Notes for Stars and Planets I: Stellar Structure and Evolution
UC Santa Cruz, Winter 2023

Aditya Sengupta

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Contents

Stars and Planets I: Stellar Structure and Evolution **Winter 2023**

Lecture 1: Distances and gravitational free-fall

Lecturer: Ryan Foley *11 January* *Aditya Sengupta*

Note: *LaTeX format adapted from template for lecture notes from CS 267, Applications of Parallel Computing, UC Berkeley EECS department.*

(the actual lecture 1 was on 9 January, but there's no notes from that)

1.1 Why do stars matter?

We can list just a few reasons:

- Chemical enrichment: we can understand where elements come from, and from that how stars and galaxies work, how planetary climates form, and so on.
- They're the most fundamental extrasolar thing we can observe/the "building blocks of the Universe".
- We can measure distances and get the distance ladder.
- They produce large amounts of dust.

We'll start by talking about aspects of measuring stellar positions and motion: *parallax*, *proper motion* and distance.

1.2 Parallax

Parallax is the relative angle to an object induced by a changing viewpoint, like what we can induce by looking at something through just one eye, and then just the other. If we draw a diagram of the Earth-Sun-object system, we can note that the object subtends an angle, call it p for parallax, relative to the Earth and the Sun. We can use trigonometry to find the distance, using the small angle approximation because $d \gg 1$ AU:

$$d = \frac{1\text{AU}}{\tan p} \approx \frac{1}{p}\text{AU} \tag{1.1}$$

Radians are inconvenient when dealing with very small angles, so we convert to arcseconds and introduce parsecs, defined as $1\text{pc} = 2.063 \times 10^5\text{AU} = 3.086 \times 10^{18}\text{cm}$.

$$d = \frac{2.063 \times 10^5}{p''}\text{AU} = \frac{1}{p''}\text{pc} \tag{1.2}$$

Here, a star at 1 pc has a parallax angle of 1 arcsecond.

We can't make this measurement with just one position of the Earth, so we make them by waiting six months and measuring from opposite positions of the Earth. That means we're usually measuring $2p$, because the distance between opposite positions is 2 AU.

The distance to a star can be written in terms of the apparent magnitude (observed flux, depends on distance) and absolute magnitude (intrinsic luminosity, doesn't depend on distance).

$$d = 10^{\frac{m-M+5}{5}}\text{pc} \tag{1.3}$$

where m is the apparent magnitude and M is the absolute magnitude.

We define the *distance modulus* as

$$\mu = m - M = 5\log d - 5 = 5\log\frac{d}{10\text{pc}}. \tag{1.4}$$

We can equivalently write

$$M = m - 5\log d + 5 = m - 5\log\frac{1}{p''} + 5 = m + 5\log p'' + 5. \tag{1.5}$$

All of these are log base 10.

1.3 Proper motion

Proper motion is the actual movement in the plane of the sky that we can see. If we don't see the star in the same place when we observe it over the span of a year, when the Earth is in the same place, it must have inherently moved. If it moves a distance Δd, that is the result of a velocity over time: $\Delta d = v_\theta \Delta t$. What we observe is an angular difference:

$$\Delta\theta = \frac{\Delta d}{d} = \frac{v_\theta}{d}\Delta t. \tag{1.6}$$

The *proper motion* is the change in this angle over time: $\mu = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{v_\theta}{d}$. (There's a lot of μs in this class.)

We can relate proper motion to velocity as follows:

$$v_\theta[\text{km/s}] = 4.74\frac{\mu[\text{arcsec/yr}]}{p''} \tag{1.7}$$

There's also a radial component to velocity, which we can measure with spectra. The total velocity is therefore

$$V = \left(V_r^2 + \left(4.74\frac{\mu}{p''}\right)\right)^{1/2}. \tag{1.8}$$

1.4 Gravitational free-fall

Suppose there's a cloud of gas working under gravity and no other forces. Let the cloud have a radius R and mass M. We want to think about a small parcel of gas that has mass m' and at a radius r. The force of gravity on this parcel is

$$F_g(r) = \frac{Gm(r)m'}{r^2} \tag{1.9}$$

where $m(r)$ is the enclosed mass. We can sketch what $m(r)$ should look like because we know it has to be monotonically increasing, that $m(0) = 0$, and that $m(R) = M$. Gauss' law tells us that at a radius r', we can ignore mass at $r > r'$. Since the force is a function of $m(r)$, no mass can "get past" any other mass that's exterior to it, and so the enclosed mass at any given time is the enclosed mass at *every* time. (Remember this only holds if you don't have any forces except gravity!)

We can write down an acceleration,

$$a(r) = \frac{Gm(r)}{r^2} = \frac{Gm_0}{r^2} \tag{1.10}$$

where m_0 is $m(r)$ at $t = 0$. This is a positive inward acceleration that's constant no matter what the time is. This implies the cloud will collapse: everything falls into $r = 0$.

How long does collapse take? We'll call this the *free-fall time*. Figuring this out from the acceleration is hard, so we won't do that. Instead, we'll look at energy. Free-fall is taking gravitational potential energy and changing it to kinetic energy.

$$|\Delta E_K| = |\Delta E_g|. \tag{1.11}$$

Assume the system starts at rest. We start with the gravitational potential energy at time zero: $E_{K,0} = 0$. Look at a shell at $r = r_0$: here, we have a potential energy $E_{g,0} = -\frac{Gm_0m'}{r_0}$. At later times, this becomes

$$E_g = \frac{Gm_0m'}{r} - \frac{Gm_0m'}{r_0} \tag{1.12}$$

as the particle falls in from r_0 to r.

The kinetic energy at later times is

$$E_K = \frac{1}{2}m'v^2 = \frac{1}{2}m'\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2. \tag{1.13}$$

The energy balance at any given time is

$$\frac{1}{2}m'\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{Gm_0m'}{r} - \frac{Gm_0m'}{r_0} \tag{1.14}$$

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{Gm_0}{r} - \frac{Gm_0}{r_0}. \tag{1.15}$$

We need to integrate this!

$$t_{ff} = \int_{r_0}^{0} \frac{\mathrm{d}t}{\mathrm{d}r}\mathrm{d}r = -\int_{r_0}^{0}\left(\frac{2Gm_0}{r} - \frac{2Gm_0}{r_0}\right)^{-1/2}\mathrm{d}r \tag{1.16}$$

$$= -\int_{r_0}^{0}\left(\frac{2Gm_0r_0 - 2Gm_0r}{rr_0}\right)^{-1/2}\mathrm{d}r \tag{1.17}$$

$$= -(2Gm_0)^{-1/2}\int_{r_0}^{0}\left(r_0^{-1}\frac{1-r/r_0}{r/r_0}\right)^{-1/2}\mathrm{d}r. \tag{1.18}$$

Substitute $x = \frac{r}{r_0}$ and $\mathrm{d}x = \frac{\mathrm{d}r}{r_0}$ to get

$$t_{ff} = (2Gm_0)^{-1/2}\int_0^1 r_0\left(r_0^{-1}\frac{1-x}{x}\right) \tag{1.19}$$

$$= \left(\frac{r_0^3}{2Gm_0}\right)^{1/2}\int_0^1\left(\frac{x}{1-x}\right)^{1/2}\mathrm{d}x. \tag{1.20}$$

This is a standard integral we can solve by setting $x = \sin^2\theta$, and it comes out to $\frac{\pi}{2}$. This gives us

$$t_{ff} = \frac{\pi}{2}\left(\frac{r_0^3}{2Gm_0}\right)^{1/2} \tag{1.21}$$

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Lecture 2: Stellar diagrams, hydrostatic balance

Lecturer: Ryan Foley *13 January* *Aditya Sengupta*

2.1 H-R diagrams and CMDs

A Hertzsprung-Russell diagram plots log-luminosity as a function of log-effective temperature going left (the temperature it'd have if it were radiating like a blackbody). For stars, the effective temperature is the surface temperature. (get H-R diagram from phone)

A key feature of the H-R diagram is that we can observe that not all regions are filled up, and there's physics reasons that should be the case. Note that this is a snapshot in time: there's regions in the diagram that stars remain in for 10,000 years, but we'll see far fewer of those compared to the main sequence or red giant branch, where stars remain for millions of years.

A color-magnitude diagram plots magnitude against color, and it's effectively the same thing as the H-R diagram but with observable quantities. How we measure color is interesting: if you have a blackbody peaking at some wavelength, the flux goes up as you reduce that wavelength at *all* wavelengths, not just the peak. At low temperatures, $B - V$ is large (e.g. for a red object: V has more flux, more negative magnitude, and it gets subtracted, and red things are cooler because of Wien's law), and at high temperatures, $B - V$ is small or even negative.

2.2 Gravitational free-fall

Last time, we saw that the gravitational free-fall time is given by

$$t_{ff} = \frac{\pi}{2}\left(\frac{r_0^3}{2Gm_0}\right)^{1/2}. \tag{2.1}$$

If we say density is proportional to m/r^3, we can say

$$t_{ff} = \frac{\pi}{2}\left(\frac{3}{8\pi G\rho}\right)^{1/2} = \left(\frac{3\pi}{32G\rho}\right)^{1/2}. \tag{2.2}$$

Example 2.1. How much longer does it take for the Sun to collapse under only gravity than the Earth?

$$t_{ff,\odot} = \left(\frac{3\pi}{32G\rho_\odot}\right)^{1/2}$$
$$t_{ff,\oplus} = \left(\frac{3\pi}{32G\rho_\oplus}\right)^{1/2}$$
$$\frac{t_{ff,\odot}}{t_{ff,\oplus}} = \left(\frac{\rho_\oplus}{\rho_\odot}\right)^{1/2}$$

Plug in $\rho_\odot = 1.4\text{g}/\text{cm}^3$ and $\rho_\oplus = 5.5\text{g}/\text{cm}^3$, and we get $\frac{t_{ff,\odot}}{t_{ff,\oplus}} \approx 2$.

□

This is contrived, because neither the Sun nor the Earth will collapse like this. What stops this from happening? There's a bunch of pressures in each self-gravitating body keeping it intact: for the Sun, it's thermal pressure, and for the Earth, it's electrostatics. Neutron stars and white dwarfs have degeneracy pressures.

2.3 Hydrostatic balance

The Sun is not collapsing, so there must be other forces. When a gas collapses, pressure increases. What does this mean for the overall force?

Take a cylinder with length scale $\mathrm{d}r$, mass $\mathrm{d}m$ and area $\mathrm{d}A$ within a star at a radius r. This has a force F_g and a pressure force from above $F_{p,a}$ pulling it in, and a pressure force from below $F_{p,b}$ pushing it out. This means the pressure force from below has to be greater than the one from above.

Say there's a pressure P at the bottom and $P + \mathrm{d}P$ at the top. The force due to gravity is $F_g = \frac{Gm_0\mathrm{d}m}{r^2}$. The force due to pressure from below is $F_{p,b} = P\mathrm{d}A$, and the force due to pressure from above is $F_{p,a} = (P + \mathrm{d}P)\mathrm{d}A = \left(P + \frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r\right)\mathrm{d}A$. Therefore, the net pressure force is

$$F_p = F_{p,a} - F_{p,b} = \left(P + \frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r - P\right)\mathrm{d}A = \frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r\mathrm{d}A. \tag{2.3}$$

We can write $\mathrm{d}m = \rho(r)\mathrm{d}V = \rho(r)\mathrm{d}r\mathrm{d}A$. The acceleration is

$$-\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{(F_g + F_p)}{\mathrm{d}m} = a_g + \frac{\mathrm{d}P}{\mathrm{d}r}\frac{\mathrm{d}r\mathrm{d}A}{\rho(r)\mathrm{d}r\mathrm{d}A} = a_g + \frac{1}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}. \tag{2.4}$$

If we want to talk about hydrostatic balance, we should have an acceleration of 0.

$$0 = a_g + \frac{1}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r} \tag{2.5}$$

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}}. \tag{2.6}$$

This is the equation of *hydrostatic balance*. Hydrostatic equilibrium holds if we have hydrostatic balance at all radii.

We integrate this equation over the radius of the star after multiplying by $4\pi r^3$.

$$\int_0^R 4\pi r^3 \frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r = -\int_0^R \frac{Gm(r)}{r} \underbrace{4\pi r^2 \rho(r)\mathrm{d}r}_{\mathrm{d}m}. \tag{2.7}$$

The right-hand side is therefore

$$RHS = -\int_0^M \frac{Gm(r)}{r}\mathrm{d}m.$$

This is the gravitational potential energy for the star, E_g.

For the left-hand side, we integrate by parts:

$$LHS = \int_0^R 4\pi r^3 \frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r = P(r)4\pi r^3\Big|_0^R - 3\int_0^R P(r)4\pi r^2\mathrm{d}r. \tag{2.8}$$

Both the limits on the first term go to 0, because $P(R) = 0$, so we just have to look at the second term.

$$LHS = -3\int_0^R P(r)4\pi r^2\mathrm{d}r = -3\langle P\rangle V \tag{2.9}$$

where $\langle P\rangle$ is the volume-averaged pressure.

Putting these sides together, we get

$$\boxed{\langle P\rangle = -\frac{1}{3}\underbrace{\frac{E_g}{V}}_{\text{potential energy density}}}. \tag{2.10}$$

This is a *virial theorem* for stars.

This gives us an overall picture of the star, but we need to fill in some more details. What's causing the pressure force we assumed has to exist?

Inside a star, we have ions, consisting of electrons and nuclei, which exert "gas pressure" according to the ideal gas law:

$$P = n_e kT + \sum n_i kT \tag{2.11}$$

We want to think about the number densities n_i, n_e. We can say

$$n_i = \frac{X_i \rho}{A_i m_p} \tag{2.12}$$

where X_i is the mass fraction, A_i is the number of nucleons, and m_p is the mass of a proton. We can write down the ion pressure:

$$P_i = \frac{kT\rho}{m_p} \sum \frac{X_i}{A_i}. \tag{2.13}$$

Define $\mu_i \equiv \sum \frac{X_i}{A_i}$.

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Lecture 3: Mean molecular weights, atmospheres

Lecturer: Ryan Foley *18 January* *Aditya Sengupta*

3.1 Mean molecular weights

Last time, we decomposed gas pressure into the pressure caused by ions and electrons. The pressure due to ions can be written as

$$P_i = \frac{kT\rho}{m_p}\frac{1}{\mu_i} \tag{3.1}$$

and due to electrons is

$$P_e = n_e kT = \sum Z_i n_i kT \tag{3.2}$$

where Z_i is the number of protons. So rewriting gives us

$$P_e = \frac{kT\rho}{m_p}\sum\frac{Z_i X_i}{A_i} = \frac{kT\rho}{m_p}\frac{1}{\mu_e} \tag{3.3}$$

so overall, we get

$$P = P_i + P_e = \frac{kT\mu}{m_p}\left(\frac{1}{\mu_i} + \frac{1}{\mu_e}\right) \tag{3.4}$$

$$= \frac{kT\rho}{m_p}\underbrace{\sum\frac{(1+Z_i)X_i}{A_i}}_{\mu} \tag{3.5}$$

Here, μ is the mean molecular weight.

For pure hydrogen, the mean molecular weight is $\frac{1}{2}$, as we can verify by plugging in 1 electron, 1 nucleus, and a weight of m_p.

For helium, there are 3 particles (2 electrons, 1 nucleus) and the weight is $4m_p$.

For carbon, there are six electrons and one nucleus and the weight is $12m_p$, so $\mu \sim 2$. As you go higher and higher up, μ gets closer to 2. So generically, we can treat just hydrogen, helium, and metals. We say the mass fraction of hydrogen is X, that of helium is Y, and that of metals is Z. We must have that $X + Y + Z = 1$. In terms of these, the mean molecular weight is

$$\mu = \frac{4}{8X + 3Y + 2Z}. \tag{3.6}$$

We can combine these to eliminate Z and write

$$\mu = \frac{4}{6X + Y + 2}. \tag{3.7}$$

Let's think about some scenarios here.

- If we have pure hydrogen, $X = 1$ and $Y = Z = 0$, we get $\mu = \frac{4}{6+2} = \frac{1}{2}$, which is what we expected. It's impossible for μ to get smaller than this.
- The "cosmic mix": this is a pretty good match for how stars form. This has $X = 0.7$ and $Y = 0.3$, and $\mu = 0.64$.
- Pure helium, like at the end of the main sequence, has $X = 0, Y = 1$, and $\mu = \frac{4}{3}$.

If μ goes up, pressure goes down, and the star starts to contract. But this contraction makes T go up.

