathbf{r}}}{r^2} \quad (29.3)$$

This field is called a magnetic monopole.

The vector potential for a pure magnetic dipole is given by

$$\mathbf{A}_{dip}(\mathbf{r}) = \frac{\mu_0 m \sin \theta}{4\pi r^2} \hat{\phi} \quad (29.4)$$

The magnetic monopole vector potential suggests the existence of a so-called Dirac string. Look at \mathbf{A}_+ defined on $\mathbb{R}^3 - \{z\text{-axis} \leq 0\}$. Here $\theta = \pi$ so the small angle approximation does not allow us to resolve the denominator going to zero.

Consider the expression $r \sin \theta \hat{\phi}$ from the dipole vector potential. Using spherical coordinate identities, we can say

$$\mathbf{A}_{dip}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3} (x\hat{\mathbf{y}} - y\hat{\mathbf{x}}) \quad (29.5)$$

Consider a loop oriented counterclockwise as viewed from above. Then there does not exist an \mathbf{A} defined on $\mathbb{R}^3 - \{(0, 0, 0)\}$ that uniquely specifies the magnetic field on any surface bounded by this. We can see this by drawing northern and southern hemispheres with this loop as the equator: $\int \mathbf{B} \cdot d\mathbf{a} > 0$ on the northern hemisphere, and $\int \mathbf{B} \cdot d\mathbf{a} < 0$ on the southern hemisphere. But both of them have to be equal to $\oint \mathbf{A} \cdot d\mathbf{l}$.

Poynting's theorem: $u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2$. From this we get

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J}) d\tau \quad (29.6)$$

We use product rule 6 on the curl of $\mathbf{E} \times \mathbf{B}$ to get

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}) \quad (29.7)$$

From this and Maxwell's equations, we get

$$\frac{dW}{dt} = -\frac{d}{dt} \int_{\mathcal{V}} u d\tau - \oint_S \mathbf{S} \cdot d\mathbf{a} \quad (29.8)$$