3.2 Isothermal atmospheres

In atmospheres, two equations govern the pressure structure:

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\rho g \tag{3.8}$$

$$P = \frac{\rho k T}{\mu m_p}. \tag{3.9}$$

Substituting in P to the derivative, and taking T constant, we get

$$\frac{kT}{\mu m_p}\frac{\mathrm{d}\rho}{\mathrm{d}z} = -\rho g. \tag{3.10}$$

The solution to this is an exponential:

$$\rho(z) = \rho(0)\exp\left(-\frac{\mu m_p g z}{kT}\right) \propto \exp\left(-\frac{z}{H}\right) \tag{3.11}$$

where $H = \frac{kT}{\mu m_p g}$ is the *scale height* of the atmosphere. We can also think about this as the Boltzmann factor. Looking at the numerator and denominator, we can also say it's the thermal energy divided by the force of gravity. If you go down by one scale height, you release kT of energy.

Let's compare the scale height to the radius.

$$\frac{H}{R} = \frac{kTR}{\mu m_p GM} = \frac{kT}{GM\mu m_p/R} = \frac{\text{thermal energy}}{\text{gravitational potential energy}}. \tag{3.12}$$

How thick the atmosphere is is telling us something about the relative strengths of thermal and gravitational potential energies. Usually gravitational potential energy is larger; the Sun has $\frac{H}{R} \sim \frac{1}{1000}$. A low $\frac{H}{R}$ implies very sharp edges.

The column density is $y(z) = \int_z^\infty \rho(z')\mathrm{d}z'$. The units are g/cm^2. For Earth, the column density is about 1000g/cm^2. This lets us integrate Equation 3.8:

$$\int_z^\infty \mathrm{d}P = -\int_z^\infty \rho(z) g \mathrm{d}z \tag{3.13}$$

$$P(\infty) - P(z) = -g\int_z^\infty \rho(z')\mathrm{d}z' = -gy(z) \tag{3.14}$$

$$P(z) = gy(z). \tag{3.15}$$

So far we haven't stipulated that the atmosphere should be isothermal, so adding in that constraint we get

$$y(z) = \int_z^\infty \rho(z) e^{-(z'-z)/H}\mathrm{d}z' = \rho(z)H. \tag{3.16}$$

This gives us $P(z) = gH\rho(z)$. We know that $\rho(z)$ is exponential, so this tells us that the pressure fall-off is proportional to the density fall-off and therefore is also exponentially decaying.

We can calculate the pressure from a non-relativistic gas. Suppose there's a gas in a box with N particles, each carrying velocity $\vec{v}$ and momentum $\vec{p} = m\vec{v}$. The box is a cube of side l.

Consider a specific particle that is only moving along some plane. It bounces off the wall and transfers some of its momentum. If it comes in with momentum p_x and leaves with p_x in the other direction, there's a momentum transfer of $2p_x$ to the wall. How often does this happen? It takes $\frac{l}{v_x}$ to make it to the other end of the box and do the same thing, so the time between collisions with the *same* wall is $\frac{2l}{v_x}$.

The collision rate is $\frac{v_x}{2l}$, and so the rate of momentum transfer is the product of these: $\frac{2p_x v_x}{2l} = \frac{p_x v_x}{l}$, which is a force on the wall. The pressure is $P = \frac{F}{A} = \frac{p_x v_x}{l^3}$.

The pressure on the wall is therefore $P_{x,tot} = \frac{N\langle p_x v_x\rangle}{l^3}$. If the gas is isotropic, then $\langle p_x v_x\rangle = \langle p_y v_y\rangle = \langle p_z v_z\rangle = \frac{1}{3}\langle \vec{p}\cdot\vec{v}\rangle$. If we say the number density is just $n = \frac{N}{l^3}$, then the total pressure is equal to

$$P = \frac{n}{3}\langle \vec{p}\cdot\vec{v}\rangle. \tag{3.17}$$

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Lecture 4:

Lecturer: Ryan Foley *20 January* *Aditya Sengupta*

4.1 Pressure from non-relativistic gas

Last time, we saw that the total pressure in a box was given by

$$P_{tot} = \frac{n}{3}\langle \vec{p}, \vec{v} \rangle \tag{4.1}$$

and we can say $\vec{p} = m\vec{v}$, so

$$P_{tot} = \frac{n}{3}\langle mv^2 \rangle = \frac{2}{3}n\langle \frac{1}{2}mv^2 \rangle \tag{4.2}$$

and we can say $n\langle \frac{1}{2}mv^2 \rangle$ is the kinetic energy density, i.e.

$$\langle P \rangle = \frac{2}{3}\frac{E_K}{V}. \tag{4.3}$$

We also saw that in hydrostatic equilibrium,

$$\langle P \rangle = -\frac{1}{3}\frac{E_g}{V}. \tag{4.4}$$

Setting these equal to each other, we can say that in the case of nonrelativistic gas in hydrostatic equilibrium, $-E_g = 2E_K$, or

$$\boxed{0 = E_g + 2E_K}. \tag{4.5}$$

We must also have that the total energy for the system is $E_{tot} = E_g + E_K$, which lets us find that $E_{tot} = -E_K$ and $E_{tot} = \frac{1}{2}E_g$. If we have a tightly bound system, we should expect a very strongly negative E_g, which means we have a very large E_K: tightly bound systems are hot.

Stars change their state over millions/billions of years, so they can't be in *perfect* hydrostatic equilibrium, but these changes are small and accumulate over time. Any change in E_g will correspond to a change that's twice as large for E_K.

4.2 Pressure from relativistic gas

A relativistic gas is made up of particles moving close to c. For instance, a star with enough photons scattering around to create a pressure will naturally create a relativistic gas. Neutron stars and heavy white dwarfs have relativistic gases.

In relativistic gases, the particle velocity $v \approx c$, and the momentum $p \gg mc$. The energy is given by $E^2 = p^2c^2 + m^2c^4 \approx p^2c^2$, so we can take $E = pc$.

The pressure is

$$P_{tot} = \frac{n}{3}\langle \vec{p} \cdot \vec{v} \rangle = \frac{1}{3}n\langle pc \rangle = \frac{1}{3}\frac{E_K}{V}. \tag{4.6}$$

Therefore, for relativistic gases, we have

$$\boxed{0 = E_g + E_K}. \tag{4.7}$$

This says the only way you can have hydrostatic equilibrium in a relativistic gas is for the binding energy (the difference between the gravitational potential energy and the kinetic energy) to be exactly 0. So if you give one of these stars any extra energy, it is no longer bound. Massive stars drive strong winds due to their photon pressure.

4.3 Polytropes and adiabatic indices

Recall that the adiabatic index is given by $\gamma = \frac{C_P}{C_V}$ and is equal to $\frac{5}{3}$ for a classical ideal gas and $\frac{4}{3}$ for a relativistic gas.

The equation of state is the relation between pressure and density. An *incompressible* fluid has P constant with ρ, or rephrased slightly, $P = K\rho^0$. Generally, you can describe a family of equations of state using

$$P = K\rho^{\frac{n+1}{n}}. \tag{4.8}$$

Here, n is the *polytropic index*.

For situations where you can define the adiabatic index, we have

$$n = \frac{1}{\gamma - 1}. \tag{4.9}$$

Finally, we need to specify a criterion for convection: this occurs when $\frac{\mathrm{d}\log T}{\mathrm{d}\log P} > \frac{\gamma-1}{\gamma}$. The quantity on the left is independent of the base of the log we choose. Everything is adiabatic before this. As soon as you hit this limit, convection starts and the gas expands if necessary. In practice, we check for when this becomes an equality.

Under convection, a gas acts as an ideal gas.

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Lecture 5: Energy transport

Lecturer: Ryan Foley *23 January* *Aditya Sengupta*

Generally, energy starts in the center and moves away from the center. There's a lot of different ways of talking about this. Note that we might talk about "cooling", which refers to transporting energy away but not necessarily to temperature going down. Also, the definition of "heat" isn't necessarily heat as we understand it day-to-day: heat is thermal energy. Heat transport refers to energy transport that's transporting thermal energy. Heating and cooling both refer to changing thermal energy, and you can balance these.

There's three main ways we can transport energy inside a star:

1. Radiation (really the same as conduction)
2. Conduction
3. Convection

Conduction refers to heat transport particle-by-particle: there's energy transport via collisions. You can do this with electrons, protons, or anything else. Conduction is the bulk transport of material to transport heat. It refers to moving heat by physically moving something hot somewhere else. Radiation is conduction with photons.

Consider a gas with a temperature gradient. This is a requirement of energy transport: in thermal equilibrium, particles may run into each other but they won't transfer any energy. There's a collection of particles with velocity v and a mean free path λ. Suppose this is isotropic and particles are only moving in one of the Cartesian directions: $\pm x, y, z$. So $\frac{1}{6}$ of the particles are moving in $+x$. (If you averaged this over 3D space it'd come out to the same thing).

The temperature of the gas is related to the thermal energy density, which we'll call $u(x)$. Set up a surface at some particular x, below which the temperature is higher and above which it's lower, so there's transport across it. Say the temperature is $T(x)$. Consider particles moving through this surface from below. If a particle lands exactly at the surface, on average it must have started from $x - \lambda$, and it'll move to $x + \lambda$. At these points we have $T(x \pm \lambda), u(x \pm \lambda)$. Let's think about how much energy passes through a fixed area in a fixed time, i.e. the energy flux, which we can call $j(x)$. The energy transport through the surface from below is proportional to $u(x - \lambda)$.

$$j(x) = [u(x-\lambda) - u(x+\lambda)]\frac{\lambda}{t} \qquad = [u(x-\lambda) - u(x+\lambda)]\lambda\frac{1}{6}\frac{v}{\lambda} \tag{5.1}$$

where we have a characteristic timescale t. The inverse of this is $\frac{1}{6}\frac{v}{\lambda}$, where $v = v_{th}$. This is the only combination we know gives us a timescale, and it's diminished by $+x$ being one of six preferred directions.

We can Taylor expand u to get

$$j(x) \approx \frac{1}{6}v\left[\left(u(x) - l\frac{\mathrm{d}u}{\mathrm{d}x}\right) - \left(u(x) + l\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right] = -\frac{1}{3}vl\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{5.2}$$

We can write the energy density gradient $\frac{\mathrm{d}u}{\mathrm{d}x}$ as

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}x} = C\frac{\mathrm{d}T}{\mathrm{d}x} \tag{5.3}$$

where C is the heat capacity per volume. This is constant for a material and describes how the energy density changes per change in the temperature.

We see that the flux density of heat j is proportional to $\frac{\mathrm{d}T}{\mathrm{d}x}$, so a steeper temperature gradient has more flux. Further, it's proportional to λ: if you go further between collisions, you go up the temperature gradient more effectively. Finally, $j \propto C$: if particles can hold more energy per volume, they can transport more energy per volume too.

5.1 Conduction by charged particles

Consider electrons in a classical, thermalized ideal gas. The concentration of electrons is n_e and the temperature is T. We know the kinetic energy is $E_K = \frac{3}{2}kT$, so the corresponding energy density is $u_e = \frac{3}{2}n_e kT$. We can find the heat capacity:

$$C_e = \frac{\mathrm{d}u_e}{\mathrm{d}T} = \frac{3}{2}n_e k. \tag{5.4}$$

We also have

$$v_e = \left(\frac{3kT}{m_e}\right)^{1/2} \tag{5.5}$$

and the mean free path is $\lambda = \frac{1}{n_i \sigma}$. We use n_i for ions and the cross-section for electron-ion interacions since the gas is thermalized.

5.2 Radiative diffusion

This describes conduction by photons. We know from the theory of radiative processes that a gas of thermalized photons has an energy density of

$$u_r = \underbrace{\frac{8\pi^5 k^4}{15h^3c^3}}_{a} T^4. \tag{5.6}$$

The heat capacity is

$$C_r = \frac{\mathrm{d}u_r}{\mathrm{d}T} = 4aT^3. \tag{5.7}$$

We can consider $v \sim c$, so

$$j(x) = -\kappa_r \frac{\mathrm{d}T}{\mathrm{d}x}, \tag{5.8}$$

where $\kappa_r = \frac{4}{3}c\lambda aT^3$. In a plasma, photons diffusing comes from electron (Thompson) scattering, which has a mean free path of

$$\lambda = \frac{1}{n_e \sigma_r} \tag{5.9}$$

where σ_r is the Thompson cross-section.

In general, we can rewrite the MFP as $\lambda = \frac{1}{\rho\kappa}$, which lets us write

$$j(r) = -\frac{4ac}{3}\frac{T^3}{\rho\kappa}\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{5.10}$$

If we introduce the radiation pressure $P_r = \frac{1}{3}aT^4$ and the electron pressure $P_e = n_e kT$, this lets us compare the coefficients of radiative diffusion and electron conduction.

$$\frac{\kappa_r}{\kappa_e} = \sqrt{3}z\left(\frac{m_e c^2}{kT}\right)^{5/2}\frac{P_r}{P_e}. \tag{5.11}$$

For the Sun's interior, $\frac{\kappa_r}{\kappa_e} \sim 2 \times 10^5$, so radiative diffusion dominates electron conduction as the dominant form of energy transport. This is generally true for most stars.

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Lecture 6: Convection in stellar interiors

Lecturer: Ryan Foley *25 January* *Aditya Sengupta*

Last time, we saw that

$$j(r) = -\frac{4ac}{3}\frac{T^3}{\rho\kappa}\frac{\mathrm{d}T}{\mathrm{d}r} \tag{6.1}$$

assuming electron scattering. In the interiors of stars, we care about two other types of interactions:

1. free-free absorption (free electron, free photon), aka inverse bremsstrahlung
2. bound-free absorption (bound electron, free photon), aka photoionization. This is the more important effect.

The mean free path for these interactions is frequency dependent, so we need to write an energy density in terms of frequency:

$$u_\nu \mathrm{d}\nu = \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} 8\pi \frac{\nu^2}{c^3}\mathrm{d}\nu. \tag{6.2}$$

The heat capacity is

$$C_\nu \mathrm{d}\nu = \frac{\partial u_\nu}{\partial T}\mathrm{d}\nu; \tag{6.3}$$

the thermal conductivity (which we integrate over all frequencies) is

$$\kappa_r = \int_0^\infty \frac{1}{3} c l_\nu C_\nu \mathrm{d}\nu \tag{6.4}$$

and the mean free path is

$$\lambda = \frac{3}{4acT^3}\kappa_r = \frac{3}{4acT^3}\int_0^\infty \frac{1}{3} c l_\nu C_\nu \mathrm{d}\nu, \tag{6.5}$$

i.e.

$$\lambda = \frac{\int_0^\infty l_\nu C_\nu \mathrm{d}\nu}{4aT^3}. \tag{6.6}$$

This is the *Rosseland mean free path*. This is a hard integral, so skipping over some details, we can relate opacity to temperature and density:

$$\kappa \propto \rho T^{-7/2} \tag{6.7}$$

which is *Kramer's law*. The proportionality constant in CGS for free-free absorption gives us

$$\kappa_{ff} = (0.64 \times 10^{23}) \rho T^{-7/2} \frac{\text{cm}^2}{\text{g}}. \tag{6.8}$$

The constant is what's different between different types of interaction.

For Thompson scattering, we have

$$\kappa_T = \frac{n_e \sigma_T}{\rho} = \frac{(1+X)\rho}{2m_p} \frac{\sigma_T}{\rho} = \frac{(1+X)\sigma_T}{2m_p}. \tag{6.9}$$

In CGS, this averages out to $\kappa_T \approx 0.2(1+X)\frac{\text{cm}^2}{\text{g}}$. When looking inside stars, we can compare these two opacities to see which one dominates.

For instance, for a main-sequence core, we can take $X = 0$ and the central density is about 150g/cm^3. Let's find the point at which $\kappa_{ff} = \kappa_T$.

$$0.64 \times 10^{27} \rho T^{-7/2} = 0.2(1+X) \tag{6.10}$$

which gives us $T = 2 \times 10^7$K. The central temperature of the Sun is about 1.6×10^7K, so electron scattering as the dominant source of opacity is a fair assumption.

6.1 Main-sequence luminosity-mass relations

We know that $L = 4\pi R^2 j = 4\pi R^2 \frac{1}{3} C \frac{1}{n_e \sigma_T} \frac{\text{d}}{\text{d}r}\left(nT^4\right)$. From here, we want to OOM and put L in terms of things we can measure. We can do this using the virial theorem,

$$kT \sim \frac{GMm_p}{R} \implies T \sim \frac{M}{R}. \tag{6.11}$$

Dropping constants and taking $\frac{\text{d}}{\text{d}r} \sim \frac{1}{R}$, we get

$$L \sim R^2 \frac{R^3}{M} \frac{1}{R} \left(\frac{M^4}{R^4}\right) \tag{6.12}$$

where we've taken $n_e \sim \frac{M}{R^3}$. This gives us $L \sim M^3$. We've found that L is explicitly independent of R. To make this equivalence into an equality, we can put these in a ratio:

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^3 \tag{6.13}$$

6.2 Convection

This refers to energy transport through bulk motion of a fluid. It depends on density/buoyancy, and the criterion depends on the temperature gradient of the fluid. Start with an ideal gas in a gravitational field (Ryan: "sounds like a star"; Thummim: "new spherical cow unlocked"). We have the ideal gas law, $PV = NkT$, and we can make it per-volume:

$$P\mu m_p = \rho kT \implies \rho = \frac{\mu m_p}{k}\frac{P}{T}. \tag{6.14}$$

Using the quotient rule, we have

$$\Delta\rho \propto \frac{T\Delta P - P\Delta T}{T^2}, \tag{6.15}$$

so

$$\frac{\Delta\rho}{\rho} = \left(\frac{\Delta P}{T} - \frac{P\Delta T}{T^2}\right)\frac{T}{P} = \frac{\Delta P}{P} - \frac{\Delta T}{T}. \tag{6.16}$$

This tells us that any fractional change in density is accompanied by a fractional change in pressure and temperature.

To find the convective criterion, we perturb a parcel of fluid. In the environment, we have a state (x, P, T, ρ) and just above it a state $(x + \Delta x, P + \Delta P, T + \Delta T, \rho + \Delta\rho)$. Within this, the fluid rises and has a new state $(x + \Delta x, P + \delta P, T + \delta T, \rho + \delta\rho)$; the δ states aren't necessarily the same as the Δ ones.

The blob will be buoyant if it's less dense than its environment: $\delta\rho < \Delta\rho$. Let's say the pressure responds quickly so that $\delta P = \Delta P$ and let's also say the temperature response is adiabatic, so we have $P \propto \rho^\gamma$. This tells us that

$$\delta P \propto \gamma\rho^{\gamma-1}\delta\rho \tag{6.17}$$

$$\frac{\delta\rho}{\rho} = \frac{1}{\rho}\frac{\delta P}{P}. \tag{6.18}$$

We can rewrite the buoyancy condition as

$$\frac{\delta\rho}{\rho} < \frac{\Delta\rho}{\rho} \tag{6.19}$$

$$\underbrace{\frac{1}{\gamma}\frac{\delta P}{P}}_{Eq6.18} < \underbrace{\frac{\Delta P}{P} - \frac{\Delta T}{T}}_{Eq6.16}. \tag{6.20}$$

Math tells us that

$$\Delta T < \frac{\gamma - 1}{\gamma}\frac{T}{P}\Delta P. \tag{6.21}$$

Letting the changes be infinitesimal, we get

$$\frac{\mathrm{d}T}{\mathrm{d}x} < \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}x}. \tag{6.22}$$

Rewriting $\mathrm{d}y/y = \mathrm{d}\ln y$, we get the condition for convection,

$$\boxed{\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} > \frac{\gamma - 1}{\gamma}}. \tag{6.23}$$

Let's further constrain this condition by plugging hydrostatic equilibrium back into this. We get

$$\frac{\mathrm{d}T}{\mathrm{d}r} < \frac{1-\gamma}{\gamma}\frac{T}{P}\delta\rho = \frac{1-\gamma}{\gamma}\frac{T}{P}\frac{Gm\rho}{r^2} \tag{6.24}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} < \frac{1-\gamma}{\gamma}\frac{Gm\mu m_p}{kr^2}. \tag{6.25}$$

This tells us that convection is very good at transporting energy, but going super-convective, i.e. having a temperature/pressure gradient that exceeds $\frac{\gamma-1}{\gamma}$, isn't too helpful, and it's more useful to reduce the temperature gradient by expanding adiabatically.

The useful equation in deciding between these two is comparing the temperature/pressure gradient for adiabatic expansion to that for convective motion. (why do these act against each other?)

$$\frac{\Delta T}{T} = \left[\left.\frac{\partial \ln T}{\partial \ln P}\right|_{ad} - \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right]\frac{\delta P}{P} \tag{6.26}$$

$$= -\frac{\mathrm{d}\ln P}{\mathrm{d}r}\mathrm{d}r. \tag{6.27}$$

In the cores of massive stars, the gradient that dominates is $\nabla = \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}$. (what does this mean?)

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Lecture 7: Heat capacities, the action of convection

Lecturer: Ryan Foley *27 January* *Aditya Sengupta*

7.1 Heat capacities

When we cook in a pot, we heat the bottom and set up a temperature gradient from the bottom to the top via conduction. The gradient gets steeper over time, and after a certain gradient, conduction isn't as efficient as convection. At this point, the bulk motion of water in the pot creates convective eddies that transport heat more efficiently.

We can relate pressure and density in a fluid using the adiabatic index. The energy density of a fluid is

$$U = \frac{3kT}{2\mu m_p}. \tag{7.1}$$

Recall the first law of thermodynamics, $\mathrm{d}U = \mathrm{d}Q - \mathrm{d}W$. For adiabatic expansion, we have $\mathrm{d}W = P\mathrm{d}V$. Define

$$C_p = \left.\frac{\mathrm{d}Q}{\mathrm{d}T}\right|_P, C_V = \left.\frac{\mathrm{d}Q}{\mathrm{d}T}\right|_V. \tag{7.2}$$

The adiabatic index is $\gamma = \frac{C_P}{C_V}$.

Let's think about the first law of thermodynamics for constant volume. If $\mathrm{d}V = 0$, we have $\mathrm{d}U = \mathrm{d}Q = C_V \mathrm{d}T$. For an ideal gas,

$$\mathrm{d}U = \frac{3}{2}\frac{k}{\mu m_p}\mathrm{d}T, \tag{7.3}$$

i.e. $C_V = \frac{3}{2}\frac{k}{\mu m_p}$.

Repeating this for constant pressure, we get

$$\mathrm{d}U = \mathrm{d}Q - P\mathrm{d}V = \left.\frac{\mathrm{d}Q}{\mathrm{d}T}\right|_P \mathrm{d}T - P\left.\frac{\mathrm{d}V}{\mathrm{d}T}\right|_P \mathrm{d}T. \tag{7.4}$$

We can use the ideal gas law, $PV = \frac{k}{\mu m_p}T \implies P\frac{\mathrm{d}V}{\mathrm{d}T} = \frac{k}{\mu m_p}$. Plugging this back in, we get

$$\frac{3}{2}\frac{k}{\mu m_p}\mathrm{d}T = \left(C_p - \frac{k}{\mu m_p}\right)\mathrm{d}T. \tag{7.5}$$

This gives us

$$C_p = \frac{5}{2}\frac{k}{\mu m_p}. \tag{7.6}$$

This gives us the adiabatic index for a monatomic ideal gas, $\gamma = \frac{C_p}{C_V} = \frac{5}{3}$. More generally, $\gamma = \frac{C_p}{C_V} = \frac{1+s/2}{s/2}$, where s is the number of degrees of freedom. As $s \to \infty$, $\gamma \to 1$. With more degrees of freedom, $\frac{\mathrm{d}T}{\mathrm{d}r}$ does not have to be steep for convection to take place.

7.2 The action of convection

We'd like to try and define an equation of motion for the blob of gas we displaced to derive the convection condition. Say the bubble moves from position 1 to position 2. If the star is a polytrope with adiabatic index γ, we have

$$\rho_{2,b} = \rho_{1,b}\left(\frac{P_2}{P_1}\right)^{1/\gamma} \tag{7.7}$$

and the change in the star is

$$\rho_{2,\star} = \rho_1 + \Delta r \left.\frac{\mathrm{d}\rho}{\mathrm{d}r}\right|_\star . \tag{7.8}$$

We compare these to see if the bubble is buoyant. Let $\Delta\rho = \rho_{2,\star} - \rho_{2,b}$. If it's positive, the bubble is buoyant and keeps moving, so it's unstable. If it's negative, the bubble is stable so it'll sink/go back to 1.

If we think about hydrostatic equilibrium, we can combine that with these conditions to get

$$\Delta\rho = \rho\Delta r\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} + \frac{\rho g}{P\gamma}\right). \tag{7.9}$$

The acceleration is

$$\vec{a} = \frac{\Delta\rho}{\rho}g = -\Delta r {-g}\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} + \frac{\rho g}{P\gamma}\right)_{N_{BV}^2}, \tag{7.10}$$

where N_{BV} is the *Brunt-Väisälä* frequency. If $N_{BV}^2 > 0$, we have simple harmonic oscillation.

When things are stable, we have $\vec{a} \approx 0$, so N_{BV} is also close to 0:

$$\left|\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right| \sim \left|\frac{\mathrm{d}\rho g}{\mathrm{d}P\gamma}\right|. \tag{7.11}$$

This implies $N_{BV}^2 \approx \frac{g}{H}$, where $H = \frac{kT}{m_p g}$ is the scale height. So

$$N = g\left(\frac{m_p}{kT}\right)^{1/2} \approx \frac{g}{v_{th}}. \tag{7.12}$$

Near the center of a star, we have $N^2 \approx \frac{g}{R}$ because the scale height is roughly R near the center. This gives us $N^2 \approx \frac{GM}{R^3}$. This is the inverse-squared *dynamical time* of the star, so

$$N \approx \frac{1}{t_{dyn}}. \tag{7.13}$$

The core oscillations of a star are inversely related to its dynamical time, as we might hope.

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Lecture 8: Convection, energy generation

Lecturer: Ryan Foley *30 January* *Aditya Sengupta*

8.1 Convection

Previously, we saw that the acceleration from convection goes as $a = -N_{BV}^2 \Delta r$, where N_{BV}^2, the Brunt-Vaisala frequency (squared), is given by

$$N_{BV}^2 = -g\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} + \frac{\rho g}{P\gamma}\right). \tag{8.1}$$

This acts like a simple harmonic oscillator. For the stable case, we said that $N \sim \frac{g}{v_{th}}$, and for the center of the star, we said $N \sim \frac{1}{t_{dyn}}$. In the unstable case, the equation of motion is

$$\ddot{x} = x\frac{1}{\tau^2}, \tag{8.2}$$

where τ is a timescale. This tells us that $x(t) = x_0 e^{t/\tau}$, and $v = \dot{x}(t) = \frac{x_0}{\tau}e^{t/\tau} = \frac{x(t)}{\tau}$. Plugging in our timescale and taking $x = l$ for our specific situation, we get

$$v = l\sqrt{g\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} + \frac{\rho g}{P\gamma}\right)} \tag{8.3}$$

To proceed, consider the fractional change in density:

$$\left.\frac{\Delta\rho}{\rho}\right|_l = l\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} + \frac{\rho g}{P\gamma}\right). \tag{8.4}$$

In terms of this, the velocity is

$$v = (gl)^{1/2}\left(\left.\frac{\Delta\rho}{\rho}\right|_l\right)^{1/2}. \tag{8.5}$$

If you increase the density gradient, the velocity will be faster due to greater buoyancy.

What if we move a scale height? Let $l = H = \frac{kT}{m_p g}$. The velocity is

$$v = \underbrace{\left(\frac{kT}{m_p}\right)^{1/2}}_{c_s} \left(\frac{\Delta\rho}{\rho}\bigg|_H\right)^{1/2}. \tag{8.6}$$

The first term is the sound speed, and the second is the density gradient. That is, if you move a scale height, you're moving at the sound speed. This almost comes directly out of the definition of the sound speed.

Whenever $\frac{\Delta\rho}{\rho} \sim 1$, we're moving really fast; convection is really efficient at moving energy around.

8.2 Energy generation

We've now got almost everything we need to set up a set of equations describing stellar structure in terms of just mass, radius, and so on. The one missing element is energy generation. We won't do nuclear fusion just yet; we'll start with luminosity and temperature.

Let the luminosity (energy per time) crossing a spherical surface of radius r interior to the star be $L(r)$. Say $L = L(R)$; then on the plot of $L(r)$ against r, we have the points $(0, 0)$ and (R, L). Since we're pushing energy outward, $L(r)$ has to be strictly non-decreasing. If it's ever flat, we can say there's no energy generation in that region (the outer region where this is the case is called the *envelope*.)

Consider $\epsilon(r)$, the power generated per unit volume at radius r. $\epsilon(R) = 0$ because there's no power being generated at the surface. In the center, ϵ starts off high, and drops off to 0 at the start of the envelope. The reason for this is at a given radius, the power added is $\epsilon(r)4\pi r^2 \mathrm{d}r$. So we have

$$\mathrm{d}L = 4\pi r^2 \epsilon(r)\mathrm{d}r \tag{8.7}$$

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \epsilon(r). \tag{8.8}$$

When energy generation stops, we have $\epsilon(r_0) = 0$, so $\frac{\mathrm{d}L}{\mathrm{d}r}\Big|_{r \geq r_0} = 0$.

8.3 Equations of stellar structure

We can now build up a system. For radiative diffusion, we have

$$L(r) = 4\pi r^2 j(r) \tag{8.9}$$

$$\frac{L(r)}{4\pi r^2} = -\frac{4ac}{3}\frac{T^3(r)}{\rho(r)\kappa(r)}\frac{\mathrm{d}T}{\mathrm{d}r} \tag{8.10}$$

$$\boxed{\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4ac}\frac{\kappa(r)\rho(r)}{T^3(r)}\frac{L(r)}{4\pi r^2}}. \tag{8.11}$$

Now, we can talk about the first real stellar structure model, which is *Eddington's standard model*. He made this model without knowing about nuclear fusion, instead just working off knowledge of energy transport.

We know stars are gas-pressure dominated; suppose the ideal gas law holds. Then

$$P(r) = \frac{\rho(r)kT(r)}{\mu m_p}. \tag{8.12}$$

To this we add hydrostatic equilibrium,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\rho(r)Gm(r)}{r^2}. \tag{8.13}$$

This gets rid of one unknown but adds another in $m(r)$, so we need to add in mass continuity:

$$m(r) = \int_0^r \rho(r')4\pi r'^2 \mathrm{d}r'. \tag{8.14}$$

We can also describe pressure in terms of density by a power law,

$$P(r) = K\rho^{\frac{n+1}{n}}(r). \tag{8.15}$$

Here, K is a constant and n is the polytropic index. This equation is called a polytrope.

We can convert between the adiabatic and polytropic indices by $n = \frac{1}{\gamma-1}$ and $\gamma = \frac{n+1}{n}$.

For Eddington's model, we assume radiative transport to relate the density to the temperature.

$$j = -\frac{1}{3}c\frac{1}{n_e\sigma}\frac{\mathrm{d}}{\mathrm{d}r}\left(aT^4\right). \tag{8.16}$$

We'll assume the relevant cross-section is Thomson scattering, and $\kappa = \frac{\sigma_T}{m_p}$. From this, the luminosity is

$$L(r) = -4\pi r^2 \frac{4}{3}\frac{acT^3}{\rho}\frac{1}{\kappa}\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{8.17}$$

We also know that the radiation pressure is equal to $P_{rad} = \frac{1}{3}aT^4$. So we can rewrite the luminosity equation to be in terms of the pressure gradient:

$$L(r) = -4\pi r^2 \frac{c}{\rho\kappa}\frac{\mathrm{d}}{\mathrm{d}r}P_{rad}. \tag{8.18}$$

Now, using hydrostatic equilibrium, we can say

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)}{r^2}. \tag{8.19}$$

Let $-\rho(r)\mathrm{d}r = \mathrm{d}y$, the column density. Then we get

$$L(r) = 4\pi r^2 \frac{c}{\kappa}\frac{\mathrm{d}}{\mathrm{d}y}P_{rad}, \tag{8.20}$$

or

$$\frac{\mathrm{d}P_{rad}}{\mathrm{d}y} = \frac{L(r)}{4\pi r^2}\frac{\kappa}{c} \tag{8.21}$$

and also

$$\frac{\mathrm{d}P}{\mathrm{d}y} = \frac{Gm(r)}{r^2}. \tag{8.22}$$

We can take a ratio:

$$\frac{\mathrm{d}P}{\mathrm{d}P_{rad}} = \frac{4\pi Gm(r)c}{\kappa L(r)} = \frac{4\pi Gc}{\kappa}\frac{M}{L}\frac{m(r)}{M}\frac{L}{L(r)}. \tag{8.23}$$

Here, M is the total mass and L is the surface luminosity. If we take $m(r = R) = M$ and $L(r = R) = L$, we get

$$L_{Edd} = \frac{4\pi GcM}{\kappa} = \frac{4\pi GcMm_p}{\sigma_T}. \tag{8.24}$$

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Lecture 9: Luminosity and pressure balance, stellar structure equations

Lecturer: Ryan Foley *1 February* *Aditya Sengupta*

9.1 Luminosity and pressure balance

Last time, we found the Eddington luminosity, where the total pressure was equal to the radiation pressure. We can write down the Eddington luminosity in units:

$$L_{Edd} = 1.2 \times 10^{38} \frac{\text{erg}}{\text{s}} \frac{M}{\text{M}_\odot} = 3.13 \times 10^4 \text{L}_\odot \frac{M}{\text{M}_\odot}. \tag{9.1}$$

We see that $\frac{\text{L}_\odot}{L_{\odot,Edd}} \sim 10^{-4}$. We can combine this with a previously-known scaling relation, that $L \propto M^3$, to see that

$$\frac{L}{L_{Edd}} \approx 3 \times 10^{-5} \left(\frac{M}{\text{M}_\odot}\right)^2. \tag{9.2}$$

This is called the *Eddington ratio*. For $M \geq 100\text{M}_\odot$, we get $\frac{L}{L_{Edd}} \sim 1$, so the total pressure is comparable to the radiation pressure, and as the photon pressure at the surface of the star is balancing out gravity, it's easy for individual particles to become unbound. Massive stars drive strong winds.

Define

$$\eta(r) = \frac{m(r)}{M} \frac{L}{L(r)}. \tag{9.3}$$

Assume that this is constant with radius. We can say

$$\int_R^r \mathrm{d}P = \frac{L_{Edd}}{L} \eta \int_R^r \mathrm{d}P_{rad}. \tag{9.4}$$

Integrating this, we get

$$P(r) = \frac{L_{Edd}}{L} \eta P_{rad}. \tag{9.5}$$

Using the ideal gas law and the definition of radiation pressure, we get

$$\frac{\rho(r)kT(r)}{\mu m_p} = \frac{L_{Edd}}{L}\eta a T^4(r). \tag{9.6}$$

This tells us that $\rho(r) \propto T^3(r)$, and if gas pressure is dominant,

$$P = P_{gas} \propto \rho(r)T(r) \propto \rho^{4/3}(r). \tag{9.7}$$

Let's analyze a wider range of systems by introducing a parameter β such that $P_{gas} = \beta P$ and $P_{rad} = \frac{1-\beta}{\beta}P_{gas}$. This lets us write temperature and pressure profiles in terms of density:

$$T(r) = \left(\frac{3k}{a\mu m_p}\frac{1-\beta}{\beta}\right)^{1/3}\rho^{1/3}(r) \tag{9.8}$$

$$P(r) = \left[\left(\frac{K}{\mu m_p}\right)^4 \frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{1/3}\rho^{4/3}(r). \tag{9.9}$$

Inverting this and applying some scaling relations, we get

$$\frac{1-\beta}{\beta^4} = 3\times 10^{-3}\mu^4\left(\frac{M}{\mathrm{M}_\odot}\right)^2. \tag{9.10}$$

To get a more concrete idea of this, let's look at various values of $\mu^2 M/\mathrm{M}_\odot$ and see what the corresponding β is.

$\mu^2 M/\mathrm{M}_\odot$	1	2	5	10	50
β	0.997	0.9885	0.9412	0.8463	0.5

and for a normal star with $\mu = 0.6$, we can compare the mass to the ratio of radiation to gas pressure:

$M/\mathrm{M}_\odot$	2.8	14	138
P_{rad}/P_{gas}	3×10^{-3}	0.062	1

138 solar masses is about the upper limit for how high a star's mass can go.

Note that since $L_{Edd} \propto M$ and $L \leq L_{Edd}$, at very high masses, the $L \propto M^3$ scaling relation falls off a bit and becomes more like $L \propto M$.

9.2 Stellar structure equations

Now, we can put everything together and come up with a system of equations to solve!

This system is sufficient to describe a star, if you can solve it. We also need sufficient boundary conditions. These are $m(0) = 0, L(0) = 0, T(R), P(R)$.

Hydrostatic equilibrium	$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}$
Conservation of mass	$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho(r)$
Energy transport, radiative diffusion	$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4ac}\frac{\kappa(r)\rho(r)}{T^3(r)}\epsilon(r)$
Energy generation	$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \epsilon(r)$

Table 9.1: Equations of stellar structure

We'd like to figure out a constraint to put on $T(R), P(R)$, but this isn't that easy to do. We can rearrange this to get

$$\frac{r^2}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r} = -Gm(r), \tag{9.11}$$

and take a derivative to get

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -G\frac{\mathrm{d}m}{\mathrm{d}r} = -4\pi r^2 \rho(r) \tag{9.12}$$

or

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho(r). \tag{9.13}$$

We enforce the polytrope equation, $P = K\rho^{\frac{n+1}{n}}$, and get rid of pressure:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}}{\mathrm{d}r}\left(K\rho^{\frac{n+1}{n}}\right)\right) = -4\pi G\rho(r). \tag{9.14}$$

This is a second-order differential equation, so we need two boundary conditions; we say $\rho(0) = \rho_c$ and $\left.\frac{\mathrm{d}\rho}{\mathrm{d}r}\right|_0 = 0$. If we set ρ_c, we have all of $\rho(r)$. As soon as we have the density profile, we also have the pressure profile. From there, we can get to the temperature profile, and from that we can get everything.

We can solve this by making the substitution $\rho = \rho_c\theta^n$, which implies $P = P_c\theta^{n+1}$. The radius is $r = \alpha\xi$, where $\alpha = \frac{K(n+1)\rho_c^{\frac{n+1}{n}}}{4\pi G}$. Here, θ is the "polytropic temperature" and ξ is a radius-like variable.

The equation is

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n. \tag{9.15}$$

On this, we impose the boundary conditions $\theta(0) = 1$ and $\left.\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_0 = 0$. We're interested in (among other things) the ξ_1 such that $\theta(\xi_1) = 0$, as this is the surface of the star.

There are three analytic solutions to this equation:

- For $n = 0$, we have $\theta(\xi) = 1 - \frac{\xi^2}{6}$ and $\xi_1 = \sqrt{6}$.
- For $n = 1$, we have $\theta(\xi) = \frac{\sin \xi}{\xi}$ and $\xi_1 = \pi$.
- For $n = 5$, we have $\left(1 + \frac{\xi^2}{3}\right)^{-1/2}$ and $\xi_1 = \infty$.

This is called the *Lane-Emden equation*. For any other n, there is no analytic solution, so we have to solve this numerically.

Stars and Planets I: Stellar Structure and Evolution **Winter 2023**

Lecture 10: Scaling relations, pre-main-sequence evolution

Lecturer: Ryan Foley *3 February* *Aditya Sengupta*

10.1 Lane-Emden continued

Last time, we saw the Lane-Emden equation. The $n = 0$ case has constant density, so it describes an incompressible fluid, like water or rock, so it's useful as a description for planet interiors. For $n = 0$, we also have a pressure profile, $P = P_c\theta = Pc\left(1 - \left(\frac{\xi}{\xi_1}\right)^2\right)$. The pressure vanishes at the surface.

For $n = 5$, we plug into the equation,

$$\theta = \left(1 + \frac{\xi^2}{3}\right)^{-1/2}. \tag{10.1}$$

For the inside of the parentheses to be infinite ($\theta = 0$), we need $\xi_1 = \infty$, so the radius is infinite. In fact, for $n \geq 5$, we always have an infinite radius. It's noteworthy that $n = 5$ still has finite mass, but $n > 5$ has infinite mass as well.

For stars, we're interested in the cases we already talked about: an ideal classical gas, $n = 1.5$, and a relativistic gas, $n = 3$. Notably, neither of these cases have analytic solutions. This is a big part of why stellar structure has relied on computation for so long.

10.2 Scaling relations

We know from hydrostatic equilibrium that $P_c \propto \frac{M^2}{R^4}$, and that $\rho_c \propto \langle\rho\rangle \propto \frac{M}{R^3}$. To this, let's add in the polytrope equation: $P \propto \rho^{1+\frac{1}{n}}$. We can combine these things and get a relation between mass and radius.

$$\frac{M^2}{R^4} \propto \left(\frac{M}{R^3}\right)^{\frac{n+1}{n}} \tag{10.2}$$

$$M^{\frac{n-1}{n}} \propto R^{\frac{n-3}{n}} \tag{10.3}$$

or in terms of the adiabatic index,

$$M^{2-\gamma} \propto R^{4-3\gamma}. \tag{10.4}$$

If we have $n = 3$, this tells us something like "there is a mass limit for relativistic stars.".

10.3 Pre-main-sequence evolution

In the first week, we saw that the timescale for gas cloud collapse was given by

$$t_{ff} = \left(\frac{3\pi}{32G\rho}\right)^{1/2}. \tag{10.5}$$

Can we try and compute this in a more realistic situation, to predict when a star will evolve? We can do this by looking at energies. For polytropes, we can compute

$$E_g = -\frac{3}{5-n}\frac{GM^2}{R}. \tag{10.6}$$

For convenience, we'll let $n = 2$. Compare this to

$$E_K = \frac{3}{2}\frac{k}{\mu m_p}MT. \tag{10.7}$$

We have collapse if $E_g > E_K$, which we can rewrite in terms of a critical mass:

$$M_J = \frac{3kT}{2G\mu m_p}R. \tag{10.8}$$

This is the *Jeans mass*. For $M > M_J$, the star will collapse.

We can write

$$M_J = 500\text{M}_\odot\left(\frac{T}{10^4}\right)^{3/2}\left(\frac{1}{n}\right)^{1/2}. \tag{10.9}$$

and there's an equation that I didn't get from there.

The virial theorem (for $n = 0$, to give us an upper limit) tells us that

$$E_{tot} = \frac{1}{2}E_g = \frac{3GM^2}{10R}. \tag{10.10}$$

This gives us the timescale $t_{KH} = \frac{E_{tot}}{L}$. This is the *Kelvin-Helmholtz timescale*. This describes cooling: we're removing energy through luminosity (via photons), i.e. kinetic energy leaves and the temperature drops.

For the luminosity, we'll do the easiest thing possible:

$$L = 4\pi R^2 \sigma T_{eff}^4. \tag{10.11}$$

The timescale for collapse is

$$t_{KH} = \frac{3GM^2}{10R} \frac{1}{4\pi R^2 \sigma T_{eff}^4} = \frac{3GM^2}{40\pi R^3 \sigma T_{eff}^4}. \tag{10.12}$$

For the Sun, this is about 10 million years. But we know the Sun is older than that, and we knew this when we first wrote these equations down. What's powering the Sun?

Put a pin in that – we'll finish up handling the Jeans mass first. Consider the radiative luminosity

$$L_{rad} = 4\pi R^2 \epsilon \sigma T_{eff}^4. \tag{10.13}$$

Other than photons, where does the star's energy go? Some fraction of it will break up molecules.

We want to compare L_{rad} to the free-fall luminosity,

$$L_{ff} = \frac{E_g}{t_{ff}} \approx G^{3/2} \left(\frac{M_J}{R_J} \right)^{5/2}. \tag{10.14}$$

We can combine these to find that

$$M_J^{5/2} = \frac{4\pi}{G^{3/2}} R_J^{9/2} \epsilon \sigma T^4. \tag{10.15}$$

Stars and Planets I: Stellar Structure and Evolution **Winter 2023**

Lecture 11: Fully convective stars, the Hayashi track

Lecturer: Ryan Foley *8 February* *Aditya Sengupta*

We're used to the H-R diagram, where we plot luminosity against effective temperature (which increases to the left). In a protostar, we start with dust as the dominant opacity source, but this gets destroyed at $T_{\text{eff}} \sim 1000$ K. This causes the photosphere to recede inwards until we've hit a new opacity source. If we have the same luminosity, but a much smaller radius, the effective temperature increases a lot. So once dust is destroyed we go very far left on the H-R diagram very fast, until we reach a fully convective star.

What happens to the evolution of a protostar when it's fully convective? First, here's how we get to the fully convective state. We get gas cloud collapse, and faster center collapse. This creates a strong $\frac{\mathrm{d}T}{\mathrm{d}R}$ that leads to the star becoming fully convective.

We say the surface is where $\tau = \frac{2}{3}$. Here, $T_{\text{eff}} = T(R)$ and $P_{Ph} = \frac{g}{\kappa}$. Since this is a convective star, we also have that $T \propto P^{2/5}$, so filling in the proportionality using the photosphere as our known point, we get

$$P_c = 0.77\frac{GM^2}{R^4} \tag{11.1}$$
$$T_c = 0.54\frac{GM\mu m_p}{kR} \tag{11.2}$$

and so

$$kT_{\text{eff}} = 0.6\frac{GM\mu m_p}{R}\left(\frac{R^2}{M\kappa}\right)^{2/5}. \tag{11.3}$$

We're almost done, but we need to understand the opacity source. This isn't Thomson scattering, because it's too cold to have free electrons. Since the star is convective, we have $\gamma = \frac{5}{3}$ and $\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = 0.4$. Multiplying across, we get

$$\gamma \mathrm{d}\ln T = (\gamma - 1)\mathrm{d}\ln P \tag{11.4}$$

and so

$$P^{1-\gamma}T^{\gamma} = K \tag{11.5}$$

where K is the polytropic constant (and there's some work left out here).

Therefore

$$P^{1-\gamma}T^{\gamma} = M^{2-\gamma}R^{3\gamma-4}. \tag{11.6}$$

Taking logs, we get

$$\log P = \frac{2-\gamma}{1-\gamma}\log M + \frac{3\gamma-4}{1-\gamma}\log R - \frac{\gamma}{1-\gamma}\log T. \tag{11.7}$$

How do we find the radius? We have the optical depth at the photosphere, so

$$\frac{2}{3} = \int_R^\infty \kappa\rho \mathrm{d}r = \kappa \int_R^\infty \rho \mathrm{d}r \tag{11.8}$$

if we assume κ is constant with radius. This isn't a nice integral, but we can relate it to something else. Going back to hydrostatic equilibrium, we get

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g \tag{11.9}$$

$$P(R) = \int_R^\infty \rho g \mathrm{d}r \underbrace{\approx}_{\text{mass above surface is negligible}} \frac{GM}{R^2}\int_R^\infty \rho \mathrm{d}r \tag{11.10}$$

and therefore

$$P(R) = \frac{2GM}{3\kappa R^2}. \tag{11.11}$$

If we have an opacity relation $\kappa = \kappa_0 P^a T_{\text{eff}}^b$, we get

$$P(R) = \left(\frac{2GM}{3\kappa_0 R^2} T_{\text{eff}}^{-b}\right)^{\frac{1}{1+a}}. \tag{11.12}$$

Now, we want to relate these to luminosity so we can figure out movement on the H-R diagram.

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4 \tag{11.13}$$

$$\log R = \log K' + 0.5\log L - 2\log T_{\text{eff}} \tag{11.14}$$

Let's eliminate pressure and assume $\gamma = \frac{5}{3}$. We get

$$\log K + \frac{1}{1+a}(\log M - 2\log R - b\log T_{\text{eff}}) = -\frac{1}{2}\log M - \frac{3}{2}\log R - \frac{5}{3}\log T_{\text{eff}} \tag{11.15}$$

and doing even more math, we get

$$\log T_{\text{eff}} = A \log L + B \log M + C. \tag{11.16}$$

for some constants A, B, C. We can substitute Equation 11.14 in to get A, B in terms of a, b:

$$A = \frac{0.75a - 0.25}{5.5a + b + 1.5}, B = \frac{0.5a + 1.5}{5.5a + b + 1.5}. \tag{11.17}$$

To keep going, we need to know the opacity source, which is H^-: a proton with two bound electrons. The ionization energy is 0.75 eV, a small fraction of the 13.6 eV ionization energy for neutral hydrogen. This has an opacity that goes as $\kappa = \kappa_0 P^a T^b Z^c$, where $a \approx 1, b \approx 3, c \approx 0.5$. This yields $A \approx 0.05$ and $B \approx 0.2$. Now, we have a relationship between luminosity and temperature for a given mass.

$$T_{\text{eff}} = 4000\text{K}\mu^{\frac{3}{51}} m^{\frac{7}{51}} l^{\frac{1}{102}}. \tag{11.18}$$

(Here, m is $M/M_\odot$ and similarly for l.)

The exponent on l is *really* small in practical terms, so the effective temperature is basically fixed. So as you collapse, luminosity goes down while temperature stays constant. This vertical line on the H-R diagram is the *Hayashi track*.

The Hayashi forbidden zone is to the right of the Hayashi track: it's not actually forbidden, but stars move over it so fast that you're unlikely to see any stars there.

Stars and Planets I: Stellar Structure and Evolution **Winter 2023**

Lecture 12: Nuclear fusion

Lecturer: Ryan Foley *10 February* *Aditya Sengupta*

Last time, we found that for fully convective stars,

$$T_{\text{eff}} \approx 4000\text{K}\mu^{13/51} m^{7/51} l^{102}. \tag{12.1}$$

or alternatively

$$T_{\text{eff}} \approx 2500\text{K} m^{7/51} r^{1/49}. \tag{12.2}$$

How do we get out of the Hayashi track? One way is to stop collapsing by starting fusion, and the other is to stop being fully convective. If you stop being fully convective, you still get some movement in parallel with the Hayashi track because gravitational collapse and increasing effective temperature still happen. This happens along the *Henyey track*. Along either track, nuclear fusion has to start at some point. The curve on the H-R diagram along which the stars reach the point where they have to start fusion is called the *zero-age main sequence* (ZAMS).

If nucleons combine, their binding energy per nucleon goes up until they reach iron-56.

Luminosity is set by energy transport, which is set by the sources of energy. What could this source be for the Sun? It can't be gravitational collapse, because the timescale for that is too short.

$$t_{KH} \approx \frac{GM^2}{RL} \approx 10^7 \text{ years} \tag{12.3}$$

At the Big Bang, about 75% hydrogen and 25% helium-4 was created. Fusing this hydrogen into helium provides enough energy to power a star for billions of years.

How does fusion work? First, we combine two protons into deuterium, a positron (to balance charge), and an electron neutrino (to balance lepton number).

$$\text{p} + \text{p} \longrightarrow \text{D} + \text{e}^+ + \nu_e$$

This is slow because it's a weak interaction. Once we have deuterium, we fuse another proton:

$$\text{p} + \text{D} \longrightarrow {}^3\text{He} + \gamma$$

where we've released some energy γ. Finally, we fuse two helium-3s:

$${}^3\text{He} + {}^3\text{He} \longrightarrow {}^4\text{He} + 2\,\text{p}$$

This is the *p-p chain*. If you go through and count up how many of each thing you need, you find that overall it fuses 4 protons to get helium-4.

We might ask: if it's energetically favoured, why isn't the entire universe iron? Or why doesn't all hydrogen instantly fuse to helium? The reason is electromagnetic forces. The strong nuclear force is much stronger than anything else, but that only holds at small separations, around 1 fm. Above that, electromagnetism dominates. We can get particles close to each other by heating them up.

The electrostatic energy is

$$V = \frac{Z_1 Z_2 e^2}{r} \tag{12.4}$$

which we balance against $E_{Th} = kT$. Using this conversion, we get 1 eV = 11,600 K. So near the surface of the Sun, fusion isn't possible. At the center, we have $T_c \sim 10^7$ K, so $kT \sim 1$ keV, but this still isn't of the order of the 1 MeV we need. We fix this with quantum tunneling.

Let's review basic quantum mechanics! The Schrödinger equation tells us

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(r) = E\psi(r) \tag{12.5}$$

where m is the reduced mass $\frac{m_1 m_2}{m_1+m_2}$. If $V \approx 0$ except at a potential barrier, we can solve this for the wavefunction outside the barrier

$$E = -\frac{\hbar^2 k^2}{2m}; \psi = e^{ikr} \tag{12.6}$$

and inside the barrier

$$\psi \approx e^{\pm kr}; |\psi(0)|^2 = (e^{-k\Delta r})^2 |\psi_\infty|^2. \tag{12.7}$$

If Δr increases, the probability we can tunnel through it goes down.

For our case, we have

$$-\frac{\hbar^2 k^2}{2mr} = \frac{Z_1 Z_2 e^2}{r} + E. \tag{12.8}$$

The probability of tunnelling is

$$P \approx \left|e^{-k_i \Delta r}\right|^2 = \exp\left(-2\int k(r)\mathrm{d}r\right). \tag{12.9}$$

Solving this, we get that the wavenumber is

$$k = \left(\frac{2m_r}{\hbar^2}\right)^{1/2}\left[\frac{Z_1 Z_2 e^2}{r} - E\right]^{1/2} \tag{12.10}$$

$$= \left(\frac{2m_r E}{\hbar^2}\right)^{1/2}\left(\frac{r_c}{r} - 1\right)^{1/2} \tag{12.11}$$

where r_c is the radius at which the Coulomb force is felt,

$$r_c = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 E}. \tag{12.12}$$

We want to do the integral

$$I = \int k\mathrm{d}r = \left(\frac{2m_r E}{\hbar^2}\right)^{1/2}\int_{r_N}^{r_c}\left(\frac{r_c}{r} - 1\right)^{1/2}\mathrm{d}r. \tag{12.13}$$

Here, r_N is the radius of the nucleus. If we substitute $x = \frac{r}{r_c}$, we get

$$I = \left(\frac{2m_r E}{\hbar^2}\right)^{1/2} r_c \int_{\frac{r_n}{r_c}\approx 0}^{1}\left(\frac{1}{x} - 1\right)^{1/2}\mathrm{d}x. \tag{12.14}$$

Looking this up in an integral table, we get

$$I = \frac{\pi}{2}\alpha Z_1 Z_2\left(\frac{2m_r c^2}{E}\right)^{1/2} \tag{12.15}$$

and so

$$P(0) = \exp\left(-\pi\alpha Z_1 Z_2\left(\frac{2m_r c^2}{E}\right)^{1/2}\right). \tag{12.16}$$

If we increase the number of positive charges, either Z_1 or Z_2 goes up, so the probability goes down. The Coulomb barrier is larger as more protons are present, so it's harder to fuse.

If we rewrite this in terms of $E_G = (\pi\alpha Z_1 Z_2)^2(2m_r c^2)$ and $E = -\frac{\hbar^2 k^2}{2m}$, we can say

$$P(0) = \exp\left(-\left(\frac{E_G}{E}\right)^{1/2}\right). \tag{12.17}$$

Stars and Planets I: Stellar Structure and Evolution **Winter 2023**

Lecture 13: Fusion, reaction rates

Lecturer: Ryan Foley *13 February* *Aditya Sengupta*

Last time, we were looking at the probability of starting fusion using quantum tunnelling. For the *p*-*p* chain, E_G is about 494 keV, and the Sun's thermal energy is $kT \sim 1$ keV. So

$$P(0) = \exp\left(-\left(\frac{494}{1}\right)^{1/2}\right) = \exp(-22) = 2 \times 10^{-10}. \tag{13.1}$$

Another way to think about energy is in terms of velocity, $E = \frac{1}{2}m_r v^2$, where v is the relative velocity between the two particles. In these terms, we have

$$P(0) = \exp\left(-2\pi\frac{e^2 Z_1 Z_2}{\hbar v}\right) \tag{13.2}$$

or, in terms of the Coulomb energy $E_{\text{Coul}} = \frac{e^2 Z_1 Z_2}{\lambda}$, where λ is the de Broglie wavelength,

$$P(0) \propto \exp\left(-\frac{E_{\text{Coul}}(\lambda)}{\frac{1}{2}mv^2}\right). \tag{13.3}$$

Now, to get a reaction rate or a probability density over time, we need to think about time. We'll get this from nuclear cross-sections because that'll give us known inputs to scattering theory. Nuclear cross-sections measure how "sticky" particles are. We'll ignore resonance for this class.

The only length scale we have to start estimating these is the de Broglie wavelength $\lambda = \frac{h}{p} \approx 10^{-10}$ cm. The reaction rate depends on the energy of the system, for which we can use the center of mass energy,

$$E_{\text{COM}} = \frac{1}{2}m_r|\vec{v_1} - \vec{v_2}|^2. \tag{13.4}$$

So the cross-section is

$$\sigma(E) = 4\pi\lambda^2 K(E) P(0) \tag{13.5}$$

which comes from the effective surface area, an energy-dependent constant, and the probability that the wavefunctions actually overlap. (For now, E and E_{con} are the same.) Here, $K(E)$ is a dimensionless constant. We can rewrite the prefactor in terms of other known factors:

$$4\pi\lambda^2 = \frac{4\pi}{k^2} = \frac{2\pi\hbar}{m_r E_{\text{con}}} = 2000 \text{ barns } \frac{keV}{E_{\text{com}}}. \tag{13.6}$$

If $K(E) \sim 1$, then σ is large. We don't actually have a way to analytically figure out $K(E)$, but we can get it from extrapolating from human scales and matching models of the Sun/other stars to observations.

Let's try and get a reaction time. We can get this from $t = \frac{1}{n\sigma v}$ by dimensional analysis/remembering fluids class. Our reaction rate is the inverse of this, or $\frac{1}{t} = n\sigma v$. We want to figure out the rate per volume to put it in the context of stellar cores. If we have species with concentrations n_1, n_2 reacting, we can say

$$\frac{\text{rate}}{\text{volume}} = R_{12} = n_1 n_2 \sigma v. \tag{13.7}$$

We can take a thermal average of σv:

$$\langle \sigma v \rangle = \int_0^\infty v_r \sigma(v_r) P(v_r) \mathrm{d}v_r \tag{13.8}$$

where $P(v_r)\mathrm{d}v_r$ represents the probability that two particles have a relative velocity between v_r and $v_r + \mathrm{d}v_r$. We know this probability follows the Maxwell distribution:

$$P(v_r)\mathrm{d}v_r = \left(\frac{m_r}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m_r v_r^2}{2kT}\right) \mathrm{d}^3 v_r \tag{13.9}$$

so, substituting this in, we can say

$$\langle \sigma v \rangle = \left(\frac{m_r}{2\pi kT}\right)^{3/2} \int_0^\infty \exp\left(-\frac{m_r v_r^2}{2kT}\right) 4\pi v_r^2 \mathrm{d}v_r v_r \sigma(v_r). \tag{13.10}$$

This is the third moment of a half-normal distribution, so it's a known integral, but there's another way to simplify it. Make the substitution $\mathrm{d}E = m_r v_r \mathrm{d}v_r$. We get

$$4\pi v_r^3 \mathrm{d}v_r = \frac{8\pi}{m_r^2} E \mathrm{d}E. \tag{13.11}$$

So the integral becomes

$$\langle \sigma v \rangle = \left(\frac{m_r}{2\pi kT}\right)^{3/2} \int_0^\infty \frac{8\pi}{m_r^2} E \mathrm{d}E \exp\left(-\frac{E}{kT}\right) \frac{1}{E} S(E) \exp\left(-\left(\frac{E_G}{E}\right)^{1/2}\right). \tag{13.12}$$

Here, we've defined $S(E)$ so that it plays the role of $K(E)$ in the expansion of σ. We can rewrite this as

$$\langle \sigma v \rangle = \frac{1}{(kT)^{3/2}} \left(\frac{8}{\pi m_r}\right)^{1/2} \int_0^\infty S(E) \underbrace{\exp\left(-\frac{E}{kT} - \left(\frac{E_G}{E}\right)^{1/2}\right)}_{P(E)} \mathrm{d}E. \tag{13.13}$$

We can understand some of the behaviour of this function by looking at $P(E)$. At large energies, the $-\frac{E}{kT}$ dominates, so the overall curve looks like a decaying exponential. Physically, we can interpret this as our particles following the thermal distribution. At small energies, the $-\left(\frac{E_G}{E}\right)^{1/2}$ dominates, so the overall curve looks like an increasing exponential. Physically, we can interpret this as there being a high probability of tunnelling. If we multiply these two together, we get $P(E)$, a function with a peak and some spread around that peak.

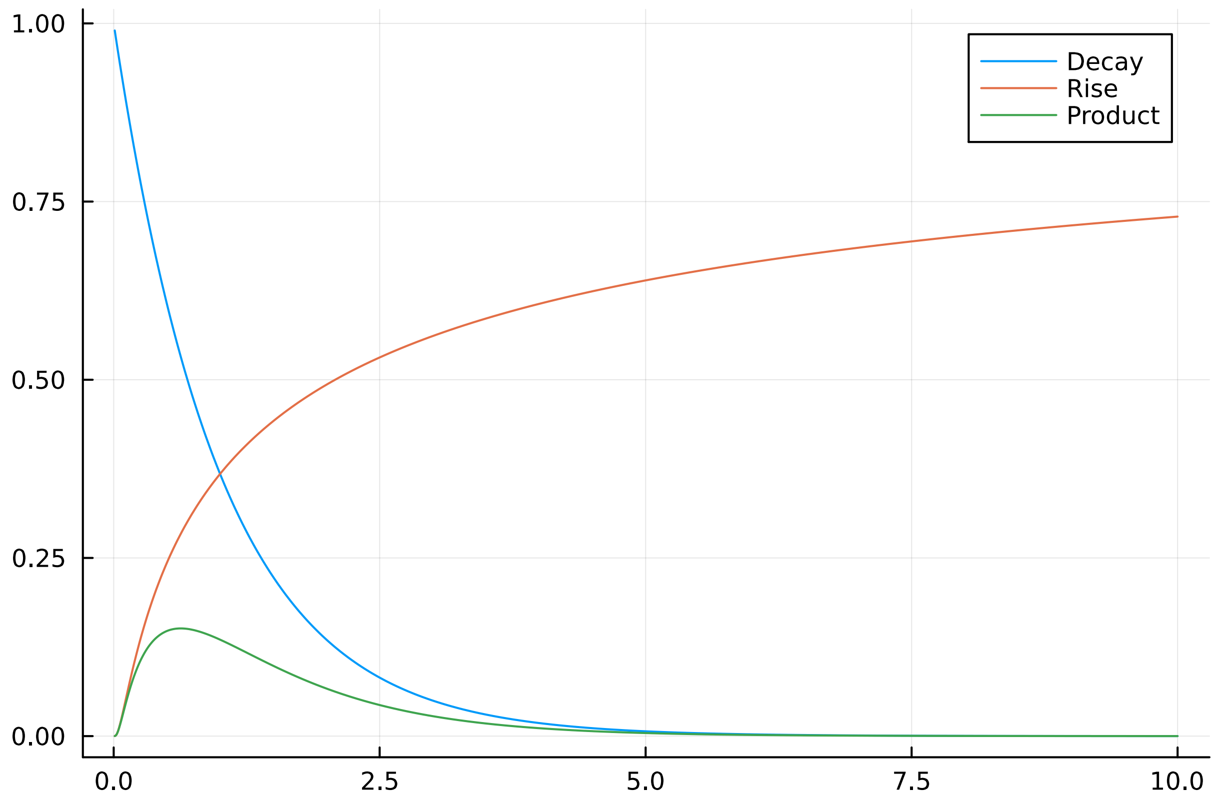

Figure 13.1: Plots of $\exp(-x)$, $\exp\left(-\sqrt{1/x}\right)$, and their product, to show it forms a peak with some spread

How do we find out what this peak is? We can take a derivative and set it to zero, and since $\exp$ is monotonic we can just do this with the argument. We know $P = \exp(-f)$, where

$$f = \frac{E}{kT} + \left(\frac{E_G}{E}\right)^{1/2}. \tag{13.14}$$

Taking a derivative, we get

$$\frac{\partial f}{\partial E} = \frac{1}{kT} - \frac{1}{2}\frac{E_G^{1/2}}{E^{3/2}}. \tag{13.15}$$

The peak energy is attained at $E = E_0$ such that $\left.\frac{\partial f}{\partial E}\right|_{E=E_0} = 0$. This comes out to

$$E_0 = \left(\frac{E_G(kT)^2}{4}\right)^{1/3}. \tag{13.16}$$

To find the spread of this peak, we Taylor expand f around it.

$$f(E) = f(E_0) + \frac{1}{2}(E - E_0)^2 \frac{\partial^2 f}{\partial E^2} + \mathcal{O}\big((E - E_0)^4\big). \tag{13.17}$$

Substituting in gives us $f(E_0) = 3\left(\frac{E_G}{4kT}\right)^{1/3}$. We get

$$P = \exp\left(-\left(\frac{3E_G}{4kT}\right)^{1/3}\right) \exp\left(-\left(\frac{E - E_0}{\Delta/2}\right)^2\right). \tag{13.18}$$

We get

$$\Delta = 1.83 E_G^{1/6} (kT)^{5/6} \tag{13.19}$$

so the peak is governed more heavily by the thermal energy than the Gamow energy.

We can plug in some typical numbers for the *p*-*p* and *p*-*C* reactions to get an idea of how large these peaks are.

	p + p	p + ^{12} C
E_G	494 keV	32.6 MeV
E_0	4.5 keV $\left(\frac{T}{10^7\,\mathrm{K}}\right)^{2/3}$	18.2 keV $\left(\frac{T}{10^7\,\mathrm{K}}\right)^{2/3}$
$\frac{\Delta}{E_0}$	$\left(\frac{T}{10^7\,\mathrm{K}}\right)^{1/6}$	0.5 $\left(\frac{T}{10^7\,\mathrm{K}}\right)^{2/3}$

Table 13.2: Gamow energy, peak energy, and spread for two reactions

We see that *p* + *C* has a smaller peak than *p* + *p* does.

Returning to $\langle \sigma v \rangle$ and treating $S(E)$ as a constant at around the energy of fusion,

$$\langle \sigma v \rangle = 2.6 \frac{E_G^{1/6}}{m_r^{1/2}} \frac{S(E_0)}{(kT)^{3/2}} \exp\left(-3\left(\frac{E_G}{4kT}\right)^{1/3}\right). \tag{13.20}$$

The Gamow energy and $S(E_0)$ are constant for each type of reaction, so we can consider this to be just a function of T.

We can plug this into the reaction rate to get a final result:

$$\boxed{R_{12} \propto n_1 n_2 S(E_0) \exp\left(-3\left(\frac{E_G}{4kT}\right)^{1/3}\right)} \tag{13.21}$$

We can qualitatively interpret each term: the reaction rate goes up if there's more of each reactant, if the particles are more sticky, or if the Gamow energy "wins" over kT. How sensitive is this to temperature?

$$\frac{\mathrm{d}R_{12}}{\mathrm{d}T} \approx \left(\frac{E_G}{4kT}\right)^{1/3} \frac{R_{12}}{T}. \tag{13.22}$$

For *p* + *d*, we have $T \sim 2 \times 10^7$ K, so

$$\frac{\mathrm{d}R_{\text{pd}}}{\mathrm{d}T} \approx 4.6 \frac{R_{\text{pd}}}{T} \implies R_{\text{pd}} \propto T^{4.6}. \tag{13.23}$$

Detailed calculations show $R_{\text{pd}} \approx T^4$, so these are highly dependent on temperature. We'll deal with reactions with proportionalities around T^{20}.

Stars and Planets I: Stellar Structure and Evolution **Winter 2023**

Lecture 14: Nuclear fusion and halting collapse

Lecturer: Ryan Foley *15 February* *Aditya Sengupta*

Last time, we saw that reaction rates had the relationship

$$R_{12} \propto n_1 n_2 S(E_0) \exp\left(-3\left(\frac{E_G}{4kT}\right)^{1/3}\right) \tag{14.1}$$

and

$$\frac{\mathrm{d}R_{12}}{\mathrm{d}T} \approx \left(\frac{E_G}{4kT}\right)^{1/3} \frac{R_{12}}{T}. \tag{14.2}$$

For *p-d*, we find that $R_{pd} \propto T^4$.

Let's think about how this relates to proton capture onto carbon. If we say $R \propto T^\nu$, we're interested in

$$\frac{\nu_{pC}}{\nu_{pd}} = \frac{\left(\frac{E_{G,pC}}{4kT}\right)^{1/3}}{\left(\frac{E_{G,pd}}{4kT}\right)^{1/3}} = \left(\frac{E_{G,pC}}{E_{G,pD}}\right)^{1/3} = \left(\frac{32.6}{0.494}\right)^{1/3} \approx 66^{1/3} \approx 4. \tag{14.3}$$

Therefore, $R_{pc} \propto T^{17}$ (approximately 16 and more careful analysis finds 17.) This tells us that all fusion reactions are very temperature-sensitive. This is because as you go up to heavier elements, the Gamow energy keeps rising. At some level, we're getting to a situation where to produce even close to the same amount of luminosity for stars, the temperature remains roughly the same.

We need to push this back into an energy generation rate ϵ.

$$\epsilon = \frac{n_1 n_2 \langle \sigma v \rangle E_{\text{Nuc}}}{\rho}. \tag{14.4}$$

For *p-p*, we look at the rate-limiting step, i.e. the thing that happens slowest. This is because the timescale is set by $t = \frac{1}{n\sigma v}$, so we're interested in the slowest σv as what dominates this. For the *p-p* chain, this is p + p $\longrightarrow$ d + e^+ + ν_e where a weak interaction converts a proton to a neutron. The ϵ we get here is

$$\epsilon = 2.8 \times 10^{30} \frac{S}{S_{\text{strong}}} \frac{\rho}{T_7^{2/3}} \exp\left(-\frac{15.70}{T_7^{1/3}}\right) \tag{14.5}$$

where $T_7 = \frac{T}{10^7\,\text{K}}$. Here, $\frac{S}{S_{\text{strong}}} \sim 10^{-25}$ is the weak interaction suppression factor, which we should intuitively understand as accounting for the fact that protons don't convert to neutrons very often.

When do we halt collapse? I don't know because I'm very annoyed at my TeX distribution now.

$$L = 3 \times 10^{28} \frac{\text{erg}}{\text{s}} m^{1/2} r^4 \tag{14.6}$$

where $m = M/0.1M_\odot$ and similar for R.

We can now find the luminosity due to nuclear fusion:

$$L_{\text{Nuc}} = \int \epsilon \text{d}M \approx \epsilon_c M_{\text{core}} \tag{14.7}$$

where ϵ_c is the energy rate in the core.

For *p*-*p*, $S = 10^{-25} S_{\text{strong}} = 4 \times 10^{-22}$keV barn.

For fully convective stars on the Hayashi track,

$$T_c = 0.54 \frac{GM\mu m_p}{kR} \tag{14.8}$$

so

$$L_{\text{nuc}} = 2 \times 10^{40} \frac{\text{erg}}{\text{s}} \frac{m^{4/3}}{r^{7/3}} \exp\left(-17.35 \frac{r^{1/3}}{m^{1/3}}\right). \tag{14.9}$$

When collapse is halted, $L = L_{\text{nuc}}$, so

$$10^{12} m^{4/3} r^{-7/3} \exp\left(-17.35 m^{-1/3} r^{1/3}\right) = m^{1/2} r^4. \tag{14.10}$$

We can solve this and find that $r \sim 1$ and $m \approx 3$.

We can halt collapse when we have fusion from $\sim 0.3M_\odot$ in $\sim 0.1R_\odot$. At $0.1M_\odot$, we have $T = 3.4 \times 10^6$ K, which is consistent with *p*-*p*.

Can the other steps in the *p*-*p* chain that aren't rate-limiting also halt collapse? Let's look at deuterium burning to find out.

$$\langle \sigma v \rangle = 2.6 \frac{E_G^{1/6}}{m_p} \frac{S(E_0)}{(kT)^{2/3}} \exp\left(-3\left(\frac{E_G}{4kT}\right)^{1/3}\right) \tag{14.11}$$

$$E_G = (\pi \alpha Z_1 Z_2)^2 (2 m_r c^2) = 0.988 \text{MeV} (Z_1 Z_2)^2 \frac{m_r}{m_p}. \tag{14.12}$$

Almost all fusion reactions have similar S, so is there anything else in here that might help us compete with hydrogen burning? For the p + d $\longrightarrow$ ^{3}He + γ reaction, we have a reduced mass of $\frac{1m_p \times 2m_p}{1m_p + 2m_p} = \frac{2}{3} m_p$, and

$Z_1 = Z_2 = 1$. We get that the Gamow energy is 655 keV, higher than the 494 for *p*-*p*, and the nuclear energy we get is 5.5 MeV. We have $\langle \sigma v \rangle = 8 \times 10^{-20} \text{cm}^3/\text{s} \frac{1}{T_7^{2/3}} \exp\left(\frac{-17.24}{T_7^{1/3}}\right)$. The timescale for deuterium burning is

$$t_D = \frac{1}{n_p \langle \sigma v \rangle} = \frac{2 \times 10^{-5} T_7^{2/3}}{\rho} \exp\left(\frac{17.24}{T_7^{1/3}}\right). \tag{14.13}$$

We use n_p here because the reaction is limited by when a deuterium atom will run into a proton.

For comparison, the collapse time is $\frac{R}{\frac{dR}{dt}}$ or $\frac{E_g}{L} = \frac{3GM^2}{7R}\frac{1}{L}$. If we want to compare these, we'll need more information. We can use the scaling relations of the Hayashi track:

$$T_{\text{eff}} \propto M^{7/51} L^{1/102} \propto 4000\text{K}\left(\frac{M}{M_\odot}\right)^{7/51} \tag{14.14}$$

i.e.

$$L = 9 \times 10^{32} \frac{\text{erg}}{\text{s}} \left(\frac{R}{R_\odot}\right)^2 \left(\frac{M}{M_\odot}\right)^{28/51} \tag{14.15}$$

so the collapse time is

$$t_{\text{collapse}} = 1.8 \times 10^{15} \text{s} \left(\frac{M}{M_\odot}\right)^{74/51 \approx 3/2} \left(\frac{R}{R_\odot}\right)^3. \tag{14.16}$$

We're still not done because we need to relate M, R to ρ, T. We can do this using an $n = \frac{3}{2}$ polytrope, which gives us $\rho_c = 6\langle\rho\rangle = 8.3 \frac{\text{g}}{\text{cm}^3} \left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1}$. So we have a central temperature of

$$T_c = 0.54 \frac{GMm_p}{kR} = 7.4 \times 10^6 \text{K} \left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1} \tag{14.17}$$

Now, we can set the two timescales equal. Let $m = M/M_\odot$ and $r = R/R_\odot$; then

$$1.8 \times 10^{15} m^{3/2} r^3 = \frac{2 \times 10^{-5}}{8.3 m/r^3} 0.74^{2/3} \left(\frac{m}{r}\right)^{2/3} \exp\left(\frac{17.24}{0.74^{1/3}} \left(\frac{r}{m}\right)^{1/3}\right) \tag{14.18}$$

and so

$$r = m(2.53 + 0.096 \ln m - 0.28 \ln r)^3. \tag{14.19}$$

This happens at $m \sim 1, r \sim 1$ (no it doesn't, unless they're close to 1 in opposite directions; see the table below) so generally we burn all deuterium about when we reach MS.

m	r	$\frac{T_c}{10^5\text{K}}$	$0.13\frac{r}{m}$	t_D (Myr)
0.03	0.43	5.2	1.86	7.5
0.1	1.17	6.3	1.53	1.7
0.3	2.86	7.7	1.24	0.5

Table 14.3: Deuterium-burning timescales for convective stars

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Lecture 15: CNO luminosity, stellar structure, the Saha equation

Lecturer: Ryan Foley *22 February* *Aditya Sengupta*

15.1 CNO luminosity

Last time, we saw that the CNO cycle's rate-limiting step was proton capture onto 14-nitrogen. For this step, the Gamow energy is 48.1 MeV. The rate is $n_p\langle\sigma v\rangle_{pN}$ and the energy generation in the core is is

$$\epsilon = 28\text{ MeV}\,\frac{n_p n_{14}\langle\sigma v\rangle}{\rho}. \tag{15.1}$$

Combining this with $S = 2.75$ keV barn and $n_{14} = 10^{-3}n_p$, we get

$$\epsilon = 2.5\times 10^{25}\frac{\text{erg}}{\text{g}\cdot\text{s}}\rho T_7^{-2/3}\exp\left(-\frac{72.19}{T_7^{1/3}}\right). \tag{15.2}$$

The luminosity provided by this is $L = \int \epsilon \mathrm{d}M$. If we have $\epsilon \propto T^\nu$, we can say

$$\begin{aligned} \mathrm{d}\ln\exp(x) &= \mathrm{d}\ln T^\nu \\ \mathrm{d}x &= T^{-\nu}\nu T^{\nu-1}\mathrm{d}T = \nu\frac{\mathrm{d}T}{T} \\ \nu &= \frac{\mathrm{d}x}{\mathrm{d}T}T. \end{aligned} \tag{15.3}$$

Taylor expanding around T_7, we can say

$$\nu = \frac{\mathrm{d}x}{\mathrm{d}T}T = \frac{1}{3}\frac{72.19}{T_7^{4/3}}T \approx 24T_7^{-1/3}. \tag{15.4}$$

Therefore $\epsilon \propto \exp(-72.19)T_7^{24}$ and the luminosity generated by the CNO cycle is

$$L = \int \epsilon \mathrm{d}M = 5 \times 10^{58} \frac{\mathrm{erg}}{\mathrm{s}} \frac{M^2}{R^3} T_7^{-2/3} \exp(-72.19) T_7^{24} \approx 10^{26} \frac{M^2}{R^3} T_7^{23}. \tag{15.5}$$

If we want to match the usual scaling relation for luminosity as a function of mass, we need

$$L = L_\odot \left(\frac{M}{M_\odot} \right)^3 \tag{15.6}$$

$$3.9 \times 10^{33} M^3 = 10^{26} \frac{M^2}{R^3} T_7^{23} \tag{15.7}$$

so we find that $T_7 \sim 2.1$ pretty much independent of mass. However, this depends on the Taylor expansion we did around $T_7 \sim 1$, so we should re-expand around $T_7 = 2$.

$$\nu = \frac{1}{3} \frac{2^{-1/3} \times 72.19}{(T_7/2)^{1/3}} \approx 19 \left(\frac{T_7}{2} \right)^{-1/3} \tag{15.8}$$

$$3.9 \times 10^7 = M^{-1} R^{-3} \left(\frac{T_7}{2} \right)^{18}. \tag{15.9}$$

We can eliminate R using hydrostatic balance:

$$3.9 \times 10^7 = M^{-1} \left(\frac{GM\mu m_p}{kT_c} \right)^{-3} \left(\frac{T_7}{2} \right)^{18} \tag{15.10}$$

or

$$T_7 = 1.83 M^{4/21}. \tag{15.11}$$

It's important to remember that this is only a statement about CNO.

The central temperature of CNO stars is roughly independent of mass.

On the main sequence, we can apply previously-known scaling relations and find that for CNO stars, $T_{\text{eff}} \propto M^{0.34}$. This works for $1 \lesssim M/M_\odot \lesssim 10$.

15.2 The structure of massive stars

With this understanding of nuclear reactions, we can start to understand the structure of massive stars. For CNO, ϵ is centrally located, meaning $L(r) = L$ even for small radii. To figure out how that gets transported outwards, we need to figure out where the star is convective, i.e. where $\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} > \frac{2}{5}$ holds. For an ideal gas, $\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g$. We can write

$$F(r) = -\frac{4}{3}\frac{ac}{\kappa\rho}T^3\frac{\mathrm{d}T}{\mathrm{d}r} \tag{15.12}$$
$$= \frac{4}{3}\frac{acg}{\kappa}T^3\frac{\mathrm{d}r}{\mathrm{d}P}\frac{\mathrm{d}T}{\mathrm{d}r} \tag{15.13}$$
$$= \frac{4}{3}\frac{acg}{\kappa}P_{\text{tot}}^{-1}T^4\frac{P}{\mathrm{d}P}\frac{\mathrm{d}T}{T} \tag{15.14}$$
$$= \frac{4}{3}\frac{acg}{\kappa}P_{\text{tot}}^{-1}T^4\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}. \tag{15.15}$$

Therefore, using $L(r) = 4\pi r^2 F(r)$ and $g = \frac{Gm(r)}{r^2}$,

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = \frac{P_{\text{tot}}}{aT^4}\frac{3}{4}\frac{L(r)}{L_{\text{Edd}}}\frac{M}{M(r)}. \tag{15.16}$$

This becomes large at small r, so CNO stars have convective cores. How big is this convective core? We can find scaling relations in terms of mass for each term:

$$\frac{P_{\text{tot}}}{P_{\text{rad}}} = 2600\left(\frac{M}{M_\odot}\right)^{-2} \tag{15.17}$$
$$\frac{L}{L_{\text{Edd}}} = 4\times 10^{-5}\left(\frac{M}{M_\odot}\right)^{2} \tag{15.18}$$

so

$$\frac{2}{5} = \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = 0.1\frac{M}{m(r)} \tag{15.19}$$

i.e. $m(r) = \frac{1}{4}M$.

If you generate all your energy in $< \frac{1}{4}M$, you have a convective core. CNO stars have a convective core of about $\frac{1}{4}M$. Outside the core, radiative diffusion dominates, so there's a radiative envelope.

$\frac{M}{M_\odot}$	T_7	$q_c = \frac{M_{\text{conv}}}{M}$	$q_e = \frac{M_{\text{conv,env}}}{M}$
60	3.9	0.73	0
15	3.3	0.4	0
5	2.6	0.23	0
1.5	1.9	0.07	0
1.0	1.5	0	0.0035
0.6	0.9	0	0.7
0.3		1	1

Table 15.4: Convective core and envelope sizes for different stars

Roughly, at $M > M_\odot$, we have the CNO cycle so there's a convective core and no envelope, and at $M < M_\odot$ we have the *p-p* chain so there's no convective core and a convective envelope. As we get smaller, we go towards being fully convective. This gives us three broad types of stars:

- For $M/M_\odot \gtrsim 1.5$, the energy source is the CNO cycle. It has a convective core and a radiative envelope.
- For $0.5 \lesssim M/M_\odot \lesssim 1.5$, the energy source is the *p-p* chain. It has a radiative core and a convective envelope.
- For $M/M_\odot \lesssim 0.5$, the energy source is the *p-p* chain. It is fully convective.

15.3 The Saha equation

When we look at a stellar spectrum, we see a blackbody with absorption features. We need to understand the ionization states of elements in the star to understand those absorption features. Let's start with hydrogen – when is it ionized? Hydrogen's ionization energy is 13.6 eV, which happens at about $T \sim 10^5$ K.

Stars and Planets I: Stellar Structure and Evolution **Winter 2023**

Lecture 16: The Saha equation

Lecturer: Ryan Foley *24 February* *Aditya Sengupta*

Stars and Planets I: Stellar Structure and Evolution **Winter 2023**

Lecture 17: Spectral sequences, evolution on the main sequence

Lecturer: Ryan Foley *27 February* *Aditya Sengupta*

17.1 Excitation states

If we want to think about hydrogen in stars using traditional optical instrumentation, we want to look at the lines of hydrogen at optical wavelengths, which are the Balmer lines. These are transitions from or to the $n = 2$ state, and the energy difference from $n = 1$ to $n = 2$ is 10.2 eV. Therefore,

$$\frac{n_{n=2}}{n_{n=1}} = \exp\left(-\frac{10.2 \text{ eV}}{k}T\right) = 10^{-5} \text{ at } T = 10^4\text{K}. \tag{17.1}$$

The cross-section is $\sigma_\gamma = 10^{-16}\text{cm}^2$. For comparison, the Thomson cross-section is $\sigma_T = 0.6 \times 10^{-24}\text{cm}^2$, so there's more scattering through this process even if there's less abundance. The ratio of these cross-sections is $\sigma_\gamma/\sigma_T = 1.7 \times 10^8$, and $n_H/n_p \approx 0.1$, so we can compare their activation energies to get a critical temperature:

$$\exp\left(-\frac{10.2\text{eV}}{kT}\right) > 6 \times 10^{-8} \implies T < 11100\text{K}. \tag{17.2}$$

So we see Balmer lines for $T \lesssim 10,000$K and no Balmer lines above that.

17.2 Spectral sequences

The original spectral sequence came from the relative strength of Balmer lines: from the strongest to the weakest Balmer lines, the sequence was ABFGKMO. Annie Jump Cannon discovered that the order in temperature was OBAFGKM. Why is there this inversion? More specifically, why do the hotter stars not appear to have hydrogen?

This was resolved using the Saha equation. Cecilia Payne-Gaposchkin determined that stars were composed primarily of hydrogen. The reason O stars have nonexistent hydrogen lines is because its hydrogen is all ionized. In the other direction, you have more hydrogen in the ground state and the opacity source switches to molecular absorption.

There are gradients within each of these categories. You can define these as O1 through O10, followed by B0 through B10, and so on. Increasing numbers are cooler. In recent years, we've added L, T, Y stars, which are even colder. There are also luminosity classes from I to V. V is a dwarf and I is a supergiant.

Hotter stars are called "early-type" stars and cooler stars are called "late-type" stars. This originates from when we thought luminosity came from Kelvin-Helmholtz contraction, but it doesn't actually have anything to do with age.

Type	T_{eff} (K)	$\frac{M}{M_\odot}$ (ZAMS)
O3	52,000	120
O8	36,000	23
B0	30,000	17
B5	15,400	6
A0	9520	3
G0	6030	1.05
M0	3850	0.5

Table 17.5: Effective temperatures and ZAMS masses of stars of different types

17.3 H^- opacity for cool stars

At low temperatures, hydrogen is neutral and alkali are ionized. This allows for the formation of H^-, and the second electron has $E = 0.75$ eV.

$$e^- + H \longleftrightarrow H^- + \gamma$$

We can plug this into the Saha equation:

$$\frac{n_H}{n_{H^-}} = \frac{n_{o,e}}{n_e} \exp\left(-\frac{0.75}{kT}\right). \tag{17.3}$$

If all alkali metals are singly ionized, $n_e = n_{alk} = 10^{13}\text{cm}^{-3}$, which gives us

$$\frac{n_H}{n_{H^-}} = 10^{-8} \exp\left(\frac{8700}{T}\right). \tag{17.4}$$

This is satisfied at $T = 470$K, but alkali atoms are ionized at 3000K, so this isn't actually feasible. We can do a more detailed calculation of where H^- abundance and free-electron abundance overlap, and we find that we actually get $T \sim 4000$ K.

17.4 Evolution on the main sequence

How long does a star spend on the main sequence? We can get this from $t_{MS} = \frac{E_{\text{nuc}}}{L}$: energy divided by energy per time. We can rewrite this:

$$t_{MS} = \frac{E_{\text{nuc}}/m_p}{E_{th}/m_p} t_{KH}. \tag{17.5}$$

Since $E_{\text{nuc}}/m_p \gg E_{th}/m_p$, the main sequence timescale is much longer than the Kelvin-Helmholtz timescale. Therefore, main sequence stars always have to be at hydrostatic equilibrium, because if they weren't, that would be a change on a timescale much shorter than the main sequence timescale we just calculated.

So what's changing on the main sequence? We're converting hydrogen to helium. This turns 8 particles into 3, which changes the pressure.

$$P = \frac{\rho k T}{\mu m_p} = \frac{\rho k T}{m_p}\left(2X + \frac{3}{4}Y\right). \tag{17.6}$$

For 25% helium (the cosmic mix), $\mu = 0.6$, and for 100% helium, $\mu = \frac{4}{3}$. So μ is changing by roughly a factor of 2.

Throughout this, we must have that $kT = \frac{GM\mu m_p}{R}$. For CNO, T is about constant, so we can say $R \propto \mu$. This affects the luminosity:

$$L = 4\pi R^2 \frac{c}{3\kappa\rho}\frac{1}{R}\sigma T^4. \tag{17.7}$$

Here, $\kappa = 0.2(1+X)\text{cm}^2\text{g}^{-1}$. Therefore

$$L \propto R^2 \frac{1}{1+X}\frac{R^3}{M}\frac{1}{R}T^4 \tag{17.8}$$

$$= \frac{1}{1+X}\frac{R^4}{M}\frac{M^4\mu^4}{R^4} \tag{17.9}$$

$$= M^3 \frac{\mu^4}{1+X}. \tag{17.10}$$

This reproduces the result we knew already, that $L \propto M^3$, but with abundances taken into account.

If we could change a star entirely to helium, it'd be 43 times as luminous as one with the cosmic mix of helium. In fact, helium only gets formed in the core, so luminosity increases but not this much.

CNO-burning stars move up and to the right (increased L, decreased Teff) on the H-R diagram. This doesn't hold for p-p stars. For these, we have $L_{\text{nuc}} \propto T^{4-6}$. As μ increases, T does too, and $L \propto \mu^4 M^3$ and $R \propto \mu T^{-1}$. As μ goes up, R is constant. This produces movement along the zero-age main sequence.

Age dating in open clusters is easy and in globular clusters is hard.

Stars and Planets I: Stellar Structure and Evolution **Winter 2023**

Lecture 18: Central hydrogen depletion, the S-C limit

Lecturer: Ryan Foley *1 March* *Aditya Sengupta*

18.1 Central hydrogen depletion

As you burn through hydrogen, it all turns into helium, and in high-mass stars, this helium is well-mixed on timescales much shorter than the burning. So high-mass stars run out of all of their hydrogen at once. At this point, the core continues to collapse. This happens on a Kelvin-Helmholtz timescale, and eventually hydrogen in the shell above the core will ignite. On the H-R diagram, this makes the star go through what's called the *Henyey hook*, where L increases slightly and T_{eff} decreases significantly.

In low-mass stars, the same change happens but more gradually.

18.2 Transition to the red giant branch

We have an inert helium core that contracts, and so the temperature rises. The hydrogen-burning shell equilibriates with the core, but the surface temperature remains low, so there's a large temperature gradient. Before convection starts, this gradient drives adiabatic expansion, so the envelope expands and the star becomes redder.

Eventually, the surface gets cold enough that alkali metals start to recombine. At the star's surface, we no longer have H^- opacity. So photons from further in that have been interacting with H^- in the envelope no longer have to do that, and so they escape fast. The surface is very efficient at removing energy from the system. So we need to supply it with energy somehow. This is achieved by creating a convective layer just below the surface.

The radius increases, and so does the luminosity. This means the convective layer grows, and eventually the convective zone reaches all the way down to the shell. These become big stars with a tiny core, and they're almost fully convective but for the core. The radius is set by H^- opacity. This is essentially the same as the Hayashi track.

The transition from the main sequence to the red giant branch is very fast, so it's very rare to find any stars in the space between them (the *Hertzsprung gap*.)

18.3 The Schönberg-Chandrasekhar limit

This is not the same as the Chandrasekhar limit, but it also has to do with masses of degenerate objects. The basic idea here is to say: once you've stopped hydrogen burning, your core is inert and has to be thermally supported. The weight of the envelope is significant for this. There's some maximum pressure corresponding to some critical core radius after which thermal support falls off.

If $M_c/M < 0.08$, there's a stable solution and the thermal pressure is sufficient. Above this, the core becomes degenerate. For these, $P = K\rho^{5/3}$, and a degenerate core is always sufficient to hold up the system.

For $M > 6\mathrm{M}_\odot$, we get that $M_c/M > 0.08$ at the end of hydrogen burning: the core is a substantial fraction of the star. The core "discovers" a lack of pressure support and quickly contracts over a timescale of

$$t_{\text{contract}} = \frac{GM^2/R}{L} \approx 10^6 \text{ years for } 6\mathrm{M}_\odot, 3 \times 10^4 \text{ years for } 30\mathrm{M}_\odot. \tag{18.1}$$

For $2 < \frac{M}{\mathrm{M}_\odot} < 6$, we meet the condition $M_c/M < 0.08$ at TAMS, so the core is isothermal during shell burning. For $\frac{M}{\mathrm{M}_\odot} < 2$, the core becomes degenerate before the limit. Eventually, every star gets a degenerate core, either one that'll explode or one that'll become a white dwarf.

The key thing for understanding our stars' degeneracy condition is the virial theorem, $T \propto M/R$, and combining this with the definition of density, we can say

$$T \propto M^{2/3}\rho^{1/3}. \tag{18.2}$$

The Fermi energy is proportional to n, which is proportional to ρ. With this in mind, we can think about the temperature of the core as a function of the density of the core. On this plot, you can plot a line over which $E_F \approx kT$. To its left the star is non-degenerate and to its right it's degenerate. We're interested in curves that cross this line, where there's helium burning to the left and hydrogen burning to the right.

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Lecture 19: The red giant branch, helium burning

Lecturer: Ryan Foley *3 March* *Aditya Sengupta*

19.1 The red giant branch

The red giant branch is defined by shell hydrogen burning. Red giant stars have an inert helium core and CNO burning. Their helium core mass grows and creates a big, puffy, convective envelope.

For a degenerate core, we know that $P_c \propto \frac{M^2}{R^4}$ and $P_{\text{deg}} \propto \rho^{5/3} = \frac{M^{5/3}}{R^5}$. Setting these equal, we get that $R \propto M^{-1/3}$. For a degenerate object like a white dwarf, this actually holds.

For a relativistic degenerate object, we have that $P_{\text{deg}} \propto \rho^{4/3}$, so if we do the same calculation the radii cancel. This means there's a set maximum mass a relativistic degenerate object can have, the *Chandrasekhar* mass. More massive degenerate stars have lower luminosities because their radii go down.

The shell temperature of a red giant is linked to the core temperature, which is in turn linked to the core mass. So the luinosity is linked to the core mass and essentially unlinked from the envelope. This is very different than the main sequence, where both the core and the envelope matter. Here, just the core tells you everything you need: shell burning sets the overall temperature and core mass sets the radius, so the luminosity is set by the core.

The shell must be geometrically thick in order to be thermally stable. We can see this in terms of the scale height?

$$H = \frac{kT}{m_p g} = \frac{kTR_c^2}{m_p GM_c} \sim R_c \tag{19.1}$$

Because the size of the shell is essentially R_c, the energy balance is set by

$$kT_{\text{shell}} \approx \frac{GM_c m_p}{R_c}. \tag{19.2}$$

The shell temperature is linked to the core mass. What we want is to study the star's luminosity, which is largely radiatively defined:

$$L_{\text{rad}} = 4\pi R_c^2 F_{\text{rad}} = 4\pi R_c^2 \left(\frac{1}{3}\frac{c}{\kappa\rho_s}\frac{1}{R_c} aT_s^4\right) \tag{19.3}$$

$$= \frac{4\pi}{3}\frac{R_c}{\kappa\rho_s} acT_s^4. \tag{19.4}$$

We can see this is produced through nuclear fusion, so $L_{\text{rad}} = L_{\text{nuc}} = \int \epsilon \mathrm{d}M = \epsilon_{CNO} 4\pi R_c \rho_s$.

So what sets ϵ_{CNO}? This is given by $\epsilon_{CNO} = \epsilon_0 \rho T^{\nu}$. Given this, we can find

$$\rho_s = \left(\frac{4}{3} \frac{\sigma T_s^{4-\nu}}{\epsilon_0 R_c^2 \kappa} \right)^{1/3}. \tag{19.5}$$

Then

$$L \propto \frac{R_c}{\rho_s} T_s^4 \propto R_c \left(\frac{R_c^2}{T_s^{4-\nu}} \right)^{1/3} T_s^4 = R_c^{5/3} T_s^{(8+\nu)/3}. \tag{19.6}$$

We want to eliminate temperature, which we can do using $T_s \propto \frac{M_c}{R_c}$. Substituting this in gives us

$$L \propto M_c^{3+\frac{4\nu}{9}} \tag{19.7}$$

The luminosity of the red giant branch depends only on the core mass. The exponent in this relationship is about 12 because ν is about 20.

If we trace out tracks on the H-R diagram, we can find that stars with different masses but the same core mass end up at the same point. This is called "core convergence". It mostly happens for $M < 2\mathrm{M}_{\odot}$ stars.

We can get some scaling relations:

$$R_c = 2 \times 10^9 \text{ cm} \left(\frac{M_c}{0.1\mathrm{M}_{\odot}} \right)^{-1/3} \tag{19.8}$$

$$T_s = 2 \times 10^7 \text{ K} \left(\frac{M_c}{0.1\mathrm{M}_{\odot}} \right)^{4/3} \tag{19.9}$$

$$L = \mathrm{L}_{\odot} \left(\frac{M_c}{0.16\mathrm{M}_{\odot}} \right)^{7.3}. \tag{19.10}$$

We can combine these to get a mass increase rate:

$$\dot{M}_c = \frac{L}{E_{\text{nuc}}/m_p} = 6 \times 10^{14} \frac{\text{g}}{\text{s}} \left(\frac{M_c}{0.16\mathrm{M}_{\odot}} \right)^{6.3}. \tag{19.11}$$

The core grows faster at higher masses. So there are fewer stars at higher L because the higher the core mass gets, the faster you grow.

What's the timescale for this change?

$$t = \frac{M_c}{\dot{M}_c} = 10^{10} \text{ years} \left(\frac{M_c}{0.16\mathrm{M}_{\odot}} \right)^{-6.3}. \tag{19.12}$$

For $M_c = 0.25 M_\odot$ this is 10^9 years, and for $M_c = 0.4 M_\odot$ this is 53×10^6 years.

The helium core gets denser ($\rho \propto M^2$) and hotter until helium burning stars. The red giant branch stops at this point. This occurs when $M_c \sim 0.42\mathrm{M}_\odot$. At this point, $L = 10^{3.3}\mathrm{L}_\odot$. Since we know this luminosity pretty precisely, we can use the tip of the red giant branch (TRGB) as a standard candle to measure distances.

For $M < 2\mathrm{M}_\odot$, this is the end of the RGB. For $M > 2\mathrm{M}_\odot$, the core collapses when $M_c/M < 0.08$ and this is before TRGB, so we're never that luminous.

We created a degenerate object with a helium core. If we remove the envelope, we have a helium white dwarf. A low-mass single star has $t_{MS} > \frac{1}{H_0}$, so we can't create white dwarfs. We can get helium white dwarfs in binaries and speed this up with mass transfer. We often see helium white dwarf companions, and we often see helium white dwarf companions to neutron stars.

19.2 Helium burning

At the end of the RGB, the core is helium-4 and residuals of CNO, mostly nitrogen-14. The problem is beryllium-8 is unstable, with a half-life of about 10^{-15}s. If you can fuse two helium-4s, it'll just decay right back. Helium-4 is very stable and it's hard to energetically get past.

At high temperatures, we can have small amounts of beryllium-8. The Gamow energy is

$$E_G = (4\pi\alpha)2m_r c^2 \tag{19.13}$$

$$m_r = \frac{m_\alpha m_\alpha}{2m_\alpha} = \frac{1}{2}m_\alpha \tag{19.14}$$

This gives us $E_G = 31.4$ MeV.

$$E_0 = \left(\frac{E_G(kT)^2}{4}\right)^{1/3} = 83\ \mathrm{keV} T_8^{2/3}. \tag{19.15}$$

At $T \sim 10^8$ K, KE is comparable to...something.

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Lecture 20: Helium burning

Lecturer: Ryan Foley *6 March* *Aditya Sengupta*

20.1 Helium burning

We want to apply the Saha equation to helium burning. We have

$$\mu_\alpha + \mu_\alpha = \mu_{^8\mathrm{Be}} \tag{20.1}$$

$$\mu_\alpha = n_\alpha c^2 - kT \ln\left(\frac{g n_{Q,\alpha}}{n_\alpha}\right) \tag{20.2}$$

$$\mu_{^8\mathrm{Be}} = m_{^8\mathrm{Be}} c^2 - kT \ln\left(\frac{g n_{Q,^8\mathrm{Be}}}{n_{^8\mathrm{Be}}}\right) \tag{20.3}$$

Therefore we get

$$\frac{n_{^8\mathrm{Be}}}{n_\alpha^2} = \left(\frac{2h^2}{2\pi m_\alpha kT}\right)^{3/2} \exp\left(-\frac{(m_{^8\mathrm{Be}} - 2m_\alpha)c^2}{kT}\right) \tag{20.4}$$

Taking $\rho = 10^4$ g/cm^3 (for pure helium, like RGB cores), we can find that $n_\alpha = \frac{\rho}{4m_p}$ and we can solve for the relative concentrations of helium and beryllium. At $T_8 = 2$, we get $\frac{n_{^8\mathrm{Be}}}{n_\alpha} = 10^{-8}$, which isn't sufficient. But the energy released by fusing helium-4 and beryllium-8 is almost exactly the energy needed to reach the O$^+$ excited state of carbon-12, which we'll refer to as ^{12}C*.

The energy difference between ^{4}He + ^{8}Be and ^{12}C* is only 280 keV.

To get to the ground state of carbon-12, we can release a photon, but this process is slow relative to strong decay. But if this photon is emitted, carbon-12 is very stable/low energy, so it doesn't decay and it's taken out of the chain. So we get the *triple alpha* process,

$$^4\mathrm{He} + {}^4\mathrm{He} + {}^4\mathrm{He} \longleftrightarrow {}^{12}\mathrm{C}^*$$

We can apply the Saha equation here as well.

$$\frac{n_{^{12}C^*}}{n_\alpha} 5.2 \times 10^{-10} \left(\frac{\rho}{10^5 \mathrm{g/cm}^3}\right) T_8^{-3} \exp\left(-\frac{44}{T_8}\right). \tag{20.5}$$

The -44 comes from balancing the energies of carbon and helium with te 280 keV. After this, we undergo decay,

$$^{12}\text{C}^* \longrightarrow {}^{12}\text{C}.$$

This has a decay time of $\tau = 1.8 \times 10^{-13}$s. This step releases 7.64 MeV, so it provides

$$\epsilon = 5.3 \times 10^{21} \text{erg/g/s} \left(\frac{\rho}{10^5 \text{g/cm}^3} \right)^2 T_8^{-3} \exp\left(-\frac{44}{T_8} \right). \tag{20.6}$$

We can also consider alpha capture onto carbon:

$$^{12}\text{C} + \alpha \longrightarrow {}^{16}\text{O} + \gamma$$

This has a similar required temperature as triple-alpha. Not all of the carbon burns, though; we get a mix of carbon and oxygen. We can keep going:

$$^{16}\text{O} + \alpha \longrightarrow {}^{20}\text{Ne} + \gamma$$
$$^{20}\text{Ne} + \alpha \longrightarrow {}^{24}\text{Mg} + \gamma$$

But these require higher temperatures.

Reaction	E_G (MeV)	$E_0(T = 2T_8)$ (keV)	$\exp\left(-(\frac{E_G}{4kT})^{1/3}\right)$
3α (He+He/Be+He)	31/168	132/231	10^{-16} / 10^{-26}
^{12}C + α	424	315	10^{-24}
^{16}O + α	800	390	10^{-37}
^{20}Ne + α	1300	460	10^{-44}

Table 20.6: Reaction energies for α capture reactions

20.2 Helium core flash

For $M < 2\text{M}_\odot$, helium ignites when $M_c \sim 0.45\text{M}_\odot$. This leads to core degeneracy and thermonuclear runaway. As T increases, ρ remains constant until degeneracy is lifted. If $M_c = 0.45\text{M}_\odot$, we get $R = 10^9$ cm and $E_g = 3 \times 10^{49}$ erg. Triple-alpha can provide more than enough energy for this: we only need to burn 5% of the core to unbind it.

However, this doesn't happen because the burning happens slowly enough that the core doesn't unbind. For Type Ia supernovae and C/O white dwarfs, we need to burn 50% to unbind. A helium flash is a local runaway process whereas Type Ia supernovae is a global runaway process.

A helium flash causes temperature to drop off more drastically with increasing radius. Stars move straight up on the H-R diagram in what's called the *horizontal branch*.

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Lecture 21: The asymptotic giant branch

Lecturer: Ryan Foley *8 March* *Aditya Sengupta*

When you've gone through the main sequence (core hydrogen burning), the red giant branch (shell hydrogen burning), and the horizontal branch/red clump (core helium burning), we're left with a carbon and oxygen core, and a hydrogen shell right above. We then go through shell hydrogen burning again, creating helium that also burns. This happens in successive cycles: we burn hydrogen till there's enough helium, then helium burning turns off the hydrogen burning, and when we're out of helium we go back to hydrogen. This creates the *asymptotic giant branch*. At the end of the AGB, we have a carbon/oxygen core, a helium shell, and a hydrogen shell around that.

The helium shell burns completely when it's thin. This creates a temperature perturbation and we want it not to change the pressure.

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho\frac{Gm(r)}{r^2} \tag{21.1}$$

$$\frac{\Delta P}{R} \approx \frac{P_c}{R} \propto \frac{M^2}{R^4} \tag{21.2}$$

and since we have $T \propto \frac{M}{R}$, we can say $P_c \propto \frac{T^4}{M^2}$. For a geometrically thin shell, $r_{\text{shell}} \ll r$ and $m_{\text{shell}} \ll m(r)$, so

$$P = -\frac{GM_c}{R_c^2}\int_{R_c}^{\infty}\rho(r)\mathrm{d}r. \tag{21.3}$$

We can write down an equation for energy balance, assuming everything physical (T, P, etc.) is constant in the shell:

$$C_p\frac{\mathrm{d}T}{\mathrm{d}r} = \epsilon_{3\alpha} - \frac{acT^4}{3\kappa y^2} = \epsilon_{3\alpha} - \epsilon_{\text{cool}}. \tag{21.4}$$

Note that we can consider an appreciably-sized $\frac{\mathrm{d}T}{\mathrm{d}r}$ while still assuming the shell is approximately isothermal, because the shell is sufficiently small that the gradient doesn't have sufficient space to act over.

Substituting in known power-laws for triple-alpha and cooling, we get

$$C_p\frac{\mathrm{d}T}{\mathrm{d}r} = \epsilon_0\left(\frac{T}{T_0}\right)^{\nu} - \epsilon_0'\left(\frac{T}{T_0}\right)^4. \tag{21.5}$$

This is unstable if $\nu \gg 4$. For triple-alpha, $\nu = \frac{44}{T_8}$, so it's thermally unstable. Therefore, we see a helium shell flash.

The timescale of each hydrogen and helium burning cycle is small, about 10^5 years, and it gets shorter the longer the process runs. In this process, we "dredge up" core carbon and oxygen from the core and since the envelope is convective it may reach the surface.

For stars with $M_{\text{ZAMS}} \lesssim 6\text{M}_\odot$, we can't ignite carbon and we end up with a white dwarf. White dwarfs can't have masses greater than 1.4 $\text{M}_\odot$. Where did the mass go? We need significant mass loss - how much?

Let's say we have a luminosity of $L = 6 \times 10^4 \text{L}_\odot$. The timescale of mass loss is

$$\tau = \frac{M_{C/O}}{\dot{M}_{C/O}} = 4 \times 10^6 \text{ years} \tag{21.6}$$

AGB star surfaces are cold and can form molecules. We have carbon and oxygen. If there's equal carbon and oxygen, we can produce CO. If $O > C$, there's leftover O; if $C > 0$ we get graphite. This creates dust, which in turn creates high opacity and strong winds. Most ISM dust comes from AGB winds.

For $M < 2\text{M}_\odot$, $M_{C,\text{He}} \sim 0.45\text{M}_\odot$. We find that there's a peak at around $0.6\text{M}_\odot$, and a slight bump at $1.2\text{M}_\odot$ which we think is due to WD-WD mergers.

In the final stages of an AGB star, we blow off the last bits of the envelope and produce a planetary nebula (which has nothing to do with planets). At the center of a planetary nebula, there's a hot compact object but it's less luminous, which creates a turnoff to the white dwarf region of the H-R diagram.

How do white dwarfs cool, if they start on the AGB/as planetary nebulae where they're very hot? We start at $T \sim 10^8$ K, which is set by hydrogen/helium burning, and we cool as energy diffuses out. Applying some scaling relations,

$$R \propto M^{-1/3} \tag{21.7}$$

$$L \propto R^2 T_{\text{eff}}^4 \propto M^{-2/3} T_{\text{eff}}^4. \tag{21.8}$$

Over time the white dwarfs cool by a known cooling law.

How about higher-mass stars? Over about $6\text{M}_\odot$, we form magnesium by

$$^{12}\text{C} + {}^{12}\text{C} \longrightarrow {}^{24}\text{Mg} + \gamma.$$

Magnesium-24 decays into neon-20 + α, sodium-23 + a proton, or magnesium-23 + a neutron. This requires $T \gtrsim 7 \times 10^8$ K. Here, neutrino cooling becomes important. We do pair creation (?) and occasionally this decays into neutrinos.

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Lecture 22: Missing

Lecturer: Ryan Foley *10 March* *Aditya Sengupta*

I think I was sick

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Lecture 23: Supernovae and core-collapse

Lecturer: Ryan Foley *13 March* *Aditya Sengupta*

Last time, we wrote down this table:

Element e	$15M_\odot$, $10^4L_\odot$, neutrino lum / photon lum	Time (years)	$25M_\odot$

Table 23.7: <caption>

Neutrino luminosity is very bright for very short times. The typical supernova rate in any given galaxy is about one a century.

The dynamical timescales of massive stars are about 4 seconds, much shorter than the neutrino-burning timescale. So we're still in hydrostatic equilibrium. However, these timescales are shorter than the timescale for reaching thermal equilibrium. We need to track T closely, as it's possible to reach all burning stages at once. In order going from the inside out, we have Si, O, Ne, C, He, H burning.

There's some reactions that basically mean we keep adding 2 protons at a time via alpha capture, and eventually we get to nickel-56. The Saha equation for this step is (something).

Silicon burning is easy once we hit $T \gtrsim 5 \times 10^9$ K. What about nickel burning?

Nickel-56 is radioactive and decays through the weak process. The way this works is:

$$
\begin{aligned}
{}^{56}\text{Ni} &\longrightarrow {}^{54}\text{Co} + \gamma + \nu_e \\
{}^{56}\text{Co} &\longrightarrow {}^{52}\text{Fe} + \gamma + \nu_e.
\end{aligned}
$$

The half-life of the first reaction is about 6 days, and that of the second is about 77.1 days. At the core of the star, atoms are ionized and there are free electrons everywhere. The density is so high that decays happen very fast. We have a degenerate iron core, and as silicon burning continues we keep adding mass. Eventually, we hit the Chandrasekhar mass. For iron cores, this is $1.26M_\odot$.

The pressure for non-relativistic degenerate objects is equal to $P = \frac{2}{5}n_e E_F = \frac{2}{5}n_e \frac{p_f^2}{2m_e}$, i.e.

$$
P = n_e^{5/3}\left(\frac{3h^3}{8\pi}\right)^{2/3}\frac{1}{5m_e}. \tag{23.1}
$$

We also know that the gas is fully ionized, so $\rho = Am_p n_i$, so

$$
P = \left(\frac{\rho}{\mu_e m_p}\right)^{5/3}\left(\frac{3h^3}{8\pi}\right)^{2/3}\frac{1}{5m_e}. \tag{23.2}
$$

(something about central density)

If we assume $\rho \approx \frac{M}{R^3}$ and $\mu_e = 2$,

$$R = 2 \times 10^9 \mathrm{cm} \left(\frac{M}{0.1 \mathrm{M}_\odot} \right)^{-1/3} \frac{n_e}{n_H ?} \tag{23.3}$$

doing more proportionality things, we can say

$$M_{Ch} = 1.456 \left(\frac{2}{\mu_e} \right)^2 \mathrm{M}_\odot. \tag{23.4}$$

Lecture 24: Core collapse/explosions

Lecturer: Ryan Foley *15 March* *Aditya Sengupta*

At the end of a star's life, mass increases and temperature increases, so the star becomes relativistic. Neutrino cooling implies core contraction, which leads to the temperature going up. Iron dissociates:

$$\gamma + {}^{56}\mathrm{Fe} \longleftrightarrow 13\,\alpha + 4\,\mathrm{n}$$

For this reaction, we have $Q = (13m_\alpha + 4m_n - m_{^{56}Fe}) = 124$ MeV.

Going through the Saha equation, we have

$$\rho_9^1 6 = 9.5 \times 10^{65} T_{10}^{24} \exp\left(-\frac{144}{T_{10}}\right). \tag{24.1}$$

The combination of these shows that the temperature dependence is pretty small:

ρ_9	T_{10}
0.1	0.78
1	0.95
10	1.2

Table 24.8: Temperature dependence of ρ_9

So for all of these we have $T \sim 10^{10}$ K and $kT = 1$ MeV. We get an energy loss of about 2 MeV per nucleon. Further contraction beyond what's caused by this energy limit causes (something).

Breaking up of αs requires 7 MeV per nucleon, which implies $R \approx 100$ km. We still have electrons and protons in hydrostatic equilibrium.

As $E_F F$ increases, these react according to

$$\mathrm{e}^- + \mathrm{p} \longrightarrow \mathrm{n} + \nu_e.$$

This happens if $E_F > (m_n - m_p)c^2 = 1.3$ MeV, which we know is about where we reach. $M_{Ch} = 1.26\mathrm{M}_\odot\left(\frac{\mu_{Fe}}{\mu_e}\right)$. This drops below the mass of the iron core at some point, and the core collapses. This continues until there's other pressure support, from neutron degeneracy pressure. This happens in 0.1 seconds. Almost the entire gravitational energy, $E_G = \frac{GM^2}{R} = 10^{53}$ erg, gets released as neutrinos.

At this point, the rest of the star "discovers" that there is no pressure support, so it undergoes a shock. All the energy in this shock will be used up in breaking up iron, so the prompt shock stalls at $R = 10^8$cm. The proto-neutron star is hot so there's material right above heat, meaning that there's convection.

Neutrino timescales: (something)

SN 1987A is a core-collapse supernova, and we detected 24 neutrinos from it. This flux was one-sixth of E_G because we were detecting only one of the six neutrino flavors. We were able to measure an effective temperature of $T_{\text{eff}} = 5 \times 10^{10}$K.

Nucleosynthesis?